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Constrained Bimatrix Games with Fuzzy Goals and its Application in Nuclear Negotiations

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Abstract

Solving constrained bimatrix games in the fuzzy environment is the aim of this research. This class of two-person nonzero-sum games is considered with finite strategies and fuzzy goals when some additional linear constraints are imposed on the strategies. We consider constrained two-person nonzero-sum games with single and multiple payoffs. It is shown that an equilibrium solution of single-objective case can be characterized by solving a quadratic programming problem with linear constraints. Some mathematical programming problems are also introduced to obtain the equilibrium points in multi-objective case with crisp and fuzzy constraints. Finally, a political application of such games is presented which is about nuclear negotiations between two countries.

Keywords: Multiobjective game; Constrained game; Fuzzy constrained game; Nuclear negotiations.

1 Introduction

Many real-world problems can be modeled as game problems. When the game theory is used to analysis some conflict problems in real situations, impreci-

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sion or fuzziness is inherent in human judgments. Two types of inaccuracies of human judgments should be incorporated in the games; the players' ambiguous understanding of the payoffs and the fuzzy goals of the players for each of the objectives. Fuzzy set theory is well known for its ability to model decision making problems involving vagueness due to the lack of information and/or imprecision of the available information on the problem set-up [21, 24]. This ability has been successfully exploited for modeling different problems in various disciplines. In the field of fuzzy games, a considerable number of studies have been made (see, for example, [1,11,13,17]). Two-person nonzero-sum finite games are a kind of noncooperative games [19]. Two-person nonzero-sum games are also referred to as bimatrix games because they can be expressed by a pair of payoff matrices. These kind of games contain two-person zero-sum games as a special case. In real-world decision making problems facing humans today, people want to attain simultaneous goals; that is, they have multiple objectives. Hence, it seems natural that the game theoretic approaches to conflict resolution require to handle multiple objectives simultaneously. Wierzbicki [23] defined equilibrium solutions of multiobjective game based on order relations, using several preference cones and optimality criteria such as Pareto optimality for multiobjective noncooperative n-person games with nonlinear payoff functions. Corley [7] defined equilibrium solutions for multiobjective two-person nonzero-sum games and developed a method to compute them. Authors of [5] defined a proxy single-objective game with payoffs corresponding to a scalarizing function with weighting coefficients in multiobjective two-person nonzero-sum games and discussed the existence of equilibrium solutions for the original multiobjective two-person nonzero-sum games through the existence of the equilibrium solutions for the single objective proxy game. Fahem and Radjef [9] investigated the concept of properly efficient equilibrium for a multicriteria noncooperative strategic game. Nishizaki and Sakawa [17] studied two-person nonzero-sum game incorporating fuzzy goals in single and multiobjective environments. They defined an equilibrium solution with respect to the degree of attainment of the fuzzy goals in two-person nonzero-sum games and showed that an equilibrium obtained from solving several mathematical programming problems.

In some real-life game problems, choice of strategies for players is constrained due to some practical reason why this should be; that is, not all mixed strategies in a game are permitted for each player. These decision problems give rise to constrained games. Constrained matrix games initially were studied by Charnes [6] and then in a more general case by Kawaguchi and Maruyama [10]. Dresher [8] gave a real example of the constrained matrix game. Li and Cheng [11] presented a method based on the multiobjective programming to solve constrained matrix games with fuzzy numbers. Li and Hong [13] proposed a method for solving constrained matrix games with payoffs of triangular fuzzy numbers. They introduced the concepts of Alpha-constrained matrix games for the constrained matrix games with payoffs of triangular fuzzy numbers. Also, they [12] proposed Alpha-cut based linear programming methodology for the constrained matrix games with payoffs of trapezoidal fuzzy numbers.

The authors used multiobjective optimization in multiobjective zero-sum fuzzy matrix games in other researches [3, 4]. In this paper, we consider the case of constrained bimatrix games with single and multiple payoffs in which fuzzy goals are specified for the objectives by the players.

The remainder of the paper is organized as follows. In section 2, some preliminaries and necessary definitions about fuzzy sets and bimatrix games are presented. In section 3, constrained bimatrix games with fuzzy goals are introduced. It is shown that the equilibrium solution of this class of nonzero-sum games can be characterized by solving a quadratic programming problem. Then, multiobjective constrained bimatrix games with fuzzy goals is considered, and a mathematical programming problem to solve such games is presented. Also, for the case that additional linear constraints are imposed on the strategies are fuzzy constraints, a mathematical programming problem is introduced to obtain the equilibrium point of such games. In section 4, a political application of multiobjective constrained bimatrix games with fuzzy goals is presented in which nuclear negotiations between two countries have been discussed. Finally, conclusion is made in section 5.

2 Preliminaries

In this section, we provide some definitions and preliminaries of fuzzy sets and bimatrix games according to [1, 20].

Let X denote a universal set. A fuzzy subset $\tilde{A}$ of X is defined by its membership function $\mu_{\tilde{A}} : X \to [0,1]$ which assigns to each element $x \in X$ a real number $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$. The value of $\mu_{\tilde{A}}(x)$ represents the grade of membership of x in $\tilde{A}$. The fuzzy subset $\tilde{A}$ is denoted by a set of ordered pairs of elements x and their grades $\mu_{\tilde{A}}(x)$; that is, $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$.

A fuzzy decision making problem is characterized by a set X of possible alternatives and a set of fuzzy goals G_i, $i = 1, \dots, p$, as well as a set of fuzzy constraints C_j, $j = 1, \dots, n$, each of which is expressed by a fuzzy set on X. For such a decision making problem, Bellman and Zadeh [2] proposed that a fuzzy decision is determined by an appropriate aggregation of the fuzzy sets G_i, $i = 1, \dots, p$, and C_j, $j = 1, \dots, n$. Realizing that both the fuzzy goal and the fuzzy constraint are desired to be satisfied simultaneously, they suggested the aggregation operator to be the fuzzy intersection. Thus a fuzzy decision D could be defined as the fuzzy set $D = (G_1 \cap \dots \cap G_p) \cap (C_1 \cap \dots \cap C_n)$; that is, $\mu_D(x) : X \to [0,1]$ given by $\mu_D(x) = \min(\mu_{G_1}(x), \dots, \mu_{G_p}(x), \mu_{C_1}(x), \mu_{C_n}(x))$. Once the fuzzy decision D is known, we can define $x^* \in X$ to be an optimal decision if $\mu_D(x^*) = \max_x \mu_D(x)$.

Let two players in a two-person bimatrix game be denoted by Players I and II. In a two-person bimatrix game, each player has finite number of strategies. The payoff function is determined by two matrices $A, B \in \mathbb{R}^{m\times n}$. When Player I chooses his ith strategy and Player II his jth strategy, then a_{ij} and b_{ij} are the payoffs of Players I and II, respectively. Thus, a bimatrix game is determined by a pair of matrices (A,B). Rational behavior of players is assumed; that is, each of them attempts to maximize his reward. Mixed strategy is a probability vector over the set of pure strategies. So, bimatrix game can be represented as $BG : (X, Y, A, B)$, where

$$X = \{x \in \mathbb{R}^m | \sum_{i=1}^{m} x_i = 1, x_i \geq 0 \qquad i = 1, \dots, m\}$$

and

$$Y = \{y \in \mathbb{R}^n | \sum_{j=1}^{n} y_j = 1, y_j \geq 0,\ j = 1, \dots, n\}$$

are called the mixed strategy spaces for Players I and II, and A and B are called the payoff matrices for Players I and II, respectively.

Definition 1. A pair $(x^*, y^*) \in X \times Y$ is said to be an equilibrium solution of the bimatrix game BG if

$$x^T Ay^* \leq x^{*T} Ay^* \qquad \forall x \in X$$

and

$$x^{*T} By \leq x^{*T} By^* \qquad \forall y \in Y.$$

In other words, no player has a motivation to change his strategy.

The following theorem due to Nash guarantees the existence of an equilibrium solution of the bimatrix game.

Theorem 1. [20]*Every bimatrix game $BG : (X, Y, A, B)$ has at least one equilibrium solution.*

A Nash equilibrium solution of the bimatrix game BG can be obtained by solving an appropriate quadratic programming problem as discussed below.

Theorem 2. [15]*Let $BG : (X, Y, A, B)$ be a given bimatrix game. The pair (x^*, y^*) is an equilibrium solution of BG if and only if it is an optimal solution to the following quadratic programming problem:*

$$\begin{array}{ll} \max\limits_{x,y,\alpha,\beta} & x^T(A+B)y-\alpha-\beta \\ s.t. & Ay-\alpha e_m \leq 0, \\ & B^T x-\beta e_n \leq 0, \\ & x\in X,\ y\in Y, \end{array} \tag{1}$$

where α and β are scalars; e_m and e_n are, respectively, m- and n-dimensional column vectors whose elements are all ones. The optimal values of α and β are the expected payoffs to Players I and II, respectively. Furthermore, the optimal objective value of the problem (1) equals zero.

We will use Pareto optimality concept of multiobjective optimization in solution concept of multiobjective two-person nonzero-sum games. Thus, we review some of solutions concepts in multiobjective mathematical programming. For convenience, let us introduce the following notation: for any vectors $z, z^{'} \in \mathbb{R}^N$

$$\begin{array}{l} z=z^{'} \Longleftrightarrow z_i=z_i^{'} \quad i=1,\dots,N; \\ z\leqq z^{'} \Longleftrightarrow z_i\leqq z_i^{'} \quad i=1,\dots,N; \\ z< z^{'} \Longleftrightarrow z_i< z_i^{'} \quad i=1,\dots,N; \\ z\leq z^{'} \Longleftrightarrow z\leqq z^{'} \quad \text{and} \quad z\neq z^{'}. \end{array}$$

A multiobjective mathematical programming problem can be written as

$$\begin{array}{l} \min f(z)=(f_1(z),\dots,f_l(z)) \\ \text{s.t.} \\ \qquad z\in Z=\{z\in\mathbb{R}^N|g(z)\leqq 0,\ h(z)=0\}, \end{array}$$

where $g(z)=(g_1(z),\dots,g_{m_1}(z))$, $h(z)=(h_1(z),\dots,h_{m_2}(z))$, and 0 is zero vector with the same dimension as the left hand side vectors.
There does not generally exist a solution minimizing all of the objectives simultaneously. Therefore, Pareto optimal (efficient) solutions are introduced as follows.

Definition 2. [22] $z^* \in Z$ is said to be a Pareto optimal solution if there does not exist another $z\in Z$ such that $f(z)\leq f(z^*)$.

As a slightly weaker solution concept than Pareto optimality, weak Pareto optimal solutions are defined by replacing $\leq$ with $<$ in Definition 2.

3 Constrained bimatrix games with fuzzy goals

Nishizaki and Sakawa [18] studied two-person nonzero-sum games incorporating fuzzy goals in multiobjective environment. They defined an equilibrium solution with respect to the degree of attainment of the fuzzy goals and proved that an equilibrium solution can be obtained by solving several certain mathematical programming problems. Let us consider a case of BG in which all mixed strategies x and y must be chosen from some sets $\hat{X}$ and $\hat{Y}$ determined by some linear constraints. A constrained bimatrix game is denoted by $CBG : (\hat{X}, \hat{Y}, A, B)$ in which

$$\hat{X} = \left\{x \in \mathbb{R}^m \left| e_m^T x = 1, Dx \leq c \ , \ x \geq 0 \right.\right\}, \tag{2}$$

and

$$\hat{Y} = \left\{y \in \mathbb{R}^n \left| e_n^T y = 1, Ey \leq d \ , \ y \geq 0 \right.\right\}, \tag{3}$$

are called the mixed strategy spaces for Players I and II in the constrained game, respectively. In the above sets $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$, $D \in \mathbb{R}^{p\times m}$, and $E \in \mathbb{R}^{q\times n}$. e_m and e_n are m- and n-dimensional vectors of ones.
When Player I chooses a mixed strategy $x \in \hat{X}$ and Player II chooses a mixed strategy $y \in \hat{Y}$, the values $x^T Ay$ and $x^T By$ are the expected payoffs for Players I and II, respectively.

Definition 3. A pair $(x^*, y^*) \in \hat{X} \times \hat{Y}$ is said to be an equilibrium solution of the constrained bimatrix game CBG if

$$\begin{array}{ll} x^T Ay^* \leq x^{*T} Ay^* & \forall x \in \hat{X}, \\ x^{*T} By \leq x^{*T} By^* & \forall y \in \hat{Y}. \end{array}$$

In this paper, we incorporate fuzzy goals for objectives and introduce equilibrium solutions in terms of maximizing the degree of attainment for fuzzy goals similar to the work of [18]. In the following, we define fuzzy goals for Players I and II.

Definition 4. Let the expected payoffs of Player I be $D_1 = \{x^T Ay| \ x \in \hat{X}, y \in \hat{Y}\}$. A fuzzy goal for Player I is a fuzzy set $\tilde{G}_1$ represented by an increasing membership function $\mu_1 : D_1 \to [0, 1]$.

Definition 5. Let the expected payoffs of Player II be $D_2 = \{x^T By| \ x \in \hat{X}, y \in \hat{Y}\}$. A fuzzy goal for Player II is a fuzzy set $\tilde{G}_2$ with an increasing membership function $\mu_2 : D_2 \to [0, 1]$.

Note that μ_1 (μ_2) assigns to each expected payoff $x^T Ay \in D_1$ ($x^T By \in D_2$) a real number $\mu_1(x^T Ay)$ ($\mu_2(x^T By)$) which denotes the degree of satisfaction of Player I (Player II) from the payoff $x^T Ay$ ($x^T By$).

An equilibrium solution of a constrained bimatrix game with fuzzy goals is defined with respect to the degree of attainment of the fuzzy goals.

Definition 6. (Equilibrium solution). A pair $(x^*, y^*) \in \hat{X} \times \hat{Y}$ is said to be an equilibrium solution of the constrained bimatrix game $CBG : (\hat{X}, \hat{Y}, A, B)$ with fuzzy goals if

$$\mu_1(x^{*T}Ay^*) \geq \mu_1(x^TAy^*) \qquad \forall x \in \hat{X},$$
$$\mu_2(x^{*T}By^*) \geq \mu_2(x^{*T}By) \qquad \forall y \in \hat{Y}.$$

The membership function μ_1 (μ_2) in the above definition can be interpreted as Player I's (Player II's) satisfaction function of payoff. Thus the game with fuzzy goals can be reduced to an ordinary two-person constrained nonzero-sum game whose payoff function is its satisfaction function. Let us consider the fuzzy goal of Player I (Player II) as "the expected payoff should be substantially greater than or equal to $\bar{a}$ ($\bar{b}$)". Here, we use linear membership functions for these fuzzy goals as follows (see Figure 1 for μ_1):

$$\mu_1(x^TAy) = \begin{cases} 0, & x^TAy \leq \underline{a}, \\ 1 - \frac{\bar{a}-x^TAy}{\bar{a}-\underline{a}}, & \underline{a} \leq x^TAy \leq \bar{a}, \\ 1, & x^TAy \geq \bar{a}, \end{cases} \tag{4}$$

and

$$\mu_2(x^TBy) = \begin{cases} 0, & x^TBy \leq \underline{b}, \\ 1 - \frac{\bar{b}-x^TBy}{\bar{b}-\underline{b}}, & \underline{b} \leq x^TBy \leq \bar{b}, \\ 1, & x^TBy \geq \bar{b}. \end{cases} \tag{5}$$

In (4) and (5), $\bar{a}$ and $\bar{b}$ are the desirable values of expected payoffs for Players I and II, respectively. Also, $\underline{a}$ and $\underline{b}$ are the least allowable values for their expected payoffs. Thus $\bar{a}-\underline{a}$ ($\bar{b}-\underline{b}$) is the most allowable value for violation from $\bar{a}$ ($\bar{b}$) for Player I (Player II). Notice that $\underline{a}$, $\bar{a}$, $\underline{b}$, and $\bar{b}$ could be any values with $\bar{a} > \underline{a}$ and $\bar{b} > \underline{b}$ which is given by players. In the case that they can not give these values; we can set $\underline{a} = \min_i \min_j a_{ij}$, $\bar{a} = \max_i \max_j a_{ij}$, $\underline{b} = \min_i \min_j b_{ij}$, and $\bar{b} = \max_i \max_j b_{ij}$, in which $\underline{a}$ ($\underline{b}$) gives the worst degree of satisfaction and $\bar{a}$ ($\bar{b}$) gives the best degree of satisfaction for Player I (Player II).
Letting

$$\hat{A} = \tfrac{A}{\bar{a}-\underline{a}}, \hat{B} = \tfrac{B}{\bar{b}-\underline{b}}, c_1 = -\tfrac{\underline{a}}{\bar{a}-\underline{a}}, \text{ and } c_2 = -\tfrac{\underline{b}}{\bar{b}-\underline{b}};$$

the membership functions $\mu_1(x^TAy)$ and $\mu_2(x^TBy)$ can be rewritten as

$$\mu_1(x^TAy) = \begin{cases} 0, & x^TAy \leq \underline{a}, \\ c_1 + x^T\hat{A}y, & \underline{a} \leq x^TAy \leq \bar{a}, \\ 1, & x^TAy \geq \bar{a}, \end{cases}$$

and

$$\mu_2(x^T By) = \begin{cases} 0, & x^T By \le \underline{b}, \\ c_2 + x^T \hat{B} y, & \underline{b} \le x^T By \le \overline{b}, \\ 1, & x^T By \ge \overline{b}. \end{cases}$$

respectively.

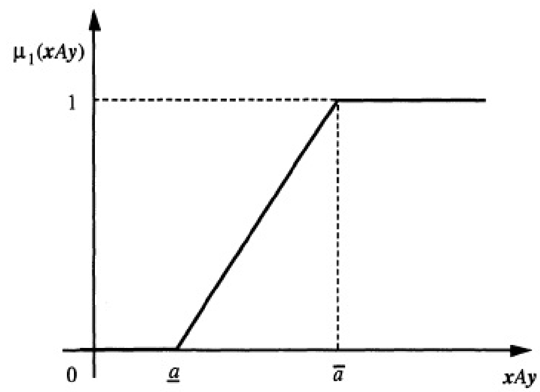

Figure 1: A linear membership function for fuzzy goal of Player I.

The following theorem states that an equilibrium solution of $CBG : (\hat{X}, \hat{Y}, \hat{A}, \hat{B})$ is equal to the equilibrium solution of $CBG : (\hat{X}, \hat{Y}, A, B)$. Moreover, it is also an equilibrium solution with respect to the degree of attainment of the fuzzy goals for $CBG : (\hat{X}, \hat{Y}, A, B)$ with fuzzy goals.

Theorem 3. *A pair of strategies (x^*, y^*) satisfies the conditions*

$$\begin{array}{ll} x^{*T} \hat{A} y^* \ge x^T \hat{A} y^* & \forall x \in \hat{X} \\ x^{*T} \hat{B} y^* \ge x^{*T} \hat{B} y & \forall y \in \hat{Y} \end{array}$$

if and only if (x^*, y^*) satisfies the following conditions

$$\begin{array}{ll} x^{*T} A y^* \ge x^T A y^* & \forall x \in \hat{X} \\ x^{*T} B y^* \ge x^{*T} B y & \forall y \in \hat{Y} \end{array}$$

furthermore, when the membership functions of the fuzzy goals are linear functions, (x^*, y^*) satisfies the following conditions

$$\begin{array}{ll} \mu_1(x^{*T} A y^*) \ge \mu_1(x^T A y^*) & \forall x \in \hat{X} \\ \mu_2(x^{*T} B y^*) \ge \mu_1(x^{*T} B y) & \forall y \in \hat{Y} \end{array}$$

Proof. The presented proof in [1] (Theorems 9.2.1 and 9.2.2) in the case of unconstrained bimatrix games is also valid for our problem. □

The following example shows that the converse of the second part of Theorem 3 is not established.

Example 1. Set

$$X = \{x | x_1 + x_2 = 1, x_1, x_2 \geq 0\},\ Y = \{y | y_1 + y_2 = 1, y_1, y_2 \geq 0\},$$
$$A = B = \begin{bmatrix} 1 & 0.75 \\ 0.5 & 0.25 \end{bmatrix},$$
$$\mu_1(x^T Ay) = \begin{cases} 0, & x^T Ay \leq 0.25, \\ 1 - \frac{0.5 - x^T Ay}{0.25}, & 0.25 \leq x^T Ay \leq 0.5, \\ 1, & x^T Ay \geq 0.5, \end{cases}$$

and

$$\mu_2(x^T By) = \begin{cases} 0, & x^T By \leq 0.25, \\ 1 - \frac{0.5 - x^T By}{0.25}, & 0.25 \leq x^T By \leq 0.5, \\ 1, & x^T By \geq 0.5. \end{cases}$$

For $x^* = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$ and $y^* = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$, we have

$$\mu_1 \left(x^{*T} Ay^* \right) \geq \mu_1 \left(x^T Ay^* \right) \qquad \forall x \in X$$

and

$$\mu_2 \left(x^{*T} By^* \right) \geq \mu_2 \left(x^{*T} By \right) \qquad \forall y \in Y.$$

Because

$$x^{*T} Ay^* = (0.2, 0.8) \begin{bmatrix} 1 & 0.75 \\ 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = [0.6 \quad 0.35] \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = 0.525$$

and

$$\mu_1 \left(x^{*T} Ay^* \right) = \mu_1 (0.525) = 1$$

which imply that

$$1 = \mu_1 \left(x^{*T} Ay^* \right) \geq \mu_1 \left(x^T Ay^* \right) \qquad \forall x \in X.$$

Similarly, $1 = \mu_1 \left(x^{*T} By^* \right) \geq \mu_1 \left(x^{*T} By \right)\ \forall y \in Y$.

On the other hand for $x = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$, we have

$$x^T Ay^* = (0.4, 0.6) \begin{bmatrix} 1 & 0.75 \\ 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = [0.7 \ \ 0.45] \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = 0.625$$

$$\mu_1 \left(x^T Ay^* \right) = \mu_1 (0.625) = 1$$

We see that, $x^{*T} Ay^* = 0.525 < x^T Ay^* = 0.625$. Therefore (x^*, y^*) is not an equilibrium point for bimatrix game (X, Y, A, B). Hence the converse of Theorem 3 is not established.

Now, we have the following theorem which provides a quadratic programming problem to find an equilibrium solution of the $CBG : (\hat{X}, \hat{Y}, A, B)$ with fuzzy goals.

Theorem 4. *(Equivalence theorem) Let $(x^*, y^*, \alpha^*, \beta^*, u^*, v^*)$ be an optimal solution of the following quadratic programming problem:*

$$\begin{array}{ll} \max\limits_{x,y,u,v,\alpha,\beta} & x^T(\hat{A}+\hat{B})y - cu - dv - \alpha - \beta \\ s.t. & \hat{A}y \leq cue_m + \alpha e_m, \\ & \hat{B}^T x \leq dve_n + \beta e_n, \\ & D^T x - c \leq 0, \\ & E^T y - d \leq 0, \\ & e_m^T x = 1, \\ & e_n^T y = 1, \\ & x, y, u, v \geq 0. \end{array} \tag{6}$$

Then (x^*, y^*) is an equilibrium solution of the $CBG : (\hat{X}, \hat{Y}, A, B)$ with fuzzy goals.

Proof. First, we show that the optimal value of objective function is zero. The constraints of the problem evidently imply that

$$x^T\hat{A}y + x^T\hat{B}y - cu - dv - \alpha - \beta \leq 0.$$

Thus the objective optimal value of the problem (6) is nonpositive. Consider the point $(\bar{x}, \bar{y}, \bar{\alpha}, \bar{\beta}, \bar{u}, \bar{v})$, where $\bar{x} = x^* \in \hat{X}, \bar{y} = y^* \in \hat{Y}, \bar{\alpha} = x^{*T}\hat{A}y^* - cu^*, \bar{\beta} = x^{*T}\hat{B}y^* - dv^*, \bar{u} = u^*$, and $\bar{v} = v^*$. We first prove that it is feasible to the problem (6). Assume that it is not feasible. Obviously, the solution does not satisfy at least one of the two first constraints of the problem (6). We assume that the first constraint does not hold. The proof of the other case is similar. Thus,

$$\hat{A}y^* > cu^*e_m + \alpha^*e_m. \tag{7}$$

Multiplying both sides of (7) $x^* \in \hat{X}$, implies that

$$x^{*T}\hat{A}y^* > cu^* + \alpha^* = x^{*T}\hat{A}y^*$$

which is incorrect.

On the other hand, it is clear that the objective value of $(\bar{x}, \bar{y}, \bar{\alpha}, \bar{\beta}, \bar{u}, \bar{v})$ is zero. Thus it means that the optimal value of objective (6) is zero.
Now, let $(x^*, y^*, \alpha^*, \beta^*, u^*, v^*)$ be an optimal solution of the problem (6); thus

$$x^{*T}\hat{A}y^* + x^{*T}\hat{B}y^* - cu^* - dv^* - \alpha^* - \beta^* = 0. \tag{8}$$

From the first and second constraints, for any $x \in \hat{X}$ and $y \in \hat{Y}$, we have

$$\begin{aligned} x^T \hat{A} y^* &\leq cu^* + \alpha^* \\ \text{and } x^{*T} \hat{B} y &\leq dv^* + \beta^*, \end{aligned} \tag{9}$$

which imply

$$x^T \hat{A} y^* + x^{*T} \hat{B} y \leq cu^* + \alpha^* + dv^* + \beta^*.$$

Also, the equation (9) implies in particular that

$$\begin{aligned} x^{*T} \hat{A} y^* &\leq cu^* + \alpha^* \\ \text{and } x^{*T} \hat{B} y^* &\leq dv^* + \beta^*. \end{aligned}$$

On the other hand, from (8) we have

$$x^{*T} \hat{A} y^* + x^{*T} \hat{B} y^* = cu^* + \alpha^* + dv^* + \beta^*.$$

Thus

$$\begin{aligned} x^{*T} \hat{A} y^* &= cu^* + \alpha^* \\ \text{and } x^{*T} \hat{B} y^* &= dv^* + \beta^*. \end{aligned}$$

Therefore, (9) implies that

$$\begin{aligned} x^{*T} \hat{A} y^* &\geq x^T \hat{A} y^* \qquad \forall x \in \hat{X} \\ \text{and } x^{*T} \hat{B} y^* &\geq x^{*T} \hat{B} y \qquad \forall y \in \hat{Y} \end{aligned}$$

This proves that (x^*, y^*) is an equilibrium pair of $CBG : (\hat{X}, \hat{Y}, \hat{A}, \hat{B})$. But, by Theorem 3, this means that (x^*, y^*) is an equilibrium solution of the $CBG : (\hat{X}, \hat{Y}, A, B)$ with fuzzy goals. □

Remark 1. When an optimal solution $(x^*, y^*, \alpha^*, \beta^*, u^*, v^*)$ of (6) has been obtained, (x^*, y^*) gives an equilibrium solution of the constrained bimatrix game $CBG : (\hat{X}, \hat{Y}, A, B)$ with fuzzy goals. The degree of attainment of fuzzy goal $\tilde{G}_1$ and $\tilde{G}_2$ can then be determined by $\mu_1(x^{*T} A y^*)$ (or $\mu_1(x^{*T} \hat{A} y^*)$) and $\mu_2(x^{*T} B y^*)$ (or $\mu_2(x^{*T} \hat{B} y^*)$).

3.1 Multiobjective constrained bimatrix games with fuzzy goals

Multiobjective two-person nonzero-sum games can be expressed as multiple $m \times n$ matrices

$$A^1 = \begin{bmatrix} a^1_{11} & \dots & a^1_{1n} \\ \vdots & \ddots & \vdots \\ a^1_{m1} & \dots & a^1_{mn} \end{bmatrix}, \dots, A^r = \begin{bmatrix} a^r_{11} & \dots & a^r_{1n} \\ \vdots & \ddots & \vdots \\ a^r_{m1} & \dots & a^r_{mn} \end{bmatrix},$$

$$B^1 = \begin{bmatrix} b^1_{11} & \dots & b^1_{1n} \\ \vdots & \ddots & \vdots \\ b^1_{m1} & \dots & b^1_{mn} \end{bmatrix}, \dots, B^s = \begin{bmatrix} b^s_{11} & \dots & b^s_{1n} \\ \vdots & \ddots & \vdots \\ b^s_{m1} & \dots & b^s_{mn} \end{bmatrix},$$

where Player I has m pure strategies and r objectives and Player II has n pure strategies and s objectives. When Player I chooses the pure strategy i and Player II chooses the pure strategy j, they respectively receive the payoff vectors $(a^1_{ij}, \dots, a^r_{ij})$ and $(b^1_{ij}, \dots, b^s_{ij})$. The mixed strategy spaces for Players I and II are, respectively, given by (2) and (3).

When Player I chooses a constrained mixed strategy $x \in \hat{X}$ and Player II chooses a constrained mixed strategy $y \in \hat{Y}$, the k-th payoff of Player I is $f^k_1(x, y) = x^T A^k y$ and the l-th expected payoff of Player II is $f^l_2(x, y) = x^T B^l y$. Assume that for each objective $f^k_1(x, y) = x^T A^k y, k = 1, \dots, r$, Player I has a fuzzy goal such as "the k-th objective value of the game should be substantially more than or equal to some value $\bar{a}^k$". This statement can be quantified by eliciting a membership function. Assume that the corresponding linear membership function is defined as follows:

$$\mu^k_1(x^T A^k y) = \begin{cases} 0, & x^T A^k y \le \underline{a}^k, \\ 1 - \frac{\bar{a}^k - x^T A^k y}{\bar{a}^k - \underline{a}^k}, & \underline{a}^k \le x^T A^k y \le \bar{a}^k, \\ 1, & x^T A^k y \ge \bar{a}^k, \end{cases}$$

where $\underline{a}^k$ is the least allowable value of the k-th payoff.
Similarly, assume that for each objective $f^l_2(x, y) = x^T B^l y, l = 1, \dots, s$, Player II has a fuzzy goal such as "the l-th objective value of the game should be substantially more than or equal to some value $\bar{b}^l$." Then the corresponding linear membership function is defined as follows:

$$\mu^l_2(x^T B^l y) = \begin{cases} 0, & x^T B^l y \le \underline{b}^l, \\ 1 - \frac{\bar{b}^l - x^T B^l y}{\bar{b}^l - \underline{b}^l}, & \underline{b}^l \le x^T B^l y \le \bar{b}^l, \\ 1, & x^T B^l y \ge \bar{b}^l, \end{cases}$$

where $\underline{b}^l$ is the least allowable value of the l-th payoff.
Letting

$$\hat{A}^k = \frac{A^k}{\bar{a}^k - \underline{a}^k}, \hat{B}^l = \frac{B^l}{\bar{b}^l - \underline{b}^l}, c^k_1 = -\frac{\underline{a}^k}{\bar{a}^k - \underline{a}^k}, \text{and } c^l_2 = -\frac{\underline{b}^l}{\bar{b}^l - \underline{b}^l},$$

the membership functions $\mu^k_1(x^T A^k y)$ and $\mu^l_2(x^T B^l y)$ can be rewritten as

$$\mu_1^k(x^TA^ky) = \begin{cases} 0, & x^TA^ky \le \underline{a}^k, \\ c_1^k + x^T\hat{A}^ky, & \underline{a}^k \le x^TA^ky \le \bar{a}^k, \\ 1, & x^TA^ky \ge \bar{a}^k, \end{cases}$$

and

$$\mu_2^l(x^TB^ly) = \begin{cases} 0, & x^TB^ly \le \underline{b}^l, \\ c_2^l + x^T\hat{B}^ly, & \underline{b}^l \le x^TB^ly \le \bar{b}^l, \\ 1' & x^TB^ly \ge \bar{b}^l. \end{cases}$$

We consider the equilibrium solutions in terms of maximizing the degree of attainment of fuzzy goals. We use the method of Bellman and Zadeh [2] to aggregate the fuzzy goals.
The aggregated fuzzy goals of Players I and II are as follows, respectively:

$$\begin{array}{l} \mu_1(x,y) = \min_{k=1,\dots,r} \mu_1^k(x^TA^ky), \\ \mu_2(x,y) = \min_{l=1,\dots,s} \mu_2^l(x^TB^ly). \end{array} \tag{10}$$

Definition 7. A pair of strategies (x^*, y^*) is an equilibrium solution with respect to the degree of attainment of the fuzzy goals aggregated by the minimum component for the multiobjective $CBG : (\hat{X}, \hat{Y}, A, B)$ with fuzzy goals if for any other mixed strategies $x \in \hat{X}$ and $y \in \hat{Y}$

$$\mu_1(x^*, y^*) \ge \mu_1(x, y^*)$$

and

$$\mu_2(x^*, y^*) \ge \mu_2(x^*, y),$$

where μ_1 and μ_2 are given by (10).

Based on the above definition, an equilibrium solution is (x^*, y^*) in which, x^* and y^* are the optimal solutions to the following two mathematical programming problems:

$$\begin{array}{ll} \text{maximize} & \min\{\mu_1^1(x^TA^1y^*), \dots, \mu_1^r(x^TA^ry^*)\} \\ \text{s.t.} & e_m^Tx = 1, \\ & D^Tx \le c, \\ & x \ge 0, \end{array}$$

and

$$\begin{array}{ll} \text{maximize} & \min\{\mu_2^1(x^{*T}B^1y), \dots, \mu_2^s(x^{*T}B^sy)\} \\ \text{s.t.} & e_n^Ty = 1, \\ & E^Ty \le d, \\ & y \ge 0. \end{array}$$

Remark 2. Similar to the proposed theorem for unconstrained bimatrix games in [17] (Lemma 4.2.1), we can say that an equilibrium point is a pair of strategies (x^*, y^*) in which x^* and y^* are the optimal solutions of the following problems, respectively:

$$\begin{array}{ll} \text{maximize} & \min_{k=1,\ldots,r}\{c_1^k + x^T \hat{A}^k y^*\} \\ \text{s.t.} & e_m^T x = 1, \\ & D^T x \leq c, \\ & 0 \leq c_1^k + x^T \hat{A}^k y^* \leq 1, \quad k = 1,\ldots,r, \\ & x \geq 0, \end{array} \tag{11}$$

and

$$\begin{array}{ll} \text{maximize} & \min_{l=1,\ldots,s}\{c_2^l + x^{*T} \hat{B}^l y\} \\ \text{s.t.} & e_n^T y = 1, \\ & E^T y \leq d, \\ & 0 \leq c_2^l + x^{*T} \hat{B}^l y \leq 1, \quad l = 1,\ldots,s, \\ & y \geq 0. \end{array} \tag{12}$$

Since the constraints of the problems (11) and (12) are separable, these problems yield the following mathematical programming problem:

$$\begin{array}{ll} \text{maximize} & (\min_{k=1,\ldots,r}\{c_1^k + x^T \hat{A}^k y^*\} + \min_{l=1,\ldots,s}\{c_2^l + x^{*T} \hat{B}^l y\}) \\ \text{s.t.} & D^T x \leq c, \\ & E^T y \leq d, \\ & e_m^T x = 1, \\ & e_n^T y = 1, \\ & 0 \leq c_1^k + x^T \hat{A}^k y^* \leq 1, \quad k = 1,\ldots,r, \\ & 0 \leq c_2^l + x^{*T} \hat{B}^l y \leq 1, \quad l = 1,\ldots,s, \\ & x \geq 0, \\ & y \geq 0. \end{array}$$

Now, we can present the following theorem.

Theorem 5. *Let $(x^*, y^*, \alpha^*, \beta^*, \lambda_1^*, \lambda_2^*, u^*, v^*)$ be an optimal solution to the following nonlinear programming problem*

(NLP1):

$$\begin{array}{ll} \max & \lambda_1 + \lambda_2 - cu - dv - \alpha - \beta \\ s.t. & \\ & \hat{A}^k y + c_1^k e_m \leq cue_m + \alpha e_m \quad \textit{for some } k \in \{1,\ldots,r\}, \qquad (c1) \\ & (\hat{B}^l)^T x + c_2^l e_n \leq dve_n + \beta e_n \quad \textit{for some } l \in \{1,\ldots,s\}, \qquad (c2) \\ & x^T \hat{A}^k y + c_1^k \geq \lambda_1, \quad k = 1,\ldots,r, \qquad (c3) \\ & x^T \hat{B}^l y + c_2^l \geq \lambda_2, \quad l = 1,\ldots,s, \qquad (c4) \\ & D^T x \leq c, \qquad (c5) \\ & E^T y \leq d, \qquad (c6) \end{array}$$

$$
\begin{aligned}
&e_m^T x = 1, && (c7)\\
&e_n^T y = 1, && (c8)\\
&0 \leq c_1^k + x^T \hat{A}^k y \leq 1, && (c9)\\
&0 \leq c_2^l + x^T \hat{B}^l y \leq 1, && (c10)\\
&0 \leq \lambda_1, \lambda_2 \leq 1, && (c11)\\
&x, y, u, v \geq 0. && (c12)
\end{aligned}
$$

Then (x^*, y^*) is an equilibrium solution with respect to the degree of attainment of the aggregated fuzzy goal.

Proof. First, we show that the optimal value of objective is zero. Let $(x, y, \alpha, \beta, \lambda_1, \lambda_2, u, v)$ be an arbitrary feasible solution of the problem (NLP1). From the constraints (c1) and (c7), it is easy to verify that

$$x^T \hat{A}^k y + c_1^k \leq cu + \alpha \qquad \text{for some}\, k \in \{1, \ldots, r\}.$$

Thus

$$\min_k \left\{ x^T \hat{A}^k y + c_1^k \right\} \leq cu + \alpha.$$

Similarly, from the constraints (c2) and (c8) we have

$$\min_l \left\{ x^T \hat{B}^l y + c_2^l, \right\} \leq dv + \beta.$$

From the constraints (c3) and (c4),

$$\min_k \left\{ x^T \hat{A}^k y + c_1^k \right\} \geq \lambda_1,$$

and

$$\min_l \left\{ x^T \hat{B}^l y + c_2^l \right\} \geq \lambda_2.$$

Therefore, we have

$$\lambda_1 + \lambda_2 - cu - dv - \alpha - \beta \leq 0.$$

In other words, the objective function value of (NLP1) is less than or equal to 0. We can verify that the solution $(\bar{x}, \bar{y}, \bar{\alpha}, \bar{\beta}, \bar{\lambda}_1, \bar{\lambda}_2, \bar{u}, \bar{v})$ in which

$$
\begin{aligned}
&\bar{x} = x^* \in \hat{X},\\
&\bar{y} = y^* \in \hat{Y},\\
&\bar{\alpha} = \min\nolimits_k \left\{ x^{*T} \hat{A}^k y^* + c_1^k \right\} - cu^*,\\
&\bar{\beta} = \min\nolimits_l \left\{ x^{*T} \hat{B}^l y^* + c_2^l \right\} - dv^*,\\
&\bar{\lambda}_1 = \min\nolimits_k \left\{ x^{*T} \hat{A}^k y^* + c_1^k \right\},\\
&\bar{\lambda}_2 = \min\nolimits_l \left\{ x^{*T} \hat{B}^l y^* + c_2^l \right\},\\
&\bar{u} = u^*, \text{and } \bar{v} = v^*,
\end{aligned}
$$

is feasible to the problem (NLP1). Assume that it is not feasible. So the solution does not satisfies at least one of the two constraints (c1) and (c2) of the problem (NLP1). We assume that the constraint (c1) does not hold. The proof of the other case is similar. Thus,

$$\hat{A}^k y^* + c_1^k e_m > cu^* e_m + \alpha^* e_m. \tag{13}$$

Multiplying $x^* \in \hat{X}$ both sides of (13), we have

$$x^{*T} \hat{A}^k y^* + c_1^k > cu^* + \alpha^*.$$

Thus,

$$\min_k \left\{ x^{*T} \hat{A}^k y^* + c_1^k \right\} > \min_k \left\{ x^{*T} \hat{A}^k y^* + c_1^k \right\}.$$

Also, its objective function value is zero. It means that the optimal value of objective of the nonlinear problem (NLP1) is zero.

Now, Let $(x^*, y^*, \alpha^*, \beta^*, \lambda_1^*, \lambda_2^*, u^*, v^*)$ be an optimal solution to the programming problem (NLP1). Thus

$$\lambda_1^* + \lambda_2^* - cu^* - dv^* - \alpha^* - \beta^* = 0. \tag{14}$$

From the constraints (c1) and (c2), for each feasible solution $(x, y, \alpha, \beta, \lambda_1, \lambda_2, u, v)$, we have

$$\begin{array}{ll} x^T \hat{A}^k y^* + c_1^k \le cu^* + \alpha^* & \text{for some k} \in \{1, \ldots, m\}, \\ x^{*T} \hat{B}^l y + c_2^l \le dv^* + \beta^* & \text{for some l} \in \{1, \ldots, n\}. \end{array}$$

Thus,

$$\begin{array}{l} \min_k \left\{ x^T \hat{A}^k y^* + c_1^k \right\} \le cu^* + \alpha^*, \\ \min_l \left\{ x^{*T} \hat{B}^l y + c_1^l \right\} \le dv^* + \beta^*. \end{array} \tag{15}$$

Also, according to the constraints (c3) and (c4),

$$\begin{array}{l} \min_k \left\{ x^{*T} \hat{A}^k y^* + c_1^k \right\} \ge \lambda_1^*, \\ \min_l \left\{ x^{*T} \hat{B}^l y^* + c_2^l \right\} \ge \lambda_2^*. \end{array} \tag{16}$$

Now, from (14) and (16) it follows that

$$\alpha^* + \beta^* + cu^* + dv^* = \lambda_1^* + \lambda_2^* \le \min_k \{x^{*T} \hat{A}^k y^* + c_1^k\} + \min_l \{x^{*T} \hat{B}^l y^* + c_2^l\}.$$

Therefore,

$$\min_k \{x^{*T} \hat{A}^k y^* + c_1^k\} \ge -\min_l \{x^{*T} \hat{B}^l y^* + c_2^l\} + \alpha^* + \beta^* + cu^* + dv^*. \tag{17}$$

On the other hand, the first inequality of (15) for $x = x^*$ and (17) imply that

$$cu^* + \alpha^* \geq \min_k \{x^{*T}\hat{A}^k y^* + c_1^k\} \geq -\min_l \{x^{*T}\hat{B}^l y^* + c_2^l\} + \alpha^* + \beta^* + cu^* + dv^*.$$

It follows that

$$\min_l \{x^{*T}\hat{B}^l y^* + c_2^l\} \geq \beta^* + dv^*. \tag{18}$$

Similarly,

$$\min_k \{x^{*T}\hat{A}^k y^* + c_1^k\} \geq \alpha^* + cu^*. \tag{19}$$

Thus, from (15), (18), and (19), we have

$$\begin{array}{l} \min_k \{x^{*T}\hat{A}^1 y^* + c_1^1\} = \alpha^* + cu^*, \\ \min_l \{x^{*T}\hat{B}^l y^* + c_2^l\} = \beta^* + dv^*. \end{array} \tag{20}$$

Consequently, (15) and (20) imply that

$$\min_k \{x^{*T}\hat{A}^k y^* + c_1^k\} \geq \min_k \{x^T\hat{A}^k y^* + c_1^k\}, \qquad \forall x \in \hat{X},$$

and

$$\min_l \{x^{*T}\hat{B}^l y^* + c_2^l\} \geq \min_l \{x^{*T}\hat{B}^l y + c_2^l\} \qquad \forall y \in \hat{Y}.$$

Hence, x^* and y^* are, respectively, optimal solutions to the problems (11) and (12). Therefore, by Remark 2 the pair (x^*, y^*) is an equilibrium solution with respect to the degree of attainment of the fuzzy goals aggregated by the minimum component for the multiobjective game. □

Using the techniques of solving nonlinear programming problems containing constraints such as (c1) and (c2), the nonlinear problem (NLP1) can be solved by solving the following mixed binary nonlinear programming problem.

$$\begin{array}{l} \max\ \lambda_1 + \lambda_2 - cu - dv - \alpha - \beta \\ \text{s.t.} \\ \quad \hat{A}^1 y + c_1^1 e_m \leq cue_m + \alpha e_m + M(1-\delta_1)e_m, \\ \quad \vdots \\ \quad \hat{A}^r y + c_1^r e_m \leq cue_m + \alpha e_m + M(1-\delta_r)e_m, \\ \quad (\hat{B}^1)^T x + c_2^1 e_n \leq dve_n + \beta e_n + M(1-\delta_1^{'})e_n, \\ \quad \vdots \\ \quad (\hat{B}^s)^T x + c_2^s e_n \leq dve_n + \beta e_n + M(1-\delta_s^{'})e_n, \\ \quad x^T\hat{A}^k y + c_1^k \geq \lambda_1, \qquad k = 1, \ldots, r, \\ \quad x^T\hat{B}^l y + c_2^l \geq \lambda_2, \qquad l = 1, \ldots, s, \end{array}$$

$$
\begin{aligned}
& D^T x \leq c, \\
& E^T y \leq d, \\
& e_m^T x = 1, \\
& e_n^T y = 1, \\
& 0 \leq c_1^k + x^T \hat{A}^k y \leq 1, \\
& 0 \leq c_2^l + x^T \hat{B}^l y \leq 1, \\
& 0 \leq \lambda_1, \lambda_2 \leq 1, \\
& \sum_{k=1}^{r} \delta_k = 1, \\
& \sum_{l=1}^{s} \delta_l^{'} = 1, \\
& \delta_k = 0 \, or \, 1, \qquad k = 1, \ldots r, \\
& \delta_l^{'} = 0 \, or \, 1, \qquad l = 1, \ldots s, \\
& x, y, u, v \geq 0,
\end{aligned}
$$

where M is an enough large number. This problem can be solved by branch and bound methods of solving mixed zero-one nonlinear problems [14], or by using optimization softwares such as Lingo [25] and Gams [26].

3.2 Multiobjective constrained bimatrix games with fuzzy goals and fuzzy constraints

Assume that the constraints imposed on mixed strategies are fuzzy constraints. In other words, the mixed strategy spaces for Players I and II are, respectively, as follows:

$$\tilde{X} = \left\{ x \in \mathbb{R}^m \left| e_m^T x = 1, Dx \tilde{\geq} c \ , \ x \geq 0 \right. \right\},$$

$$\tilde{Y} = \left\{ y \in \mathbb{R}^n \left| e_n^T y = 1, Ey \tilde{\geq} d \ , \ y \geq 0 \right. \right\}.$$

The symbol "$\tilde{\geq}$" denotes a relaxed or fuzzy version of the ordinary inequality "$\geq$". (We consider only fuzzy version of $\geq$ in order to adopt to the considered fuzzy goals. Note that a $\leq$ constraint can be transformed to a $\geq$ one). More explicitly, these fuzzy inequalities mean that "for $t = 1, \ldots, p$ and $\tau = 1, \ldots, q$, the constraints $D_t x$ and $E_\tau y$ should be substantially more than or equal to c_t and d_τ, respectively ".

For treating the fuzzy inequality $D_t x \tilde{\geq} c_t$, we use the following linear membership function

$$\mu_1(D_t x) = \begin{cases} 0, & D_t x \leq \underline{c_t}, \\ 1 - \frac{\bar{c}_t - D_t x}{\bar{c}_t - \underline{c_t}}, & \underline{c_t} \leq D_t x \leq \bar{c}_t, \\ 1, & D_t x \geq \bar{c}_t, \end{cases} \tag{21}$$

where $\bar{c_t} = c_t$ and $\underline{c_t} = c_t - \delta_t$, in which δ_t is a subjectively chosen constant expressing the limit of the admissible violation of the inequality. Letting $\hat{D}_t = \frac{D_t}{\bar{c_t}-\underline{c_t}}$ and $c_{1t}^{'} = -\frac{c_t}{\bar{c_t}-\underline{c_t}}$, the above membership function can be rewritten as follows:

$$\mu_1(D_t x) = \begin{cases} 0, & D_t x \le \underline{c_t}, \\ c_{1t}^{'} + \hat{D}_t x, & \underline{c_t} \le D_t x \le \bar{c_t}, \\ 1, & D_t x \ge \bar{c_t}. \end{cases} \tag{22}$$

Also, $\mu_2(E_\tau y)$ can be defined similar to (21) and rewritten as

$$\mu_2(E_\tau y) = \begin{cases} 0, & E_\tau y \le \underline{d_\tau}, \\ c_{2\tau}^{'} + \hat{E}_\tau y, & \underline{d_\tau} \le E_\tau y \le \bar{d_\tau}, \\ 1, & E_\tau y \ge \bar{d_\tau}, \end{cases} \tag{23}$$

where $\hat{E}_\tau = \frac{E_\tau}{\bar{d_\tau}-\underline{d_\tau}}$, $c_{2\tau}^{'} = -\frac{d_\tau}{\bar{d_\tau}-\underline{d_\tau}}$, $\bar{d_\tau} = d_\tau$, and $\underline{d_\tau} = d_\tau - \delta_\tau^{'}$.

According to these membership functions, we define an equilibrium solution with respect to the degree of attainment of the aggregated fuzzy goal as follows.

Definition 8. A pair of strategies (x^*, y^*) is an equilibrium solution with respect to the degree of attainment of the fuzzy goals aggregated by the minimum component for the multiobjective $CBG : (\tilde{X}, \tilde{Y}, A, B)$ with fuzzy goals if for any other mixed strategies $x \in \tilde{X}$ and $y \in \tilde{Y}$

$$\min_{\substack{k=1,\dots,r \\ t=1,\dots,p}} \left\{\mu_1^k(x^{*T} A^k y^*), \mu_1(D_t x^*)\right\} \ge \min_{\substack{k=1,\dots,r \\ t=1,\dots,p}} \left\{\mu_1^k(x^T A^k y^*), \mu_1(D_t x)\right\},$$

and

$$\min_{\substack{l=1,\dots,s \\ \tau=1,\dots,q}} \left\{\mu_2^l(x^{*T} B^l y^*), \mu_2(E_\tau y^*)\right\} \ge \min_{\substack{l=1,\dots,s \\ \tau=1,\dots,q}} \left\{\mu_2^l(x^{*T} B^l y), \mu_2(E_\tau y)\right\}.$$

According to the membership functions (22) and (23), we can say that an equilibrium point is a pair of strategies (x^*, y^*) in which x^* and y^* are the optimal solutions of the following problems, respectively.

$$\begin{array}{ll} maximize & \min\{c_1^1 + x^T \hat{A}^1 y^*, \dots, c_1^r + x^T \hat{A}^r y^*, c_{11}^{'} + \hat{D}_1 x, \dots, c_{1p}^{'} + \hat{D}_p x\} \\ s.t. & e_m^T x = 1, \\ & 0 \le c_1^k + x^T \hat{A}^k y^* \le 1, \quad k = 1, \dots, r, \\ & 0 \le c_{1t}^{'} + \hat{D}_t x \le 1, \qquad t = 1, \dots, p, \\ & x \ge 0, \end{array} \tag{24}$$

and

$$\begin{array}{lll} maximize & \min\{c_2^1 + x^{*T}\hat{B}^1 y, \ldots, c_2^s + x^{*T}\hat{B}^s y, c'_{21} + \hat{E}_1 y, \ldots, c'_{2q} + \hat{E}_q y\} & \\ s.t. & e_n^T y = 1, & \\ & 0 \le c_2^l + x^{*T}\hat{B}^l y \le 1, \quad l = 1, \ldots, s, & \\ & 0 \le c'_{2\tau} + \hat{E}_\tau y \le 1, \qquad \tau = 1, \ldots, q, & \\ & y \ge 0. & \end{array} \tag{25}$$

Note that the problems (24) and (25) are similar to the problems presented in [17] for solving bimatrix games with fuzzy goals in which additional fuzzy constraints imposed on the strategies becomes additional fuzzy goals. Thus it is natural to extend the presented theorems in [17]. The following theorem expresses that an equilibrium point can be obtained by solving a mathematical programming problem.

Theorem 6. *(Equivalence theorem). Let $(x^*, y^*, \alpha^*, \beta^*, \lambda_1^*, \lambda_2^*)$ be an optimal solution to the following nonlinear programming problem:*

(NLP2):

$$\begin{array}{ll} \max \lambda_1 + \lambda_2 - \alpha - \beta & \\ s.t. & \\ \left\{ \begin{array}{ll} \hat{A}^k y + c_1^k e_m - \alpha e_m \le 0 & \text{for some } k \in \{1, \ldots, r\}, \\ \hat{D}_t + c'_{1t} e_m \le \alpha e_m & \text{for some } t \in \{1, \ldots, p\}, \end{array} \right. & \\ \left\{ \begin{array}{ll} (\hat{B}^l)^T x + c_2^l e_n - \beta e_n \le 0 & \text{for some } l \in \{1, \ldots, s\}, \\ \hat{E}_\tau + c'_{2\tau} e_n \le \beta e_n & \text{for some } \tau \in \{1, \ldots, q\}, \end{array} \right. & \\ x\hat{A}^k y + c_1^k \ge \lambda_1, & k = 1, \ldots, r, \\ \hat{D}_t^T x + c'_{1t} \ge \lambda_1, & t = 1, \ldots, p, \\ x\hat{B}^l y + c_2^l \ge \lambda_2, & l = 1, \ldots, s, \\ \hat{E}_\tau^T y + c'_{2\tau} \ge \lambda_2, & \tau = 1, \ldots, q, \\ e_m^T x = 1, & \\ e_n^T y = 1, & \\ 0 \le c_1^k + x\hat{A}^k y, \le 1 & k = 1, \ldots, r, \\ 0 \le c'_{1t} + \hat{D}_t x \le 1, & t = 1, \ldots, p, \\ 0 \le c_2^l + x^T \hat{B}^l y \le 1, & l = 1, \ldots, s, \\ 0 \le c'_{2\tau} + \hat{E}_\tau y \le 1, & \tau = 1, \ldots, q, \\ 0 \le \lambda_1, \lambda_2 \le 1, & \\ x, y \ge 0. & \end{array}$$

Then (x^*, y^*) is an equilibrium solution with respect to the degree of attainment of the aggregated fuzzy goal.

Proof. The proof is similar to the proof of Theorem 5. □

Note that if the fuzzy constraints imposed on mixed strategies are not considered, the problem (NLP2) is the same as the introduced problem by Nishizaki and Sakawa [17] to solve bimatrix game with fuzzy goals.

Similar to the problem with crisp constraints in subsection 3.1, to solve this problem, it is sufficient to solve the following mixed binary nonlinear programming.

$$
\begin{aligned}
&\max \lambda_1 + \lambda_2 - \alpha - \beta \\
&s.t. \\
&\quad \hat{A}^1 y + c_1^1 e_m \le \alpha e_m + M(1-\delta_1) e_m, \\
&\quad \vdots \\
&\quad \hat{A}^r y + c_1^r e_m \le \alpha e_m + M(1-\delta_r) e_{m,'} \\
&\quad \hat{D}_1 + c_{11}^{'} e_m \le \alpha e_m + M(1-\delta_{r+1}) e_m, \\
&\quad \vdots \\
&\quad \hat{D}_p + c_{1p}^{'} e_m \le \alpha e_m + M(1-\delta_{r+p}) e_m, \\
&\quad (\hat{B}^1)^T x + c_2^1 e_n \le \beta e_n + M(1-\delta_1^{'}) e_n, \\
&\quad \vdots \\
&\quad (\hat{B}^s)^T x + c_2^s e_n \le \beta e_n + M(1-\delta_s^{'}) e_n, \\
&\quad \hat{E}_1 + c_{21}^{'} e_n \le \beta e_n + M(1-\delta_{s+1}^{'}) e_n, \\
&\quad \vdots \\
&\quad \hat{E}_q + c_{2q}^{'} e_n \le \beta e_n + M(1-\delta_{s+q}^{'}) e_n, \\
&\quad x^T \hat{A}^k y + c_1^k \ge \lambda_1, \quad k = 1, \ldots, r, \\
&\quad \hat{D}_t^T x + c_{1t}^{'} \ge \lambda_1, \quad t = 1, \ldots, p, \\
&\quad x^T \hat{B}^l y + c_2^l \ge \lambda_2, \quad l = 1, \ldots, s, \\
&\quad \hat{E}_\tau^T y + c_{2\tau}^{'} \ge \lambda_2, \quad \tau = 1, \ldots, q, \\
&\quad e_m^T x = 1, \\
&\quad e_n^T y = 1, \\
&\quad 0 \le c_1^k + x \hat{A}^k y \le 1, \quad k = 1, \ldots, r, \\
&\quad 0 \le c_{1t}^{'} + \hat{D} x \le 1, \quad t = 1, \ldots, p, \\
&\quad 0 \le c_2^l + x^T \hat{B}^l y \le 1, \quad l = 1, \ldots, s, \\
&\quad 0 \le c_{2\tau}^{'} + \hat{E} y \le 1, \quad \tau = 1, \ldots, q, \\
&\quad 0 \le \lambda_1, \lambda_2 \le 1, \\
&\quad \sum_{k=1}^{r+p} \delta_k = 1, \\
&\quad \sum_{l=1}^{s+q} \delta_l^{'} = 1, \\
&\quad \delta_k = 0 \, or \, 1, \qquad k = 1, \ldots r+p, \\
&\quad \delta_l^{'} = 0 \, or \; 1, \qquad l = 1, \ldots s+q, \\
&\quad x, y, u, v \ge 0.
\end{aligned}
$$

Again, the above problem can be solved by branch and bound methods of solving mixed zero-one nonlinear problems [14], or by using optimization softwares such as Lingo [25] and Gams [26].

In this section, We use multiobjective optimization to study multiobjective two-person constrained bimatrix games. Thus, we expect to consider Pareto optimality concept for multiobjective two-person bimatrix games. Before defining weak Pareto optimal equilibrium solution, we recall the definition of the best reply strategies as follows by using the concept of Pareto optimality in multiobjective optimization.

Definition 9. [17]. Let a payoff vector of Player I be denoted by $p_1(x,y) \in \mathbb{R}^r$ when Player I chooses a mixed strategy $x \in \hat{X}$, and let Player II chooses a mixed strategy $y \in \hat{Y}$. Player I's preference cone is defined by

$$C_1 = \left\{ z = (z^1, \dots, z^r) \in \mathbb{R}^r | z^k \geq 0,\, k = 1, \dots, r \right\}.$$

Then, given Player II's strategy $\hat{y}$, the set of payoffs for the weak Pareto best reply strategies is defined by

$$p_1(\hat{y}) = \{ p_1(x,\hat{y}) \in z_1(\hat{y}) | z_1(\hat{y}) \cap (p_1(x,\hat{y}) + intC_1) = \varnothing$$

$$for\, some\, strategy\, x \in \hat{X}\, of\, Player\, I \}$$

where $z_1(\hat{y})$ is the set of attainable payoffs of Player I against the strategy $\hat{y} \in \hat{Y}$ of Player II and $intC_1$ denotes the set of interior points of C_1. Similarly, let Player II's payoff vector be denoted by $p_2(x,y) \in \mathbb{R}^s$, and let Player II's preference cone be

$$C_2 = \left\{ z = (z^1, \dots, z^s) \in \mathbb{R}^s | z^l \geq 0,\, l = 1, \dots, s \right\}.$$

Then, given Player I's strategy $\hat{x} \in \hat{X}$, the set of payoffs for weak Pareto best reply strategies is defined by

$$p_2(\hat{x}) = \{ p_2(\hat{x},y) \in z_2(\hat{x}) | z_2(\hat{x}) \cap (p_2(\hat{x},y) + intC_2) = \varnothing,$$

$$for\, some\, strategy\, y \in \hat{Y}\, of\, Player\, II \},$$

where $z_2(\hat{x})$ is the set of attainable payoffs of Player II against the strategy $\hat{x}$ of Player I.

Definition 10. [23]. Let the payoff vectors of Players I and II be $p_1(x,y) = (p_1^1(x,y), \dots, p_1^r(x,y))$ and $p_2(x,y) = (p_2^1(x,y), \dots, p_2^s(x,y))$, respectively. For any pair of strategies $x \in \hat{X}$ and $y \in \hat{Y}$, let Player I's set of payoff vectors for the weak Pareto best reply strategies and Player II's set of payoff vectors for the weak Pareto best reply strategies be denoted by $p_1(y)$ and $p_2(x)$, respectively. Then the set of the weak Pareto optimal equilibrium solutions is defined by

$$WPE = \{ (x^*, y^*) | p_1^*(x^*, y^*) \in p_1(y^*),\, p_2^*(x^*, y^*) \in p_2(x^*) \}.$$

Pareto optimal equilibrium solution can be defined by replacing $intC_1$ and $intC_2$ with $C_1 \backslash \{0\}$ and $C_2 \backslash \{0\}$ in the above definitions, respectively.
Wierzbicki [23] explored in detail the relation between scalarizing functions and Pareto optimal equilibrium solutions. If scalarizing function is a strictly monotone function, then the obtained equilibrium point is Pareto optimal

equilibrium solution, and if it is monotone function which is not always strictly monotone, then obtained equilibrium point is weak Pareto optimal equilibrium solution. Also, suppose a player assesses that $\underline{a}^k$ or $\underline{b}^l$ is sufficiently small and assesses that $\bar{a}^k$ or $\bar{b}^l$ is sufficiently large. We have the following theorem.

Theorem 7 *An equilibrium solution with respect to the degree of attainment of the fuzzy goals aggregated by a minimum component is a weak Pareto optimal equilibrium solution.*

Proof. According to that the minimum component is not always strictly monotone, the proof is completed. □

4 A political application: investigating nuclear negotiations

The application of game theory and its importance to social science were quickly recognized. Now, game theory is used extensively in analyzing psychology, philosophy, sociology, politics, and economics. Most importantly, game theory is used to analyze international relations, specifically in countries with conflicting goals and interests. Game theory provides a logical analysis of situations of conflict and cooperation. A game is a situation in which

1) There are at least two players. A player may be an individual, but it may also be a more general entity like a company, a nation, or even a biological species.

2) Each player has a number of possible strategies, courses of action which he may choose to follow.

3) The strategies chosen by each player determine his outcome of the game.

4) Associated to each possible outcome of the game there is a collection of numerical payoffs, one to each player. These payoffs represent the value of the outcome to the different players.

Game theory can be divided into two categories: zero-sum games and nonzero-sum games. Zero-sum games are games in which one player wins and the other loses. The two players do not cooperate, and their interests are in total conflict. Each player chooses a certain set of strategies, and he does not know the choices of the other player. Nonzero-sum games, on the other hand, are games in which the interests of the players are not strictly opposed. The success of one player does not come as a result of the failure of the other player.

Because the interests of the players are not in total conflict and not strictly coincident, a nonzero-sum game allows the players to both compete and cooperate to achieve outcomes that are advantageous to both players. Nonzero-sum games can be played without communication, with communication before the game, and with cooperation. In international relations, countries compete and cooperate to maximize their national interests. These countries also have diplomatic relations, and therefore, they communicate with one another. Therefore, this study will focus on the nonzero-sum games in which the players can communicate before the game strategic moves.

It is important to note that the data in this discussion are bogus totally and is brainchild of the authors and have not specific explanation for what happened in the world of today between two countries.

Let us explain the details of the problem. Country I is sanctioned by Country II, and its allies due to nuclear activities. Now, Country I is not in a good economic position. To fix the economic problem, Country I decides to negotiate with Country II to lift nuclear sanctions and achieve the favorable economic situation in the region. However, it is possible to Country I find the weak political position among its allies and its people. It is possible to improve the political situation in the other countries due to negotiation but Country I is not optimistic to this improvement. Since Country I is interested in a win-win game, so we formulate our problem as a win-win game. In this negotiation it is assumed that after the establishment of results, the countries act on their decisions. During the negotiations, Country I's strategies are as follows:

(1) Do not reduce the number of centrifuges (even increase their numbers) and extend nuclear activities.

(2) Reduce of the number of centrifuges and limit the nuclear activities under strict inspections.

(3) Suspend the nuclear activities.

Also, Country II's strategies are as follows:

(1) Do not cancel any of sanctions.

(2) Cancel some natural or legal sanctions.

(3) Military attack.

Country II has stated that if Country I chooses the strategies (2) or (3), then he may choose second strategy. Both countries do not trust each other to do after agreement. So data should be chosen in such a way that the distrust between two countries be considered. Here we assume that both countries said that they would act after the agreement. Note that there are opponents of this agreement in congress of Country II. Thus, it is expected that after the agreement, by congressional pressure on the government some

Country II

	(50,30)	(80,5)	(45,10)
Country I	(20,70)	(20,50)	(30,15)
	(10,60)	(10,30)	(15,5)

Political position

Country II

	(60,30)	(70,40)	(5,20)
Country I	(1,50)	(50,50)	(0,10)
	(5,40)	(20,40)	(5,10)

Economic position

new sanctions are imposed against Country I way out of nuclear activities. Country II choosing the first strategy in addition to sanctions, concerned to military threats and create sedition in Country I. It is also possible, even move with second strategy to military threats and create sedition in this country. However, with Country II's commitment, Country I predicts these actions after the agreement. Although it is possible mathematical models of these issues are complex in real world, our aim of this discussion is only to show a part of applications of these games in the nuclear negotiations and the method of determining the best strategy.

Suppose that Country I will incur cost as much as 55, 70, and 80 unites if he chooses the strategies (1), (2), and (3), respectively. Country I does not want to spend more than 75 unites; that is, the mixed strategies of the country I must satisfy the constraint condition: $55x_1 + 70x_2 + 80x_3 \leq 75$.

Country II will incur cost as much as 30, 60, and 90 unites, by choosing the strategies (1), (2), and (3), respectively. However, Country II only provides 80 unites;that is, the mixed strategies of the country II must satisfy the constraint condition: $30y_1 + 60y_2 + 90y_3 \leq 80$.

Assume that the payoff matrices of this game in view of political and economic positions are given in Table 1. The numbers in these tables are the payoffs for the countries. Country I's payoffs are the first numbers, and Country II's payoffs are second numbers.

We ask with players the amounts which are satisfied with them and the limit of the admissible violation of the inequalities. Let fuzzy goals $\tilde{G}_1^1$ and $\tilde{G}_1^2$ of the Country I for the two objectives be represented by the following linear membership functions:

$$\mu_1(x^T A^1 y) = \begin{cases} 0, & x^T A^1 y \leq 20, \\ 1 - \frac{40 - xA^1 y}{40-20}, & 20 \leq x^T A^1 y \leq 40, \\ 1, & x^T A^1 y \geq 40, \end{cases}$$

and

$$\mu_1(x^TA^2y)=\begin{cases}0, & x^TA^2y\leq 15,\\ 1-\frac{50-xA^2y}{25-15}, & 15\leq x^TA^2y\leq 50,\\ 1, & x^TA^2y\geq 50,\end{cases}$$

Let fuzzy goals $\tilde{G}_2^1$ and $\tilde{G}_2^2$ of the Country II for the two objectives be represented by the following linear membership functions:

$$\mu_2(x^TB^1y)=\begin{cases}0, & x^TB^1y\leq 30,\\ 1-\frac{40-xB^1y}{40-30}, & 10\leq x^TB^1y\leq 40,\\ 1, & x^TB^1y\geq 40,\end{cases}$$

and

$$\mu_2(x^TB^2y)=\begin{cases}0, & x^TB^2y\leq 35,\\ 1-\frac{40-xB^2y}{45-35}, & 20\leq x^TB^2y\leq 40,\\ 1, & x^TB^2y\geq 40.\end{cases}$$

By solving the mixed binary mathematical programming problem (NLP1), the equilibrium point is obtained. Thus, equilibrium solutions are optimal solutions to the following problem.

$$\begin{aligned}
&\max\ \lambda_1+\lambda_2-75u-80v-\alpha-\beta\\
&\text{s.t.}\\
&\quad 2.5y_1+4y_2+2.25y_3-75u-\alpha\leq 1+M\delta_1,\\
&\quad y_1+y_2+1.5y_3-75u-\alpha\leq 1+M\delta_1,\\
&\quad 0.5y_1+0.5y_2+0.75y_3-75u-\alpha\leq 1+M\delta_1,\\
&\quad 12/7y_1+2y_2+1/7y_3-75u-\alpha\leq 3/7+M\delta_2,\\
&\quad 1/35y_1+10/7y_2+0y_3-75u-\alpha\leq 3/7+M\delta_2,\\
&\quad 1/7y_1+4/7y_2+1/7y_3-75u-\alpha\leq 3/7+M\delta_2,\\
&\quad 3x_1+7x_2+6x_3-80v-\beta\leq 3+M\delta_1^{'},\\
&\quad 0.5x_1+5x_2+3x_3-80v-\beta\leq 3+M\delta_1^{'},\\
&\quad 1x_1+1.5x_2+0.5x_3-80v-\beta\leq 3+M\delta_1^{'},\\
&\quad 6x_1+10x_2+8x_3-80v-\beta\leq 7+M\delta_2^{'},\\
&\quad 8x_1+10x_2+8x_3-80v-\beta\leq 7+M\delta_2^{'},\\
&\quad 4x_1+4x_2+2x_3-80v-\beta\leq 7+M\delta_2^{'},\\
&\quad 4/7x_1y_1+1/35x_2y_1+1/7x_3y_1+2x_1y_2+10/7x_2y_2+12/7x_3y_2\\
&\quad\ +1/7x_1y_3+0x_2y_3+1/7x_3y_3-\lambda_1\geq 3/7,\\
&\quad 6x_1y_1+10x_2y_1+8x_3y_1+8x_1y_2+10x_2y_2+8x_3y_2\\
&\quad\ +4x_1y_3+2x_2y_3+2x_3y_3-\lambda_2\geq 7,\\
&\quad 1.5x_1y_1+0.5x_2y_1+0.25x_3y_1+4x_1y_2+0.5x_2y_2+0.5x_3y_2\\
&\quad\ +0.5x_1y_3+1.5x_2y_3+0.25x_3y_3-\lambda_1\geq 1,
\end{aligned}$$

$$
\begin{aligned}
&3x_1y_1 + 7x_2y_1 + 6x_3y_1 + 0.5x_1y_2 + 5x_2y_2 + 3x_3y_2 \\
&\ +1x_1y_3 + 1.5x_2y_3 + 0.5x_3y_3 - \lambda_2 \geq 3, \\
&55x_1 + 70x_2 + 80x_3 \leq 75, \\
&30y_1 + 60y_2 + 90y_3 \leq 80, \\
&x_1 + x_2 + x_3 = 1, \\
&y_1 + y_2 + y_3 = 1, \\
&12/7x_1y_1 + 1/35x_2y_1 + 1/7x_3y_1 + 2x_1y_2 + 10/7x_2y_2 + 4/7x_3y_2 \\
&\ +1/7x_1y_3 + 0x_2y_3 + 1/7x_3y_3 - \lambda_1 \geq 3/7, \\
&2.5x_1y_1 + 1x_2y_1 + 0.5x_3y_1 + 4x_1y_2 + 0.5x_2y_2 + 0.5x_3y_2 \\
&\ +2.25x_1y_3 + 1.5x_2y_3 + 0.5x_3y_3 - \lambda_1 \geq 1, \\
&3x_1y_1 + 7x_2y_1 + 6x_3y_1 + 0.5x_1y_2 + 5x_2y_2 + 3x_3y_2 \\
&\ +1x_1y_3 + 1.5x_2y_3 + 0.5x_3y_3 - \lambda_1 \geq 3, \\
&6x_1y_1 + 10x_2y_1 + 8x_3y_1 + 8x_1y_2 + 10x_2y_2 + 8x_3y_2 \\
&\ +4x_1y_3 + 2x_2y_3 + 2x_3y_3 - \lambda_1 \geq 7, \\
&\delta_1 + \delta_2 = 1, \\
&\delta_1^{'} + \delta_2^{'} = 1, \\
&\delta_1, \delta_2 = 0 \, or \, 1, \\
&\delta_1^{'}, \delta_2^{'} = 0 \, or \, 1, \\
&x_1, x_2, x_3 \geq 0, \\
&y_1, y_2, y_3 \geq 0, \\
&u, v \geq 0.
\end{aligned}
$$

This problem is solved by Lingo software. The obtained strategies are as $(x_1^*, x_2^*, x_3^*) = (0.56, 0.38, 0.06)$ for Country I and $(y_1^*, y_2^*, y_3^*) = (0.69, 0.31, 0)$ for Country II. This means that according to the conditions of the problem the chance of success in negotiations between the two countries is that both countries choose their first strategy.

5 Conclusion

In this paper, constrained bimatrix games with single and multiple payoffs are studied, which incorporate fuzzy goals for objectives. Such fuzzy games has not been considered in previous researches, based on the best knowledge of the authors. A programming problem with linear constraints and a quadratic objective function was presented to obtain the equilibrium solution of single objective problem of such games. In multiobjective case, equilibrium points was defined and the mathematical programming problems was introduced to obtain the equilibrium points of this class of games with fuzzy goals and the fuzzy/crisp constraints. We proposed two mixed binary nonlinear programming problems to solve these mathematical programming problems. Weak Pareto optimal equilibrium solution was defined and showed that obtained equilibrium points of these problems are weak Pareto optimal equilibrium solutions. A political application of these games was presented which investigate nuclear negotiations between two countries. The equilibrium point of this game was obtained by the proposed method. Solving constrained multi-objective bimatrix games with fuzzy payoffs and fuzzy nonlinear goals needs more researches, which can be our future work.

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بازی‌های دوماتریسی مقید با آرمان‌های فازی و کاربرد آن در مذاکرات هسته‌ای

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دریافت مقاله ۴ اردیبهشت ۱۳۹۵، دریافت مقاله اصلاح شده ۳۰ بهمن ۱۳۹۵، پذیرش مقاله ۱ آذر ۱۳۹۶

چکیده : هدف این تحقیق، حل بازی‌های دوماتریسی مقید در محیط فازی است. این دسته از بازی‌های مجموع ناصفر دونفره، با راهبردهای متناهی و آرمان‌های فازی در نظر گرفته شده است که در آن مجموعه محدودیت‌های خطی روی راهبردها اعمال شده اند. بازی‌های مجموع ناصفر مقید به صورت تک‌هدفی و چندهدفی مورد بررسی قرار گرفته است. نشان داده شده که جواب تعادل در حالت تک‌هدفی از حل یک مسئله‌ی برنامه‌ریزی درجه دوم با محدودیت‌های خطی به دست می‌آید. همچنین برای یافتن نقطه‌ی تعادل در مسائل چندهدفی با محدودیت‌های قطعی و فازی، تعدادی مسئله‌ی برنامه‌ریزی ریاضی معرفی شده است. در نهایت، یک کاربرد سیاسی از چنین بازی‌هایی ارائه شده که در مورد مذاکرات هسته‌ای بین دو کشور می‌باشد.

کلمات کلیدی : بازی چندهدفی؛ بازی مقید؛ بازی مقید فازی؛ مذاکرات هسته‌ای.