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Research Article

Computation of eigenvalues of fractional Sturm–Liouville problems

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Abstract

We consider the eigenvalues of the fractional-order Sturm–Liouville equation of the form

$$-{}^{c}D^{\alpha}_{0^+} \circ D^{\alpha}_{0^+} y(t) + q(t)y(t) = \lambda y(t), \quad 0 < \alpha \leq 1, \quad t \in [0,1],$$

with Dirichlet boundary conditions

$$I^{1-\alpha}_{0^+} y(t)|_{t=0} = 0 \quad \text{and} \quad I^{1-\alpha}_{0^+} y(t)|_{t=1} = 0,$$

where $q \in L^2(0,1)$ is a real-valued potential function. The method is used based on Picard's iteration procedure. We show that the eigenvalues are obtained from the zeros of the Mittag-Leffler function and its derivatives.

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1 Introduction

Fractional Sturm–Liouville Problems (FSLPs) are generalizations of the classical Sturm–Liouville Problems in which the ordinary derivatives are replaced

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by fractional derivatives or derivatives of fractional order. As an introduction, the interested reader may wish to consult the great variety of works on the subject, starting with books such as [14, 20, 23]. Several authors have considered the numerical FSLP, for example Al-Mdallal [2, 24] applied the Adomian decomposition method for solving fractional Sturm–Liouville problems. Abbasbandy and Shirzadi [1] applied the homotopy analysis method for solving fractional Sturm–Liouville problems. Also in [8], the eigenvalue problems for the fractional ordinary differential equations have been investigated with different classes of boundary conditions including the Dirichlet, Neumann, and so on. They explained that choosing $\alpha = 2$ leads to the classical ones of the second-order differential equations. When the order α satisfies $1 < \alpha < 2$, the eigenvalues can be finitely many; see [9, 22]. It has been applied to many fields in science and engineering, such as viscoelasticity, anomalous diffusion, fluid mechanics, biology, chemistry, acoustics, control theory, and so on. In the applications mentioned above, a class of integro-differential equations with singularities, fractional differential equations have been involved; see [11, 16, 15, 17, 21, 18, 20, 5]. In [25], the authors have considered a regular fractional Sturm–Liouville problem (RFSLP) of two kinds RFSLP-I and RFSLP-II of order $\nu \in (0, 2)$ with the fractional differential operators both of Riemann–Liouville and Caputo type, of the same fractional-order $\mu = \nu/2 \in (0, 1)$. It was proved that the regular boundary-value problems RFSLP-I & -II are indeed asymptotic cases for the singular counterparts SFSLP-I & -II. The inverse Laplace transform method for obtaining analytical solutions of the FSLPs and eigenvalues has been investigated in [10]. The reproducing kernel method has also been used to calculate the eigenvalues of the FSLPs. Dehgan and Mingarelli [6, 7] have investigated the general solution of three- or two-term fractional differential equations of mixed Caputo/Riemann–Liouville type in the case of Dirichlet boundary conditions. From a numerical viewpoint, we also refer the reader for fractional differential equations to [2, 3, 4, 12, 24]. Particularly, the boundary value problem is of the form

$$-\,{}^{c}D^{\alpha}_{0+} \circ D^{\alpha}_{0+} y(t) + q(t)y(t) = \lambda y(t), \quad 0 < \alpha \leq 1, \quad t \in [0, 1], \tag{1}$$

$$I^{1-\alpha}_{0^+} y(t)|_{t=0} = 0, \quad \text{and} \quad I^{1-\alpha}_{0^+} y(t)|_{t=1} = 0.$$

For each $1/2 < \alpha < 1$, it is proved that there is a finite number of real eigenvalues, an infinite number of nonreal eigenvalues, that the number of such real eigenvalues grow without bound as $\alpha \to 1^-$. Also, they expressed asymptotically behavior of the eigenvalues as a function of α in the following form:

$$\lambda_n(\alpha) \sim \left(\frac{n\pi}{\sin(\frac{\pi}{2\alpha})}\right)^{2\alpha}, \qquad \alpha \to 1^-.$$

This corresponds exactly with the well-known classical asymptotic estimate $\lambda_n \to n^2\pi^2$ as $n \to \infty$; see [6, 7]. The remaining structure of this paper is organized as follows: In the next section, we review the Mittag-Leffler function and Laplace transform of the Riemann–Liouville and the Caputo fractional derivatives. In section 3, we discuss the zeros of the iterative method with two examples. In section 4, we discuss the analysis of the iterative method. The last section includes our conclusions.

2 Preliminaries

We recall some definitions in fractional calculus. We refer the reader to [6] for further details.

Definition 1. The left and the right Riemann–Liouville fractional integrals $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ of order $\alpha \in \mathbb{R}^+$ are defined by

$$I^{\alpha}_{a^+} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad (a, b], \tag{2}$$

and

$$I^{\alpha}_{b^-} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, \quad [a, b),$$

respectively.

Here $\Gamma(\alpha)$ denotes Euler's gamma function. The following property is easily verified.

Lemma 1. For a constant C, we have $I^{\alpha}_{a^+} C = \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \cdot C$.

Definition 2. The left and the right Caputo fractional derivatives ${}^cD^{\alpha}_{a^+}$ and ${}^cD^{\alpha}_{b^-}$ are defined by

$${}^cD^{\alpha}_{a^+} f(t) := I^{n-\alpha}_{a^+} \circ D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \ \ t > a, \tag{3}$$

and for $t < b$,

$${}^cD^{\alpha}_{b^-} f(t) := (-1)^n I^{n-\alpha}_{b^-} \circ D^n f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{\alpha-n+1} f^{(n)}(s) ds, \tag{4}$$

respectively, where f is sufficiently differentiable and $n-1 \leq \alpha < n$.

Definition 3. The left and the right Riemann–Liouville fractional derivatives $D^{\alpha}_{a^+}$ and $D^{\alpha}_{b^-}$ are defined by

$$D^{\alpha}_{a^+} f(t) := D^n \circ I^{n-\alpha}_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \ \ t > a, \tag{5}$$

and for $t < b$,

$$D_{b-}^{\alpha} f(t) := (-1)^n D^n \circ I_{b-}^{n-\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} f(s) ds, \quad (6)$$

respectively, where f is sufficiently differentiable and $n-1 \leq \alpha < n$.

2.1 The Mittag-Leffler function

The function $E_\alpha(z)$ is defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (z \in \mathbb{C}, \, R(\alpha) > 0), \quad (7)$$

which was introduced by Mittag-Leffler [20]. In particular, when $\alpha = 1$ and $\alpha = 2$, we have

$$E_1(z) = e^z, \quad E_2(z) = \cosh(\sqrt{z}).$$

The generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z, \theta \in \mathbb{C}, \, R(\alpha) > 0), \quad (8)$$

of course, when $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the Mittag-Leffler function (18):

$$E_{\alpha,1}(z) = E_\alpha(z).$$

Two other particular cases of (3) are as follows:

$$E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}.$$

Further properties of this special function may be found in [11].

Theorem 1 (see [22])**.** If $\alpha < 2$, β is an arbitrary real number, μ is such that
$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$, and c is a real constant, then

$$|E_{\alpha,\beta}(z)| \leq \frac{c}{1+|z|} \quad (\mu \leq |arg(z)| \leq \pi), \quad |z| \geq 0, \;\; z \in \mathbb{C}.$$

2.2 Laplace transform

Definition 4 (see [22, 19]). The Laplace transform of a function $f(t)$ defined for all real numbers $t \geq 0$, t stands for the time, is the function $F(s)$ that is a unilateral transform defined by

$$F(s) = \mathfrak{L}\{f(t)\} := \int_0^\infty e^{-st} f(t)dt,$$

where s is the frequency parameter.

Definition 5 (see [22, 19]). The convolution of $f(t)$ and $g(t)$ supported on only $[0, \infty)$ is defined by

$$(f * g)(t) = \int_0^t f(s)g(t-s)ds, \qquad f, g : [0, \infty) \to \mathbb{R}.$$

Property 1 (see [22]). The Laplace transform of the convolution of $f(t)$ and $g(t)$ is given by following relation:

$$\mathfrak{L}\{(f * g)(t)\} = \mathfrak{L}\{f(t)\} \times \mathfrak{L}\{g(t)\}.$$

Property 2 (see [6, 22]). The Laplace transform of the derivatives of the Mittag-Leffler function reads as follows:

$$\int_0^\infty e^{-st} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm\lambda t^\alpha)dt = \frac{k! s^{\alpha-\beta}}{(s^\alpha \mp \lambda)^{k+1}}, \qquad (Re(p) > |a|^{\frac{1}{\alpha}}).$$

Property 3 (see [13, 22]). The Laplace transform of the Riemann–Liouville fractional derivative is obtained as

$$(\mathfrak{L}_0 D_t^\alpha y)(s) = s^\alpha (\mathfrak{L}y)(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^k ({}_0I_t^{n-\alpha} y)(0),\ n-1 < \alpha \leq n, n \in \mathbb{N}.$$

If $0 < \alpha \leq 1$, then $(\mathfrak{L}_0 D_t^\alpha y)(s) = s^\alpha (\mathfrak{L}y)(s) - ({}_0I_t^{n-\alpha} y)(0)$.

Property 4 (see [13, 22]). The Laplace transform of the Caputo fractional derivative is obtained as

$$(\mathfrak{L}_0^c D_t^\alpha y)(s) = s^\alpha (\mathfrak{L}y)(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} (D^k y)(0), \quad n-1 < \alpha \leq n, n \in \mathbb{N}.$$

If $0 < \alpha \leq 1$, then $(\mathfrak{L}_0^c D_t^\alpha y)(s) = s^\alpha (\mathfrak{L}y)(s) - s^{\alpha-1} y(0)$.

3 Iterative method

If we take the Laplace transformation of equation (1), then we get the following equation:

$$\mathfrak{L}\{y(t)\} = \frac{s^{\alpha}}{\lambda + s^{2\alpha}} I_{0^+}^{1-\alpha} y(t)|_{t=0} + \frac{s^{\alpha-1}}{\lambda + s^{2\alpha}} D_{0^+}^{\alpha} y(t)|_{t=0} + \frac{1}{\lambda + s^{2\alpha}} \mathfrak{L}\{h(t)\},$$

where $h(t) := q(t)y(t)$. We use the inverse Laplace transform

$$\begin{aligned} y(t) =& c_1 t^{\alpha-1} E_{2\alpha,2}(-\lambda t^{2\alpha}) + c_2 t^{\alpha} E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) \\ &+ t^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda t^{2\alpha}) * h(t), \end{aligned} \tag{9}$$

in which, specifying the constants in (2) by setting, without loss of generality, $c_1 = 0$ and $c_2 = 1$ are given constants and $*$ is the convolution symbol. We consider the following iterative method:

$$y_m(t) = t^{\alpha} E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) + \int_0^t (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda (t-s)^{2\alpha}) q(s) y_{m-1}(s) ds.$$

Example 1. For simplicity, first we assume that $q(s) = 1$ and that

$$y_0(t) = t^{\alpha} E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}).$$

Then we get

$$\begin{aligned} y_1(t) =& t^{\alpha} E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) \\ &+ \int_0^t (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda (t-s)^{2\alpha}) s^{\alpha} E_{2\alpha,\alpha+1}(-\lambda s^{2\alpha}) ds. \end{aligned}$$

To calculate the integral term, we apply the following method:

$$\begin{aligned} &\mathfrak{L}^{-1}(\mathfrak{L}\{\int_0^t (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda (t-s)^{2\alpha}) s^{\alpha} E_{2\alpha,\alpha+1}(-\lambda s^{2\alpha}) ds\}) \\ &= \frac{1}{1!} t^{3\alpha} E^{(1)}_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}). \end{aligned}$$

Thus

$$y_1(t) = t^{\alpha} E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) + \frac{1}{1!} t^{3\alpha} E^{(1)}_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}).$$

Now, we calculate $y_2(t)$ as follows:

$$y_2(t) = y_1(t) + \frac{1}{2!} t^{5\alpha} E^{(2)}_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}),$$

as the same way, we get

$$y_3(t) = y_2(t) + \frac{1}{3!}t^{7\alpha}E^{(3)}_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}).$$

Finally, after the iteration, we have the following relation:

$$y_m(t) = \sum_{k=0}^{m+1} \frac{1}{k!}t^{(2k+1)\alpha}E^{(k)}_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}). \tag{10}$$

By choosing some terms from the approximate solution and calculating their zeros, the same results are obtained from [7] as follows:

$$\begin{aligned} y_3(t) =& t^{\alpha}E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) + \frac{1}{1!}t^{3\alpha}E^{(1)}_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) \\ &+ \frac{1}{2!}t^{5\alpha}E^{(2)}_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) + \frac{1}{3!}t^{7\alpha}E^{(3)}_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}). \end{aligned}$$

For example, if we take three terms with the boundary conditions

$$I^{1-\alpha}_{0^+}y(t)|_{t=1} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}y(s)ds = 0,$$

then we get

$$\begin{aligned} I^{1-\alpha}_{0^+}y_3(t)|_{t=1} =& \mathfrak{L}^{-1}(\mathfrak{L}\{\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{\alpha}E_{2\alpha,\alpha+1}(-\lambda s^{2\alpha})ds \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{1}{1!}s^{3\alpha}E^{(1)}_{2\alpha,\alpha+1}(-\lambda s^{2\alpha})ds \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{1}{2!}s^{5\alpha}E^{(2)}_{2\alpha,\alpha+1}(-\lambda s^{2\alpha})ds \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{1}{3!}s^{7\alpha}E^{(3)}_{2\alpha,\alpha+1}(-\lambda s^{2\alpha})ds\})|_{t=1} = 0. \end{aligned}$$

Using the inverse Laplace transform implies

$$\begin{aligned} I^{1-\alpha}_{0^+}y_3(t)|_{t=1} &= \mathfrak{L}^{-1}\{\frac{s^{2\alpha-2}}{s^{2\alpha}+\lambda}\} + \mathfrak{L}^{-1}\{\frac{s^{2\alpha-2}}{(s^{2\alpha}+\lambda)^{1+1}}\} \\ &\quad + \mathfrak{L}^{-1}\{\frac{s^{2\alpha-2}}{(s^{2\alpha}+\lambda)^{2+1}}\} + \mathfrak{L}^{-1}\{\frac{s^{2\alpha-2}}{(s^{2\alpha}+\lambda)^{3+1}}\} = 0 \\ &= \Big[tE_{2\alpha,2}(-\lambda t^{2\alpha}) + t^{2\alpha+1}E^{(1)}_{2\alpha,2}(-\lambda t^{2\alpha}) + t^{4\alpha+1}E^{(2)}_{2\alpha,2}(-\lambda t^{2\alpha}) \\ &\quad + t^{6\alpha+1}E^{(3)}_{2\alpha,2}(-\lambda t^{2\alpha})\Big]_{t=1} = 0. \end{aligned}$$

Therefore, the zeros of the above relation yield eigenvalues.

For the case of $q(s) = 0$, we refer the reader to [6, 7].

Table 1: The eigenvalues λ_n of the $FSLP$ of Example 1

k	α	0.88	0.92	0.96	0.98	0.99	1
10	λ_n	10.71568699	10.55992405	10.66410923	10.78678886	10.86333576	10.94942560
		27.42039814	31.48842150	35.71275308	38.01570185	39.23145940	40.49542871
		39.66187430	51.88983922	67.66694524	77.14409122	82.33647923	87.85647119
20	λ_n	10.71568686	10.55992405	10.66410923	10.78678886	10.86333576	10.94942560
		27.51414362	31.51192899	35.71778637	38.01801903	39.23303636	40.49650499
		60.83896648	66.90351096	76.82733019	82.94041507	86.28709241	89.83417215
		86.83800089	108.6657324	131.4598304	144.4575545	151.4897515	158.9183593
		107.4785323	147.2884927	199.9559706	231.8456925	238.3472917	249.1565859
						260.3905771	286.9178097
						275.8685836	300.6931093

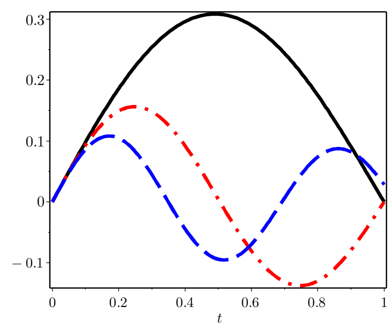

Figure 1: The curves of eigenfunctions, $k = 10$ for $n = 1$ (solid line), $n = 2$ (dash dot line), $n = 3$ (dash line), where $\alpha = 0.98$, $\lambda_1 = 10.78678886$, $\lambda_2 = 38.01570185$, and $\lambda_3 = 77.14409122$ for Example 1.

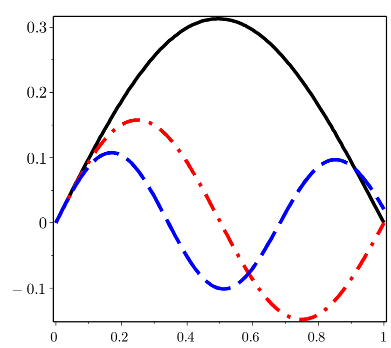

Figure 2: The curves of eigenfunctions, $k = 10$ for $n = 1$ (solid line), $n = 2$ (dash dot line), $n = 3$ (dash line), where $\alpha = 0.99$, $\lambda_1 = 10.86333576$, $\lambda_2 = 39.23145940$, and $\lambda_3 = 82.33647923$ for Example 1.

Remark 1. Note that as α approaches 1, one can see that the eigenvalues satisfy $\lambda_n = n^2\pi^2 + 1$. This shows that our results are a generalization of the classical ones.

Example 2. Now, we assume $q(s) = s^\beta$, and we have the following iterative method, analogous to the previous computation:

$$y_m(t) = t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) + \int_0^t (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda(t-s)^{2\alpha}) q(s) y_{m-1}(s) ds.$$

We define

$$\begin{aligned} y_1(t) =& t^\alpha E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha}) \\ &+ \int_0^t (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda(t-s)^{2\alpha}) \cdot s^{\alpha+\beta} E_{2\alpha,\alpha+1}(-\lambda s^{2\alpha}) ds, \end{aligned}$$

and again to calculate the integral term, we apply the following method:

$$\begin{aligned} &\mathfrak{L}^{-1}\Big(\mathfrak{L}\{t^{2\alpha-1}E_{2\alpha,2\alpha}(-\lambda t^{2\alpha})\} \cdot \mathfrak{L}\{t^{\alpha+\beta}E_{2\alpha,\alpha+1}(-\lambda t^{2\alpha})\}\Big) \\ &= \sum_{k_1=0}^{\infty} (-\lambda)^{k_1} \cdot \frac{\Gamma((2k_1+1)\alpha+\beta+1)}{\Gamma((2k_1+1)\alpha+1)} \cdot t^{(2k_1+3)\alpha+\beta} E_{2\alpha,(2k_1+3)\alpha+\beta+1}(-\lambda t^{2\alpha}). \end{aligned}$$

Thus

$$\begin{aligned} y_1(t) = y_0(t) + &\sum_{k_1=0}^{\infty} (-\lambda)^{k_1} \cdot \frac{\Gamma((2k_1+1)\alpha+\beta+1)}{\Gamma((2k_1+1)\alpha+1)} \\ &\times t^{(2k_1+3)\alpha+\beta} E_{2\alpha,(2k_1+3)\alpha+\beta+1}(-\lambda t^{2\alpha}), \end{aligned}$$

now, we calculate $y_2(t)$ as follows:

$$\begin{aligned} y_2(t) = y_1(t) + &\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} (-\lambda)^{k_1+k_2} \\ &\times \frac{\Gamma((2k_1+1)\alpha+\beta+1)}{\Gamma((2k_1+1)\alpha+1)} \cdot \frac{\Gamma((2k_2+2k_1+3)\alpha+2\beta+1)}{\Gamma((2k_2+2k_1+3)\alpha+\beta+1)} \\ &\times t^{(2k_2+2k_1+5)\alpha+2\beta} E_{2\alpha,(2k_2+2k_1+5)\alpha+2\beta+1}(-\lambda t^{2\alpha}), \end{aligned} \tag{11}$$

similarly, we have

$$\begin{aligned}
y_3(t) =& y_2(t) \\
&+ \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}(-\lambda)^{k_1+k_2+k_3}\cdot\frac{\Gamma((2k_1+1)\alpha+\beta+1)}{\Gamma((2k_1+1)\alpha+1)} \\
&\times \frac{\Gamma((2k_2+2k_1+3)\alpha+2\beta+1)}{\Gamma((2k_2+2k_1+3)\alpha+\beta+1)} \times \frac{\Gamma((2k_3+2k_2+2k_1+5)\alpha+3\beta+1)}{\Gamma((2k_3+2k_2+2k_1+5)\alpha+2\beta+1)} \\
&\times t^{(2k_3+2k_2+2k_1+7)\alpha+3\beta}E_{2\alpha,(2k_3+2k_2+2k_1+7)\alpha+3\beta+1}(-\lambda t^{2\alpha}).
\end{aligned}$$

Finally, we get the following relation for $y_m(t)$:

$$\begin{aligned}
y_m(t) =& y_{m-1}(t) + \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}(-\lambda)^{k_1+k_2+\cdots+k_m}\cdot\frac{A}{B} \\
&\times \cdot t^{(2k_m+2k_{m-1}+\cdots+2k_1+(2m-1))\alpha+m\beta} \\
&\times E_{2\alpha,(2k_m+2k_{m-1}+\cdots+2k_1+(2m-1))\alpha+m\beta+1)}(-\lambda t^{2\alpha}),
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
A =& \Gamma((2k_1+1)\alpha+\beta+1)\Gamma((2k_2+2k_1+3)\alpha+2\beta+1)\times\cdots \\
&\times \Gamma((2k_m+2k_{m-1}+\cdots+2k_1+(2m-1))\alpha+m\beta+1),
\end{aligned}$$

and

$$\begin{aligned}
B =& \Gamma((2k_1+1)\alpha+1)\Gamma((2k_2+2k_1+3)\alpha+\beta+1)\times\cdots \\
&\times \Gamma((2k_m+2k_{m-1}+\cdots+2k_1+(2m-1))\alpha+(m-1)\beta+1).
\end{aligned}$$

Now, in order to obtain eigenvalues, by choosing three terms from (37) and imposing the following boundary condition, we have

$$\begin{aligned}
&I_{0^+}^{1-\alpha}y_3(t)|_{t=1} \\
&=\mathfrak{L}^{-1}\Bigg(\mathfrak{L}\Big\{\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}s^{\alpha}E_{2\alpha,\alpha+1}(-\lambda s^{2\alpha})ds \\
&\quad+\sum_{k_1=0}^{\infty}(-\lambda)^{k_1}\frac{\Gamma((2k_1+1)\alpha+\beta+1)}{\Gamma((2k_1+1)\alpha+1)} \\
&\quad\times\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}s^{(2k_1+3)\alpha+\beta}E_{2\alpha,(2k_1+3)\alpha+\beta+1}(-\lambda s^{2\alpha})ds \\
&\quad+\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-\lambda)^{k_1+k_2}\frac{\Gamma((2k_1+1)\alpha+\beta+1)}{\Gamma((2k_1+1)\alpha+1)}\cdot\frac{\Gamma((2k_2+2k_1+3)\alpha+2\beta+1)}{\Gamma((2k_2+2k_1+3)\alpha+\beta+1)} \\
&\quad\times\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}s^{(2k_2+2k_1+5)\alpha+2\beta}E_{2\alpha,(2k_2+2k_1+5)\alpha+2\beta+1}(-\lambda s^{2\alpha})ds
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}(-\lambda)^{k_1+k_2+k_3}\frac{\Gamma((2k_1+1)\alpha+\beta+1)}{\Gamma((2k_1+1)\alpha+1)}\\
&\times \frac{\Gamma((2k_2+2k_1+3)\alpha+2\beta+1)}{\Gamma((2k_2+2k_1+3)\alpha+\beta+1)}\cdot\frac{\Gamma((2k_3+2k_2+2k_1+5)\alpha+3\beta+1)}{\Gamma((2k_3+2k_2+2k_1+5)\alpha+2\beta+1)}\\
&\times \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{(2k_3+2k_2+2k_1+7)\alpha+3\beta}\\
&\qquad\qquad \times E_{2\alpha,(2k_3+2k_2+2k_1+7)\alpha+3\beta+1}(-\lambda s^{2\alpha})ds\Big|_{t=1}\Big\}\Bigg)=0. \qquad (13)
\end{aligned}
$$

Now, by using the inverse Laplace transform, we have

$$
\begin{aligned}
&I_{0^+}^{1-\alpha}y_3(t)|_{t=1}\\
&=tE_{2\alpha,2}(-\lambda t^{2\alpha})+\sum_{k_1=0}^{\infty}(-\lambda)^{k_1}\cdot\frac{\Gamma((2k_1+1)\alpha+\beta+1)}{\Gamma((2k_1+1)\alpha+1)}\\
&\quad\times t^{(2k_1+2)\alpha+\beta+1}E_{2\alpha,(2k_1+2)\alpha+\beta+2}(-\lambda t^{2\alpha})\\
&\quad+\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-\lambda)^{k_1+k_2}\frac{\Gamma((2k_1+1)\alpha+\beta+1)}{\Gamma((2k_1+1)\alpha+1)}\cdot\frac{\Gamma((2k_2+2k_1+3)\alpha+2\beta+1)}{\Gamma((2k_2+2k_1+3)\alpha+\beta+1)}\\
&\quad\times t^{(2k_2+2k_1+4)\alpha+2\beta+1}E_{2\alpha,(2k_2+2k_1+4)\alpha+2\beta+2}(-\lambda t^{2\alpha})\\
&\quad+\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}(-\lambda)^{k_1+k_2+k_3}\frac{\Gamma((2k_1+1)\alpha+\beta+1)}{\Gamma((2k_1+1)\alpha+1)}\\
&\quad\times\frac{\Gamma((2k_2+2k_1+3)\alpha+2\beta+1)}{\Gamma((2k_2+2k_1+3)\alpha+\beta+1)}\cdot\frac{\Gamma((2k_3+2k_2+2k_1+5)\alpha+3\beta+1)}{\Gamma((2k_3+2k_2+2k_1+5)\alpha+2\beta+1)}\\
&\quad\times t^{(2k_3+2k_2+2k_1+6)\alpha+3\beta+1}E_{2\alpha,(2k_3+2k_2+2k_1+6)\alpha+3\beta+2}(-\lambda t^{2\alpha})|_{t=1}=0.
\end{aligned}
$$

Therefore, solving the above relation with respect to λ yields eigenvalues.

Table 2: The eigenvalues λ_n of the $FSLP$ for Example 2

k_i	α	0.88	0.92	0.96	0.98	0.99	1.0
$k_1,k_2,k_3=5$	λ_n	10.05385926	9.933778015	10.06348862	10.19660718	10.27791426	10.36849015
		47.56007387	32.34315896	35.63420589	37.76603821	38.92583141	40.14691848
		1126.335623	44.07433976	60.04931462	68.65397481	73.15018892	77.78582381
		2383.852442	60.98311732	79.07639037	90.47082662	96.86784919	103.7750278
		4084.543198	1672.025020	2492.533591	3047.891192	3371.626995	3730.657415
		29.99913622	3669.500947	5672.277915	7063.067903	7884.473754	8803.558665
			6468.168446	10283.17818	12984.46721	14595.72336	16410.73873
			44.95131154	62.05287730	72.95773310	79.12198365	85.81612846
$k_1,k_2,k_3=10$	λ_n	10.05563487	9.934078332	10.06355202	10.19663792	10.27793590	10.36850547
		26.86376328	30.94976951	35.18620831	37.49382726	38.71166129	39.97755111
		38.80996910	51.04646579	66.84154959	76.33005616	81.52855664	87.05489991

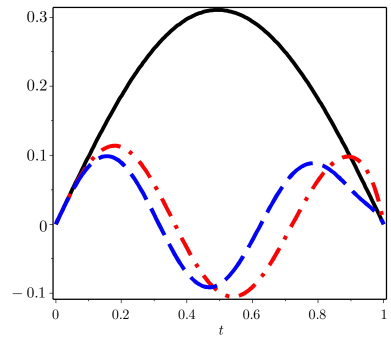

Figure 3: The curves of eigenfunctions, $k_1, k_2, k_3 = 5$ for $n = 1$ (solid line), $n = 2$ (dash dot line), $n = 3$ (dash line), where $\alpha = 0.98$, $\lambda_1 = 10.19660718$, $\lambda_2 = 68.65397481$, and $\lambda_3 = 90.47082662$ for Example 2.

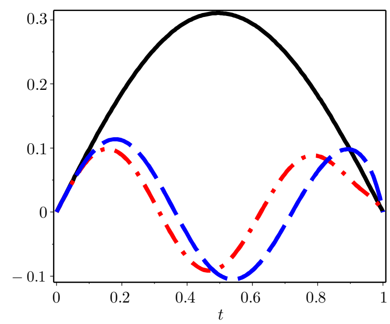

Figure 4: The curves of eigenfunctions, $k_1, k_2, k_3 = 5$ for $n = 1$ (solid line), $n = 2$ (dash dot line), $n = 3$ (dash line), where $\alpha = 0.99$, $\lambda_1 = 10.27791426$, $\lambda_2 = 73.15018892$, and $\lambda_3 = 96.86784919$ for Example 2.

4 Analysis of the iterative method

Lemma 2. The n-fold series in relation (37) is convergent for $\alpha = 1$, $\beta \in \mathbb{N}$, and $|t| < \frac{1}{\sqrt{|\lambda|}}$.

Proof. It is sufficient to show that the double series in relation (11) is convergent. By Theorem 1, we have

$$\begin{aligned} |y_2(t) - y_1(t)| \leq & \tfrac{c}{1+|\lambda t^2|} \textstyle\sum_{k_1=0}^{\infty} |\lambda|^{k_1} \tfrac{(2k_1+1+\beta)!}{[(2k_1+1)]!} \\ & \times \sum_{k_2=0}^{\infty} |\lambda|^{k_2} \frac{(2k_2 + 2k_1 + 3 + 2\beta)!}{([(2k_2 + 2k_1 + 3)] + \beta)!} t^{2k_2+2k_1+5+2\beta}. \end{aligned} \tag{14}$$

Since $\limsup_{k_2\to\infty} \sqrt[k_2]{\dfrac{(2k_2+2k_1+3+2\beta)!|\lambda|^{k_2}}{([(2k_2+2k_1+3)]+\beta)!}} = |\lambda|$, by the basic root test of convergence of nonnegative series, we conclude that for $|t| < \frac{1}{\sqrt{|\lambda|}}$, the series

$$\sum_{k_2=0}^{\infty} |\lambda|^{k_2} \frac{(2k_2+2k_1+3+2\beta)!}{([2k_2+2k_1+3]+\beta)!} t^{2k_2+2k_1+5+2\beta}$$

is absolutely convergent. Moreover, it is uniformly convergent on any compact subset of the interval $(-\frac{1}{\sqrt{|\lambda|}}, \frac{1}{\sqrt{|\lambda|}})$.

On the other hand, the series $\sum_{k_2=0}^{\infty} |\lambda|^{k_2}.t^{2k_2+2k_1+5+2\beta}$ is absolutely convergent on the interval $(-\frac{1}{\sqrt{|\lambda|}}, \frac{1}{\sqrt{|\lambda|}})$ and is uniformly convergent on any compact subset of the interval $(-\frac{1}{\sqrt{|\lambda|}}, \frac{1}{\sqrt{|\lambda|}})$. So we can take derivative of it on this interval, term by term, for β times and get

$$\begin{aligned}
&\sum_{k_2=0}^{\infty} ([2k_2+2k_1+3]+\beta+1)([2k_2+2k_1+3]+\beta+2) \\
&\quad \ldots ([2k_2+2k_1+3]+2\beta)|\lambda|^{k_2} t^{2k_2+2k_1+5+2\beta} \\
&\quad = t^{\beta}\Big(\sum_{k_2=0}^{\infty} |\lambda|^{k_2}.t^{2k_2+2k_1+5+2\beta}\Big)^{(\beta)} \\
&\quad = t^{\beta}\Big(t^{2k_1+5+2\beta} \sum_{k_2=0}^{\infty} (|\lambda|t^2)^{k_2}\Big)^{(\beta)} \\
&\quad = t^{\beta}\Big(\frac{t^{2k_1+5+2\beta}}{1-|\lambda|t^2}\Big)^{(\beta)}.
\end{aligned}$$

Now, using (14), we obtain

$$\begin{aligned}
&|y_2(t)-y_1(t)| \\
&\le \frac{c}{1+|\lambda|t^2} \sum_{k_1=0}^{\infty} |\lambda|^{k_1} \frac{(2k_1+1+\beta)!}{[2k_1+1]!} \cdot t^{\beta}\Big(\frac{t^{2k_1+5+2\beta}}{1-|\lambda|t^2}\Big)^{(\beta)} \\
&= \frac{ct^{\beta}}{1+|\lambda|t^2}\Big(\sum_{k_1=0}^{\infty} |\lambda|^{k_1} \frac{(2k_1+1+\beta)!}{[2k_1+1]!} \cdot \frac{t^{2k_1+5+2\beta}}{1-|\lambda|t^2}\Big)^{(\beta)} \\
&= \frac{ct^{\beta}}{1+|\lambda|t^2}\Big(\frac{t^{2\beta+4}}{1-|\lambda|t^2} \sum_{k_1=0}^{\infty} |\lambda|^{k_1} \frac{(2k_1+1+\beta)!}{[2k_1+1]!} t^{2k_1+1}\Big)^{(\beta)} \\
&= \frac{ct^{\beta}}{1+|\lambda|t^2}\Big(\frac{t^{2\beta+4}}{1-|\lambda|t^2} \sum_{k_1=0}^{\infty} |\lambda|^{k_1}(2k_1+2)\cdots(2k_1+1+\beta)t^{2k_1+1}\Big)^{(\beta)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{ct^{\beta}}{1+|\lambda|t^2}\Big(\frac{t^{2\beta+4}}{1-|\lambda|t^2}\sum_{k_1=0}^{\infty}|\lambda|^{k_1}\big(t^{2k_1+1+\beta}\big)^{(\beta)}\Big)^{(\beta)}\\
&= \frac{ct^{\beta}}{1+|\lambda|t^2}\Big(\frac{t^{2\beta+4}}{1-|\lambda|t^2}\Big(\sum_{k_1=0}^{\infty}|\lambda|^{k_1}t^{2k_1+1+\beta}\Big)^{(\beta)}\Big)^{(\beta)}\\
&= \frac{ct^{\beta}}{1+|\lambda|t^2}\Big(\frac{t^{2\beta+4}}{1-|\lambda|t^2}\Big(t^{\beta+1}\sum_{k_1=0}^{\infty}|\lambda|^{k_1}t^{2k_1}\Big)^{(\beta)}\Big)^{(\beta)}\\
&= \frac{ct^{\beta}}{1+|\lambda|t^2}\Big(\frac{t^{2\beta+4}}{1-|\lambda|t^2}\Big(\frac{t^{\beta+1}}{1-|\lambda|t^2}\Big)^{(\beta)}\Big)^{(\beta)}.
\end{aligned}
$$

Therefore, (14) is appeared to be the derivative of a function and hence is convergent.

It follows from the root test that $\frac{1}{R} = \limsup_{k_2\to\infty} \sqrt[k_2]{\frac{(2k_2+2k_1+3+2\beta)!|\lambda|^{k_2}}{([(2k_2+2k_1+3)]+\beta)!}} = |\lambda|$. Hence

$$
\begin{aligned}
C = &\sum_{k_2=0}^{\infty}([(2k_2+2k_1+3)]+\beta+1)([(2k_2+2k_1+3)]+\beta+2)\\
&\times\cdots\times([(2k_2+2k_1+3)]+2\beta)|\lambda|^{k_2}.t^{2k_2+2k_1+5+2\beta}\\
=&\Big(\sum_{k_2=0}^{\infty}(|\lambda|^{k_2}.t^{2k_2+2k_1+5+2\beta}\Big)^{(k)},
\end{aligned}
$$

where $k = (2k_2+2k_1+5) - [(2k_2+2k_1+5)+\beta-2]$. Then

$$
C = \Big(t^{2k_1+5+2\beta}\sum_{k_2=0}^{\infty}(|\lambda|t^2)^{k_2}\Big)^{(k)} = \Big(\frac{t^{2k_1+5+2\beta}}{1-|\lambda|t^2}\Big)^{(k)}.
$$

Noting that $(\frac{t^{2k_1+5+2\beta}}{1-|\lambda|t^2})^{(k)}$ is convergent for $\alpha = 1$ and $\beta \in \mathbb{N}$, then

$$
\begin{aligned}
&\le \frac{c}{1+|\lambda|t^2}\Big(\sum_{k_1=0}^{\infty}|\lambda|^{k_1}\frac{(2k_1+1+\beta)!}{[(2k_1+1)+1]!}\cdot\frac{t^{2k_1+5+2\beta}}{1-|\lambda|t^2}\Big)^{(k)}\\
&\le \frac{c}{1+|\lambda|t^2}\Big(\frac{t^{\beta+2}}{1-|\lambda|t^2}\sum_{k_1=0}^{\infty}([(2k_1+1)+1]+1)\times\cdots\\
&\times(2k_1+1+\beta)t^{2k_1+\beta+3}\Big)^{(k)}\\
&\le \frac{c}{1+|\lambda|t^2}\Bigg(\frac{t^{\beta+2}}{1-|\lambda|t^2}(t^{\beta+3}\sum_{k_1=0}^{\infty}t^{2k_1})^{(l)}\Bigg)^{(k)},
\end{aligned}
$$

where $l = (2k_1+\beta) - [(2k_1+3)+1]$. We have

$$\leq \frac{c}{1+|\lambda|t^2}[\frac{t^{\beta+2}}{1-|\lambda|t^2}(t^{\beta+3}\frac{1}{1-t^2})^{(l)}]^{(k)} < \infty.$$

the proof is completed. □

The reader can prove the convergence of triple series and so on in the same manner.

5 Conclusion

In this paper, we have considered the analytical and numerical solutions of the FSLPs and eigenvalue problems for the fractional differential equations with Dirichlet boundary conditions by using an iterative method. Moreover, we proved that the resulting series is convergent.

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