Iranian Journal of Numerical Analysis and Optimization
Vol. 8, No. 1, (2018), pp 19–39
DOI:10.22067/ijnao.v8i1.49742

A new dwindling nonmonotone filter method without gradient information for solving large-scale systems of equations

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Abstract

In this paper, we present a new derivative-free spectral residual method for solving large-scale systems of equations. Our algorithm is equipped with a dwindling multidimensional nonmonotone filter in which whose envelope is dwindling as the step-length of line search is decreasing. The proposed algorithm is also combined with a relaxed nonmonotone line search technique which allows the algorithm to enjoy the nonmonotone property from scratch. Under some standard assumptions, the global convergence property of the proposed algorithm is established. Numerical results on some test problems show the efficiency and effectiveness of the new algorithm in practice.

Keywords: Dwindling filter technique; Systems of equations; Nonmonotone line search; Global convergence.

1 Introduction

In this paper, we deal with the following nonlinear systems of equations:

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Received 9 September 2015; revised 12 April 2016; accepted 12 July 2017
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$$F(x) = 0, \tag{1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable map. We assume that (1) is symmetric; that is, its Jacobian is a symmetric matrix. Associated with (1), the norm function is defined by

$$f(x) = \frac{1}{2}\|F(x)\|^2.$$

Several methods in the literature have been proposed for solving (1), such as Newton and quasi-Newton methods, Gauss–Newton method, Levenberg–Marquardt method [20], trust-region methods, spectral gradient, and residual methods [10, 24] and conjugate gradient methods [9]. The main factor in these methods is how the procedure deals with large-scale settings. The spectral gradient and conjugate gradient methods are a class of methods that can suitably cope with large-scale systems of equations. The spectral gradient method was first introduced in [7] for unconstrained optimization and has been successfully extended for solving large-scale nonlinear systems of equations; see, for example, [25, 29], and the references therein.

The so-called nonmonotone line search techniques for unconstrained optimization was introduced by Grippo, Lampariello, and Lucidi in [17]. In their approach, at the point x_k, for a given positive integer M, the stepsize $\lambda \in (0, 1]$ is chosen such that

$$f(x_k + \lambda d_k) \le \max_{0\le j\le \min\{k,M\}} f(x_{k-j}) + \gamma\lambda\nabla f(x_k)^T d_k,$$

where $\gamma \in (0, 1)$ is a given scalar. An extension and adaption of this technique in the framework of a globalization technique for the Barzilai-Browein (BB) gradient method has been proposed in [18]. It is well known that the Grippo's nonmonotone technique suffers from some drawbacks; some of them have been listed in [1]. In order to overcome these difficulties, Ahookhosh, Amini, and Peyghami in [2] introduced a nonmonotone term as follows:

$$R_k = \epsilon_k f_{\ell(k)} + (1 - \epsilon_k) f_k, \tag{2}$$

where

$$f_{\ell(k)} = \max_{0\le j\le \min\{k,M\}} f_{k-j}, \tag{3}$$

$f_k = f(x_k)$, and $\epsilon_k \in [\epsilon_{\min}, \epsilon_{\max}] \subset [0, 1]$. The motivation behind this nonmonotone term is that the best convergence results are obtained by stronger nonmonotone strategy whenever the iterates are far from the minimizer and by weaker nonmonotone strategy whenever iterates are close enough to that, see, for example, [30]. Although, (2) enjoys the strong and weak nonmonotone properties; it does not allow the iterates to act as a nonmonotone iterate in the first iterations. Ataee Tarzanagh, Saeidian, Peyghami, and Mesgarani in [6] proposed a nonmonotone term by replacing R_k with $(1 + \psi_k)R_k$ in the trust region ratio, where ψ_k is determined by

$$\psi_k = \begin{cases} \eta_k & \text{if } R_k > 0, \\ 0 & \text{if } R_k \leq 0, \end{cases}$$

and $\{\eta_k\}$ is a positive sequence satisfying the following condition:

$$\sum_{k=1}^{\infty} \eta_k \leq \eta < \infty. \tag{4}$$

This substitution causes an increase in the function value in the first iterations.

Since the Jacobian of (1) might not available or requires a prohibitive amount of storage, derivative-free methods have been widely developed for solving large-scale nonlinear systems of equations. For the square systems, La Cruz, Martínez, and Raydan in [24] proposed a derivative-free spectral residual method, named as DF-SANE, based on a combination of the Li and Fukushima's line search [26] and the Grippo's nonmonotone technique. In their method, the step-length α_k is determined by one and only one of the following two conditions:

$$\begin{aligned} f(x_k - \alpha_k d_k) &\leq f_\ell(k) + \eta_k - \gamma \alpha_k \|F(x_k)\|^2, \\ f(x_k + \alpha_k d_k) &\leq f_\ell(k) + \eta_k - \gamma \alpha_k \|F(x_k)\|^2, \end{aligned}$$

where $d_k := \sigma_k F(x_k)$ and σ_k is the BB spectral coefficient, which is computed by [7]

$$\sigma_k = \frac{s_k^T s_k}{y_k^T s_k},$$

where $s_k = x_{k+1} - x_k$ and $y_k = F(x_{k+1}) - F(x_k)$.

Several variants of DFSANE algorithm for the square systems have been proposed and applied on some specific systems of equations; see, for example, [21–23]. Cheng and Li in [11] introduced another derivative-free spectral residual method for the square systems, named as NDFSANE, in which the nonmonotone term $f_{\ell(k)}$ is replaced by that of proposed in [30]. Later, the performance of DFSANE algorithm for the square systems is improved in SDFSANE algorithm [10] by using an approximation of the steepest descent direction.

Recently, La Cruz in [22] presented a residual spectral algorithm for solving monotone equations on the Hilbert space which is an extension of the SANE and DFSANE algorithms for systems of equations. This algorithm uses in a systematic way the residual $d = F(x_k)$ as a search direction, combined with a suitable step-length and a nonmonotone line search globalization strategy. The main objective of this algorithm is to guarantee the convergence without any necessity to the differentiability of F.

The concept of filter for constrained optimization problems was first introduced by Fletcher and Leyffer in [15] in order to avoid the difficulty of adjusting penalty parameter in penalty methods. Accepting/rejecting of a trial step in the filter methods is inspired from the same idea in multiobjective optimization. Indeed, a trial point is accepted by the filter if it is not dominated by any other points in the filter. In this situation, the filter is updated by removing all the points that are dominated by the new point; see, for example, [15, 16, 27], and the references therein. The fixed envelope of a multidimensional filter keeps an acceptable trial point far from iterates in the filter. However, the trial point with a tiny step-length may get close to the current iterate and be rejected by the filter due to its fixed envelope. Due to this fact, Chen and Sun in [8] proposed a dwindling multidimensional filter line search method for unconstrained optimization. Their idea is to incorporate step-length in the framework of multidimensional filter, and let the envelope of the filter become thin as the step-length approaches zero. Recently, Arzani and Peyghami in [4] proposed a dwindling filter technique for solving positive definite generalized eigenvalue problem.

Our aim in this paper is to introduce a new dwindling nonmonotone filter DFSANE method for solving large-scale nonlinear systems of equations. Our approach is a new version of DFSANE algorithm which is installed by a filter technique. The concept of filter leads to save some points that are eliminated by the DFSANE algorithm. The accumulated points in the filter helps the algorithm to reach the optimal point as quickly as possible. Moreover, our approach uses some relaxations in its structure and is equipped with the nonmonotone term as proposed in [6]. Besides, by using dwindling technique, we have more flexibility for the acceptance of the trial step. Under some standard and suitable assumptions, the global convergence property of the new proposed algorithm is constructed. Numerical results on some test problems validate that the proposed algorithm is practically efficient and robust and it has some priorities to that provided in [31].

The rest of the paper is organized as follows. Section 2 is devoted to describe the new algorithm in details. In Section 3, we establish the global convergence property of the new proposed algorithm under some suitable assumptions. Some preliminary numerical results of applying the new algorithm on some test problems are given in Section 4. Finally, we end up the paper by some concluding remarks in Section 5.

2 The new algorithm

In this section, we propose a new dwindling nonmonotone filter DFSANE algorithm for solving large-scale nonlinear systems of equations. To do so, we first recall the filter that is used in this paper and then describe the new algorithm.

Recently, Fatemi and Mahdavi-Amiri in [13] introduced a new filter that controls the size of the filter. In their proposed filter, a tentative point x_k is acceptable with respect to x_l if there exists $j \in \{1, \dots, n\}$ such that

$$|g_j(x_k)|^{\mu_2} + \theta_2 \|g(x_k)\|^{\mu_1} \leqslant |g_j(x_l)|^{\mu_2} + \theta_1 \|g(x_l)\|^{\mu_1}, \tag{5}$$

where $g(x) = \nabla f(x)$, μ_1, and μ_2 are positive constants and θ_1 and θ_2 satisfy the following relation:

$$0 \leqslant \theta_1 < \theta_2 < \frac{1}{\sqrt{n}}. \tag{6}$$

Therefore, a point x_k is accepted to the filter $\mathcal{F}$ if it is accepted with respect to every x_l with $g(x_l) \in \mathcal{F}$. In the case of acceptance of x_k, $g(x_k)$ is added to the filter and any $g(x_l) \in \mathcal{F}$ with the following property is removed:

$$|g_j(x_k)|^{\mu_2} + \theta_2 \|g(x_k)\|^{\mu_1} \leqslant |g_j(x_l)|^{\mu_2} + \theta_1 \|g(x_l)\|^{\mu_1} \qquad \forall j \in \{1, \dots, n\}.$$

In our dwindling filter, the parameters θ_1 and θ_2, satisfying (6), are considered as a function of the step-length. Since the step-length α in line search techniques is normally less than or equal to one, in our setting, we use the following parameters in the filter (5):

$$\hat{\theta}_1 = \phi(\alpha)\theta_1 \quad \text{and} \quad \hat{\theta}_2 = \phi(\alpha)\theta_2,$$

where θ_1 and θ_2 satisfy (6) and $\phi : [0, 1] \to \mathbb{R}^+$ is a monotonically increasing continuous function with the following properties:

$$\begin{aligned} &\lim_{\alpha \to 0^+} \frac{\phi(\alpha)}{\alpha} = 0, \\ &\phi(\alpha) = 0 \quad \text{if and only if} \quad \alpha = 0 \end{aligned} \tag{7}$$

such that (6) holds for $\hat{\theta}_1$ and $\hat{\theta}_2$.

Now, let us briefly describe one iteration of our new algorithm. Given x_k, if the stopping criteria do not hold, the trial step $d_k = -\sigma_k F_k$ is computed where σ_k is the so called spectral coefficient. Assuming $\alpha_+ = \alpha_- = 1$, the trial point $x^+_{k+1} = x_k + \alpha_+ d_k$ and $x^-_{k+1} = x_k - \alpha_- d_k$ are introduced and with the priority of x^+_{k+1} are verified for acceptance to the filter. In case of rejecting both trial points by the filter, these points with the same priority are checked for acceptance by a nonmonotone line search conditions. If the trial points are rejected by the filter and line search conditions, then a backtracking scheme is done on α_+ and α_- and the above mentioned procedure is repeated until one of the trial points is accepted by the filter or nonmonotone line search conditions.

The structure of the new proposed algorithm is outlined in Algorithm 1.

Algorithm 1. *: A new dwindling derivative free filter algorithm*

Step 0. *Given* $x_0 \in \mathbb{R}^n$*, integer* $M \geq 0$*,* $0 < \gamma < 1$*,* $\epsilon > 0$*,* $0 < \sigma_{\min} < \sigma_{\max} < \infty$*,* $0 \leqslant \theta_1 < \theta_2 < \frac{1}{\sqrt{n}}$*,* $0 \leq \tau_{\min} < \tau_{\max} < 1$*, and a monotonically increasing continuous function* ϕ *satisfying* (7)*; set* $\mathcal{F}_0 = \emptyset$ *and* $k := 0$*.*

Step 1. *If* $\|F(x_k)\| \leq \epsilon$*, then stop.*

Step 2. *Choose the spectral coefficient* σ_k *such that* $|\sigma_k| \in [\sigma_{\min}, \sigma_{\max}]$*, and compute* $f_{\ell(k)}$ *by using* (3)*. Set* $d_k = -\sigma_k F(x_k)$*,* $\alpha_+ = 1$ *and* $\alpha_- = 1$*.*

Step 3. *Compute* $x_{k+1}^+ = x_k + \alpha_+ d_k$*,* $x_{k+1}^- = x_k - \alpha_- d_k$*, and* R_k *by* (2)*.*
If x_{k+1}^+ *is accepted by the filter, then set* $x_{k+1} = x_{k+1}^+$ *and add* x_{k+1} *to the filter* $\mathcal{F}_k$*. Update the filter to form* $\mathcal{F}_{k+1}$*. Set* $k := k+1$*,* $\alpha_k = \alpha_+$*, and goto Step 1.*
Else if x_{k+1}^- *is accepted by the filter, then set* $x_{k+1} = x_{k+1}^-$ *and add* x_{k+1} *to the filter* $\mathcal{F}_k$*. Update the filter to form* $\mathcal{F}_{k+1}$*. Set* $k := k+1$*,* $\alpha_k = \alpha_-$*, and goto Step 1.*
Else if
$$f(x_{k+1}^+) \leq (1+\psi_k)R_k - \gamma {\alpha_+}^2 f(x_k), \tag{8}$$
then set $x_{k+1} = x_{k+1}^+$*,* $k := k+1$*,* $\alpha_k = \alpha_+$*, and goto Step 1.*
Else if
$$f(x_{k+1}^-) \leq (1+\psi_k)R_k - \gamma {\alpha_-}^2 f(x_k), \tag{9}$$
then set $x_{k+1} = x_{k+1}^-$*,* $k := k+1$*,* $\alpha_k = \alpha_-$*, and goto Step 1.*

Step 4. *Choose* $\alpha_+ \in [\tau_{\min}\alpha_+, \tau_{\max}\alpha_+]$ *and* $\alpha_- \in [\tau_{\min}\alpha_-, \tau_{\max}\alpha_-]$*, and goto Step 3.*

Remark 1. Since $\psi_k > 0$, $R_k > 0$, and $R_k \geq f(x_k)$, due to Lemma 2.2 in [2], it is easily seen that after finite number of reductions of α_+ or α_- in Step 4 of Algorithm 1, one of the conditions (8) and (9) is necessarily satisfied. This shows that the proposed algorithm is well-defined.

3 Convergence analysis

In this section, our goal is to analyze the global convergence property of Algorithm 1. For this purpose, let

$$P1 = \{k | x_k \in \mathcal{F}_k\},$$

and let

$$P2 = \{k | x_k \text{ satisfies (8) or (9)}\}.$$

It is easily seen that $P1 \cap P2 = \emptyset$. Before going further, we first consider the following assumptions on the problem (1):

A1. F is a continuously differentiable function map over $\mathbb{R}^n$.

A2. For a given η, as in (4), the level set $L^0 = \{x \in \mathbb{R}^n | f(x) \leq e^{\eta} f(x_0)\}$ is closed and bounded.

A3. The sequence $\{x_k\}$, generated by Algorithm 1, is contained in a bounded and closed set $\Omega \subset \mathbb{R}^n$. Let $\{x_k\}$ be the sequence generated by Algorithm 1. Then, Algorithm 1 either stops at a certain iteration k_0 with $\|F(x_{k_0})\| = 0$, or generates an infinite sequence. In the latter case, it is shown that there exists a subsequence of $\{x_k\}$ which converges to a stationary point of F. Therefore, from now on, we assume that $|P1 \cup P2| = \infty$. Two possibilities are recognized for the cardinality of $P1$ and $P2$, $|P1| = \infty$ or $|P2| = \infty$.

3.1 Analysis of the case $|P1| = \infty$

Consider the filter $\mathcal{F}_k$ and its acceptance criterion, which is given by (5). Due to Lemma 4 in [13], the size of $\mathcal{F}_k$ is finite. Although, $\mathcal{F}_k$ contains finite number of elements; it may happen that an infinite number of iteration points is accepted by the filter. The following lemma states the behaviour of the algorithm in this case.

Lemma 1. *Let Assumptions A1–A3 hold, and let $|P1| = \infty$. Then,*

$$\lim_{k \to \infty,\, k \in P1} \|F_{k+1}\| = 0. \tag{10}$$

Proof. The proof is similar to the proof of Lemma 5.4 in [14] by setting $\theta(x) = F(x)$ and using the fact that based on the definition of $\phi(\alpha)$ and the initial values of $\alpha_+ = \alpha_- = 1$ in Step 2, the condition $0 \leqslant \theta_1 < \theta_2 < \frac{1}{\sqrt{n}}$ implies that $0 \leqslant \hat{\theta}_1 < \hat{\theta}_2 < \frac{1}{\sqrt{n}}$. □

From Lemma 1, for the case $|P1| = \infty$, it is concluded that there exists a limit point of subsequence $\{x_k\}_{k\in P1}$ which is a stationary point of f; that is,

$$\lim_{k\to\infty,\, k\in P1} \inf \|F_k\| = 0. \tag{11}$$

3.2 Analysis of the case $|P2| = \infty$

First of all, without loss of generality, we may assume that $|P2| = \infty$ and that $|P1|$ is finite, as for the case $|P1| = \infty$ (11) holds. Under this assumption, we show that there exists a limit point of subsequence $\{x_k\}_{k\in P2}$ satisfying the first-order necessary condition; that is,

$$\lim_{k\to\infty,\, k\in P2} \inf \|F_k\| = 0.$$

Since $|P1|$ is finite and $|P2| = \infty$, there exists a positive integer M_0 such that for every $k \geq M_0$, we have $k \in P2$. Without loss of generality and in order to establish simple analysis, in what follows, we assume that $M_0 = 0$, and therefore

$$P2 = \{0, 1, 2, \ldots\}. \tag{12}$$

To construct convergence property of Algorithm 1 in this case, we first state some technical lemmas.

Lemma 2. *Suppose that $\{x_k\}_{k\in P2}$ is generated by Algorithm 1. Then, we have*

$$f_{k+1} \leq |f_0| \prod_{i=0}^{k} (1+\psi_i) - \omega_k,$$

where $\omega_k = \gamma {\alpha_k}^2 f(x_k)$.

Proof. The proof is similar to the proof of Lemma 3 in [5], and therefore we omit it here. □

The following lemma shows that all the iteration points that are generated by Algorithm 1 are contained in the level set L^0. Using Lemma 2, its proof is the same as the proof of Lemma 4 in [5] and Lemma 3.4 in [28].

Lemma 3. *For the sequence $\{x_k\}_{k\in P2}$, generated by Algorithm 1, we have $\{x_k\} \subseteq L^0$.*

Remark 2. Let $\{x_k\}_{k\in P2}$ be generated by Algorithm 1, and let $|P2| = \infty$. Then, due to the (12), for a given positive integer M, one can rewrite $P2$ as below:

$$P2 = \{LM + r \mid L \in \mathbb{N} \cup \{0\}, 0 \leq r \leq M - 1\}.$$

Lemma 4. *Assume that $\{x_k\}_{k \in P2}$ is an infinite sequence, generated by Algorithm 1 and that M is the constant as given by (3). Then, for every $k \in P2$, there exist a non-negative integer L and $0 \leq r \leq M - 1$ such that $k = LM + r$ and*

$$f_{k+1} = f_{LM+r+1} \leq |f_0| \prod_{i=0}^{LM+r} (1 + \psi_i) - \sum_{i=0}^{L} \omega_{s(i)}, \qquad L = 0, 1, 2, \ldots,$$

where $\omega_{s(i)} = \min_{iM \leq j \leq (i+1)M-1} \omega_j$ and ω_j is defined as in Lemma 2.

Proof. The proof is similar to the proof of Lemma 6 in [5]. □

Lemma 5. *Suppose that Assumptions* **A1** *and* **A2** *hold and that $\{x_k\}_{k \in P2}$ is the infinite sequence generated by Algorithm 1. Then, one has*

$$\lim_{i \to \infty} \omega_{s(i)} = 0,$$

where $\omega_{s(i)}$ is defined as in Lemma 4.

Proof. Using Lemmas 3 and 4, for $1 \leq r \leq M$ and $LM + r \in I$, we have

$$\sum_{i=0}^{L} \omega_{s(i)} \leq |f_0| \prod_{i=0}^{LM+r-1} (1 + \psi_i) - f_{LM+r} \leq e^{\eta} |f_0| - f_{LM+r}.$$

By taking limit from both side of this inequality, as $L \to \infty$, and using Assumption **A1**, we conclude that $\sum_{i=0}^{\infty} \omega_{s(i)} < \infty$, which implies that

$$\lim_{i \to \infty} \omega_{s(i)} = 0. \tag{13}$$

This completes the proof of the lemma. □

From now on, we define $K = \{s(i) | i = 1, 2, \ldots\}$. The following theorem establishes the global convergence of Algorithm 1.

Theorem 1. *Let Assumptions* **A1**–**A3** *hold, and let $\{x_k\}$ be the sequence generated by Algorithm 1. Then, for every limit point x^* of $\{x_k\}_{k \in K}$, one has*

$$\langle J(x^*)^T F(x^*), F(x^*) \rangle = 0,$$

where $\langle ., . \rangle$ stands for the inner product of two vectors in $\mathbb{R}^n$.

Proof. From Lemma 5, we have

$$\lim_{k \in K,\, k \to \infty} \omega_{s(k)} = \lim_{k \in K,\, k \to \infty} \min_{kM \leq j \leq (k+1)M-1} \gamma \alpha_j^2 \|F(x_j)\|^2 = 0,$$

which implies that

$$\lim_{k\in K,\, k\to\infty} \alpha_k^2 \|F(x_k)\|^2 = 0, \tag{14}$$

using the fact that $0 < \gamma < 1$. Since x^* is a limit point of $\{x_k\}_K$, there exists an infinite index set $K_1 \subset K$ such that $\lim_{k\in K_1,\, k\to\infty} x_k = x^*$. Besides, (14) implies that

$$\lim_{k\in K_1,\, k\to\infty} \alpha_k^2 \|F(x_k)\|^2 = 0. \tag{15}$$

To proceed, we consider the following two possible cases:
Case 1: $\lim_{k\in K_1} \alpha_k \neq 0$.
In this case, there exist an infinite sequence of indices $K_2 \subset K_1$ and a positive constant c such that $\alpha_k \geq c > 0$ for all $k \in K_2$. Therefore, (14) implies that

$$\lim_{k\in K_2} \|F(x_k)\|^2 = 0. \tag{16}$$

Now, (16) along with the continuity of F imply $F(x^*) = 0$.
Case2: $\lim_{k\in K_1} \alpha_k = 0$.
In this case, there exists $k_0 \in K_1$ such that $\alpha_k < 1$, for all $k_0 \leq k \in K_1$. Therefore, for infinitely many iterations, the line search is not immediately successful and α_+ and α_- should be updated at least once based on Step 4 of Algorithm 1. Suppose that, in step $k \in K_1$, α_+ and α_- are adapted m_k times in the line search procedure in Steps 3–4. Let α_k^+ and α_k^- be the values of α^+ and α^-, respectively, in the last unsuccessful line search process; that is, inequalities (8) and (9) are violated. Now, form Step 4 of Algorithm 1, for all $k_0 \leq k \in K_1$, we have

$$\alpha_k \geq \tau_{\min}^{m_k}.$$

This inequality together with $\lim_{k\in K_1} \alpha_k = 0$ and the fact that $\tau_{\min} < 1$ imply that

$$\lim_{k\in K_1} m_k = \infty.$$

On the other hand, Step 4 of Algorithm 1 implies that $\alpha_k^+ \leq \tau_{\max}^{m_k - 1}$ and $\alpha_k^- \leq \tau_{\max}^{m_k - 1}$. Therefore, using the fact that $\tau_{\max} < 1$, we deduce that

$$\lim_{k\in K1} \alpha_k^+ = \lim_{k\in K1} \alpha_k^- = 0.$$

Now, from the definition of α_k^+ and α_k^-, for all $k \geq k_0$, $k \in K_1$, we have

$$f(x_k - \alpha_k^+ \sigma_k F(x_k)) > (1 + \psi_k)R_k - \gamma(\alpha_k^+)^2 f(x_k), \tag{17}$$
$$f(x_k + \alpha_k^- \sigma_k F(x_k)) > (1 + \psi_k)R_k - \gamma(\alpha_k^-)^2 f(x_k). \tag{18}$$

Since $\psi_k > 0$ and $R_k > 0$, then $(1 + \psi_k)R_k > R_k \geq 0$. Besides, from Lemma 2.2 in [2], one has $R_k > f(x_k)$. Thus, (17) implies that

$$f(x_k - \alpha_k^+ \sigma_k F(x_k)) - f(x_k) > -\gamma(\alpha_k^+)^2 f(x_k).$$

From Lemma 3 and Assumptions **A1**–**A2**, $\{f(x_k)\}$ is a bounded above sequence. Thus

$$f(x_k - \alpha_k^+ \sigma_k F(x_k)) - f(x_k) > -\gamma(\alpha_k^+)^2 e^\eta f(x_0),$$

which implies that

$$\left\|F(x_k - \alpha_k^+ \sigma_k F(x_k))\right\|^2 - \|F(x_k)\|^2 > -\gamma(\alpha_k^+)^2 e^\eta f(x_0),$$

and therefore,

$$\frac{\left\|F(x_k - \alpha_k^+ \sigma_k F(x_k))\right\|^2 - \|F(x_k)\|^2}{\alpha_k^+} > -\gamma \alpha_k^+ e^\eta f(x_0).$$

Now, by Mean Value theorem, there exists $\xi_k \in [0, 1]$ such that

$$\sigma_k \langle g(x_k - \xi_k \alpha_k^+ F(x_k)), -F(x_k)) \rangle \geq -e^\eta f(x_0) \gamma \alpha_k^+. \tag{19}$$

Due to Algorithm 1, two possible cases may happen for σ_k, it might be positive or negative for infinitely many indices. If $\sigma_k > 0$ for infinitely many indices k, then (19) implies that

$$\langle g(x_k - \xi_k \alpha_k^+ F(x_k)), F(x_k)) \rangle \leq e^\eta f(x_0) \frac{\gamma \alpha_k^+}{\sigma_k} \leq e^\eta f(x_0) \frac{\gamma \alpha_k^+}{\sigma_{\min}}. \tag{20}$$

Using an analogous approach, from (18), we conclude that

$$\langle g(x_k + \xi_k{}' \alpha_k^- F(x_k)), F(x_k)) \rangle \geq -e^\eta f(x_0) \frac{\gamma \alpha_k^-}{\sigma_k} \geq -e^\eta f(x_0) \frac{\gamma \alpha_k^-}{\sigma_{\min}}, \tag{21}$$

for some $\xi_k{}' \in [0, 1]$.
Now, as $\alpha_k^+ \to 0$, $\alpha_k^- \to 0$, and $\|\sigma_k F(x_k)\|$ is bounded, taking limit from both sides in (20) and (21) leads to the following equality:

$$\langle J(x^*)^T F(x^*), F(x^*) \rangle = 0. \tag{22}$$

For the case in which $\sigma_k < 0$, for infinitely many indices, proceeding in a similar approach, we can deduce (22) as well. □

The following corollary is an immediate consequence of Theorem 1.

Corollary 1. *Assume that x^* is a limit point of the sequence $\{x_k\}_{k \in K}$ generated by Algorithm 1 and that $\langle J(x^*)v, v \rangle \neq 0$ for all $0 \neq v \in \mathbb{R}^n$. Then, $F(x^*) = 0$.*

Lemma 6. *For $k \in \mathbb{N}$, let*

$$L_k := f(x_{\ell(kM)}) = f_{\ell(kM)},$$

where $\ell(kM) \in \{(k-1)M+1, \dots, kM\}$ *and* $f_{\ell(j)}$ *is defined as in* (3)*. Then, we have*

$$L_{k+1} := f(x_{\ell((k+1)M)}) \leq \left[1 + \sum_{i=kM}^{\infty} \eta_i + \prod_{i=kM}^{\infty} \eta_i\right] L_k.$$

Proof. Using Step 3 of Algorithm 1, we have

$$\begin{aligned} f(x_{kM+1}) &\leq (1+\psi_{kM})R_{kM} \leq (1+\psi_{kM})L_k \\ f(x_{kM+2}) &= (1+\psi_{kM+1})[\varepsilon_k \max\{L_k, f(x_{kM+1})\} + (1-\varepsilon_k) f(x_{kM+1})] \\ &\leq (1+\psi_{kM+1})(1+\psi_{kM})L_k \\ &= [1+\psi_{kM}+\psi_{kM+1}+\psi_{kM}\psi_{kM+1}]L_k, \end{aligned}$$

and so on. Therefore, by an inductive argument,

$$f(x_{kM+j}) \leq \left[1 + \sum_{i=kM}^{kM+j} \eta_i + \prod_{i=kM}^{kM+j} \eta_i\right] L_k.$$

Since $\ell(k+1) \in \{kM+1, \dots, (k+1)M\}$, then, we have

$$L_{k+1} = f(x_{l((k+1)M)}) \leq \left[1 + \sum_{i=kM}^{\infty} \eta_i + \prod_{i=kM}^{\infty} \eta_i\right] L_k.$$

This completes the proof of the lemma. □

Let us define K_+ as below:

$$K_+ = \{\ell(1), \ell(2), \dots\}.$$

One can easily observe that

$$\ell(j+1) \leq \ell(j) + 2M - 1, \qquad j = 1, 2, \dots. \tag{23}$$

The following theorem states that in case of existing a limit point for the sequence generated by Algorithm 1, all the limit points are the solutions of the nonlinear system (1).

Theorem 2. *Let* x^* *be a limit point of the sequence* $\{x_k\}$*, generated by Algorithm 1, such that* $F(x^*) = 0$*. Then, we have*

$$\lim_{k\to\infty} F(x_k) = 0.$$

Proof. Let K_1 be an infinite subset of $\mathbb{N}$ such that

$$\lim_{k\in K_1,\, k\to\infty} x_k = x^*.$$

This relation together with Assumption **A1** imply that

$$\lim_{k\in K_1,\, k\to\infty} F(x_k) = 0. \tag{24}$$

Now, since $x_{k+1} = x_k \pm \alpha_k\sigma_k F(x_k)$ and $|\alpha_k\sigma_k| < \sigma_{\max}$, for all $k \in \mathbb{N}$, using (24), one can deduce that

$$\lim_{k\in K_1,\, k\to\infty} \|x_{k+1} - x_k\| = 0,$$

and therefore,

$$\lim_{k\in K_1,\, k\to\infty} x_{k+1} = x^*.$$

Proceeding by induction, one can easily show that for the fixed $j \in \{0, 1, \ldots, 2M-1\}$, we have

$$\lim_{k\in K_1,\, k\to\infty} x_{k+j} = x^*. \tag{25}$$

Now, using (23), for a given $k \in K_1$, there exists $\mu(k) \in \{0, 1, \ldots, 2M-1\}$ such that $k + \mu(k) \in K_+$. Since K_+ is infinite, there exists $\bar{i} \in \{0, 1, \ldots, 2M-1\}$ such that, $\mu(k) = \bar{i}$ for infinitely many $k \in K_1$. Assume that

$$K_2 = \{k + \mu(k) | k \in K_1 \text{ and } \mu(k) = \bar{i}\}.$$

It is easily seen that $K_2 \subset K_+$, and by (25)

$$\lim_{k\in K_2,\, k\to\infty} x_{k+1} = x^*.$$

Thus

$$\lim_{k\in K_2,\, k\to\infty} F(x_k) = 0.$$

Now, as $K_2 \subset K_+$, there exists an infinite subsequence $\{x_{\ell(j)}\}_{j\in J}$ such that

$$\lim_{j\in J} x_{\ell(j)} = x^* \quad \text{and} \quad \lim_{j\in J} f(x_{\ell(j)}) = \lim_{j\in J} L_j = 0. \tag{26}$$

Let us write $J = \{j_1, j_2, j_3, \ldots\}$, where $j_1 < j_2 < j_3 < \cdots$. Then, (26) implies that

$$\lim_{i\to\infty} L_{j_i} = 0. \tag{27}$$

Now, from Lemma 6, we have

$$L_p \le \left[1 + \sum_{t=Mj_i}^{\infty} \eta_t + \prod_{t=Mj_i}^{\infty} \eta_t\right] L_{j_i} \qquad \forall\, p \in \mathbb{N}, p > j_i.$$

Thus,

$$\sup_{p \geq j_i} L_p \leq \left[1 + \sum_{t=Mj_i}^{\infty} \eta_t + \prod_{t=Mj_i}^{\infty} \eta_t \right] L_{j_i}. \tag{28}$$

On the other hand, from (4), we have

$$\lim_{i \to \infty} \left[\sum_{t=Mj_i}^{\infty} \eta_t + \prod_{t=Mj_i}^{\infty} \eta_t \right] = 0.$$

Therefore, using (27) and (28), we conclude that

$$\lim_{i \to \infty} \sup_{p \geq j_i} L_p = 0,$$

which in turn implies that

$$\lim_{j \to \infty} L_j = 0.$$

Now, using the definition of L_j, we have

$$\lim_{k \to \infty} f(x_k) = \lim_{k \to \infty} \frac{1}{2} \|F(x_k)\|^2 = 0.$$

This completes the proof of the theorem. □

4 Numerical results

In this section, our aim is to investigate the practical performance of Algorithm 1, denoted by DF-DFSANE, along with the following algorithms on some test problems:

- DFSANE: Algorithm 1 in [24].
- NF-DFSANE: Algorithm 1 in this paper in which the concept of filter has been removed.

All the considered algorithms are implemented in MATLAB 7.10.0 (R2010a) environment on a PC with CPU Intel 2.33 GHz Quad, 4GB RAM memory, and double precision format. Test problems are mostly taken from [3, 19], and the problem dimensions vary from 2 to 10000.

All the considered algorithms are being stopped whenever $\| F(x_k) \| \leq 10^{-6}$. Moreover, we declare that an algorithm is failed whenever the number of iterations and the number of function evaluations exceed 10000 and 50000, respectively. We have also utilized the advantages of the performance profile of Dolan and Moré [12] in order to properly compare the considered algorithms. The following parameters are considered in the relevant algorithms:

$$\epsilon = 10^{-6}, \mu_1 = 0.25, \mu_2 = 0.75, \sigma_{\min} = 10^{-6}, \sigma_{\max} = 10^{+6},$$
$$M = 20, \gamma = 10^{-4}, \tau_{\min} = 0.1, \tau_{\max} = 0.5, \eta_k = \frac{1}{(1+k)^2},\ k \geq 0.$$

Moreover, we utilized $\phi(\alpha) = \alpha^{\frac{3}{2}}$ in the dwindling technique. It is worth mentioning that the parameters are chosen from a set of values in which reasonably better results are achieved while performing an algorithm. Numerical results are given in Table 1. In this table, $Problem$, n, n_i, and n_f represent the problem name, the problem size, the number of iterations, and the number of function evaluations, respectively. Based on the results of Table 1, we have plotted the performance profile of the considered algorithms in Figures 1 and 2 in terms of n_i and n_f, respectively.

At a glance to Figure 1, one can easily find out that DF-DFSANE algorithm solves roughly 93% of the test problems successfully while this percentage for NF-DFSANE and DFSANE algorithms is below 90%. Moreover, DF-DFSANE algorithm solves about 62% of problems at the lowest value of n_i while this percentage for the NF-DFSANE and DFSANE algorithms are 40% and 10%, respectively.

Figure 2 reveals that when the DF-DFSANE, NF-DFSANE, and DFSANE algorithms are applied on the test problems; the DF-DFSANE algorithm solves roughly 49% of the test problems in the lowest value of n_f while this number for the NF-DFSANE and DFSANE algorithms are 38% and 10%, respectively. Moreover, DF-DFSANE algorithm solves about 90% of the test problems without any failure. It has to be noticed that in the cases that DF-DFSANE is not the best algorithm; its performance index is roughly close to the performance index of the best algorithm.

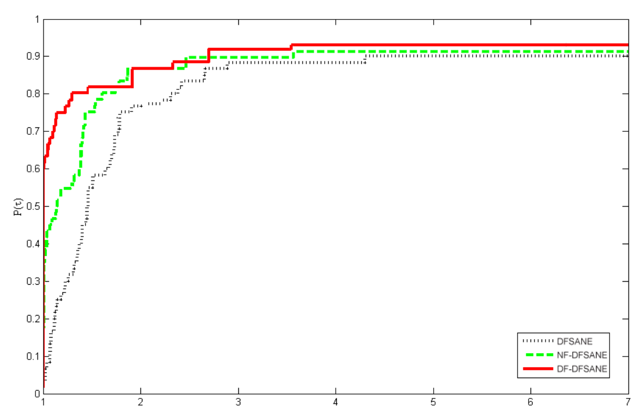

Figure 1: Performance profile of the considered algorithms in terms of n_i.

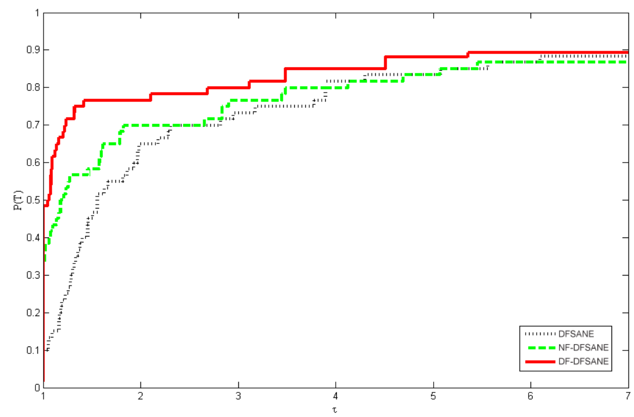

Figure 2: Performance profile of the considered algorithms in terms of n_f.

5 Conclusion

In this paper, a new dwindling nonmonotone filter DFSANE method for solving large-scale nonlinear systems of equations is proposed. In our approach, a filter is set up on the so-called DFSANE algorithm which has been developed in [24]. Indeed, the concept of filter helps us to save some appropriate points that are eliminated by the DFSANE algorithm. Moreover, the accumulated points in the filter cause the algorithm to reach the optimal point as quickly as possible. The proposed approach also uses some relaxations in its structure and is equipped with the nonmonotone term as proposed in [6]. Furthermore, we have more flexibility for the acceptance of the trial step by using a dwindling technique. Under some standard assumptions, the global convergence property is established. Numerical results on some test problems confirm that our proposed algorithm is practically efficient, robust, and has some priorities to that provided in [31].

Acknowledgements

The authors would like to thank the Research Council of K.N. Toosi University of Technology and the SCOPE research center for supporting this work.

Table 1: Test problems and numerical results.

Problem	n	DFSANE n_i/n_f	NF-DFSANE n_i/n_f	DF-DFSANE n_i/n_f
Extended Beale [3]	1000	38/52	33/46	29/39
Extended Beale [3]	5000	53/79	47/57	38/52
Extended Beale [3]	10000	62/90	48/72	38/64
Extended penalty [3]	1000	17/39	13/35	10/24
Extended penalty [3]	5000	21/49	17/45	21/47
Extended penalty [3]	10000	23/49	21/51	22/50
Extended Three Exponential [3]	1000	18/32	16/24	15/19
Extended Three Exponential [3]	5000	29/43	18/32	14/30
Extended Three Exponential [3]	10000	30/62	20/34	20/32
Generalized Tridiagonal-2 [3]	1000	58/68	58/72	58/72
Generalized Tridiagonal-2 [3]	5000	Failed	193/296	126/258
Generalized Tridiagonal-2 [3]	10000	Failed	789/3299	440/1083
Extended PSC1 Function [3]	1000	18/26	14/21	14/20
Extended PSC1 Function [3]	5000	22/42	18/26	17/23
Extended PSC1 Function [3]	10000	36/42	22/34	17/23
Extended Block Diagonal BD1 [3]	1000	17/23	17/23	15/23
Extended Block Diagonal BD1 [3]	5000	19/25	15/23	20/24
Extended Block Diagonal BD1 [3]	10000	99/131	114/213	105/285
DQDRTIC (CUTE) [3]	1000	64/116	52/74	50/70
DQDRTIC (CUTE) [3]	5000	89/209	60/90	35/61
DQDRTIC (CUTE) [3]	10000	126/301	44/66	33/47
LIARWHD function [3]	1000	1/1	1/1	1/1
LIARWHD function [3]	5000	18/90	18/90	16/74
LIARWHD function [3]	10000	87/233	111/401	105/341
Extended DENSCHNF [3]	1000	1/1	1/1	1/1
Extended DENSCHNF [3]	5000	25/41	21/49	21/49
Extended DENSCHNF [3]	10000	52/76	25/41	22/40
Generalized Quartic [3]	1000	17/23	10/12	10/10
Generalized Quartic [3]	5000	20/28	10/12	2/4
Generalized Quartic [3]	10000	50/98	21/29	12/16
Diagonal 8 [3]	1000	8/9	6/7	5/6
Diagonal 8 [3]	5000	20/37	16/17	7/9
Diagonal 8 [3]	10000	38/95	5/38	7/9
Full Hessian FH3 [3]	1000	4/24	4/24	4/22
Full Hessian FH3 [3]	5000	5/31	4/30	4/28
Full Hessian FH3 [3]	10000	Failed	Failed	Failed
SINCOS [3]	1000	18/26	14/21	11/16
SINCOS [3]	5000	27/33	18/26	17/23
SINCOS [3]	10000	36/42	30/36	17/23
HIMMELH (CUTE) [3]	1000	1/1	1/1	1/1
HIMMELH (CUTE) [3]	5000	14/18	14/18	8/10
HIMMELH (CUTE) [3]	10000	Failed	Failed	Failed

Table 1: Test problems and numerical results (Continued).

		DFSANE	NF-DFSANE	DF-DFSANE
Problem	n	n_i/n_f	n_i/n_f	n_i/n_f
Power [3]	1000	75/91	74/90	54/62
Power [3]	5000	152/176	123/142	120/134
Power [3]	10000	224/345	210/311	129/135
FLETCHCR function (CUTE) [3]	1000	750/1404	712/1201	425/689
FLETCHCR function (CUTE) [3]	5000	Failed	1012/4865	994/4752
FLETCHCR function (CUTE) [3]	10000	Failed	Failed	Failed
Problem 2 [19]	2	46/49	25/31	2/6
problem 4 [19]	5	1/2	1/2	1/2
Problem 6 [19]	2	3/7	4/15	3/6
Problem 7 [19]	5	268/271	62/74	33/42
Problem 8 [19]	10	15/18	19/25	14/16
Problem 26 [19]	5	149/157	21/21	21/21
Problem 27 [19]	2	1/1	1/1	1/1
Problem 39 [19]	5	1383/2100	24/28	20/34
Problem 40 [19]	5	Failed	Failed	Failed
Problem 42 [19]	2	1/1	1/1	1/1
Problem 46 [19]	5	Failed	Failed	Failed
Problem 47 [19]	10	324/841	260/642	222/423
Problem 48 [19]	5	423/1108	112/504	53/71
Problem 53 [19]	5	1/1	1/1	1/1
Problem 56 [19]	10	22/24	3/5	3/3
Problem 61 [19]	10	394/5784	357/5650	357/5650
Problem 63 [19]	5	22/26	47/66	21/38
Problem 77 [19]	10	149/219	51/126	17/24
Problem 78 [19]	2	Failed	Failed	Failed
Problem 79 [19]	5	970/4130	952/4025	844/3441
Problem 81 [19]	5	Failed	Failed	Failed
Problem 81 [19]	10	Failed	843/4344	432/3456
problem 111 [19]	5	344/1815	310/1012	72/770
problem 111 [19]	10	Failed	572/4567	424/3245

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یک روش فیلتر نایکنوای کاهشی جدید بدون اطلاعات گرادیان برای حل دستگاههای معادلات مقیاس بزرگ

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دریافت مقاله ۱۸ شهریور ۱۳۹۴، دریافت مقاله اصلاح شده ۲۴ فروردین ۱۳۹۵، پذیرش مقاله ۲۱ تیر ۱۳۹۶

چکیده : در این مقاله، یک روش باقیمانده طیفی مشتق آزاد جدید برای حل دستگاه های معادلات مقیاس بزرگ ارایه می دهیم. الگوریتم پیشنهادی مجهز به یک فیلتر نایکنوای چند بعدی کاهشی است که از پوشش های آن زمانی که طول گام جستجوی خطی کاهش می یابد، کاسته می شود. همچنین، روش پیشنهادی با یک تکنیک جستجوی خطی نایکنوای رهاسازی شده آمیخته می شود که به الگوریتم اجازه می دهد تا از خاصیت نایکنوایی از ابتدا بهره بگیرد. تحت برخی شرایط استاندارد، خاصیت همگرایی سراسری الگوریتم پیشنهادی اثبات می شود. نتایج عددی روی مسایل آزمونی کارایی الگوریتم جدید را در عمل نیز نشان می دهد.

کلمات کلیدی : تکنیک فیلتر کاهشی؛ دستگاه معادلات؛ جستجوی خطی نایکنوا؛ همگرایی سراسری.