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Research Article

Richardson extrapolation technique on a modified graded mesh for singularly perturbed parabolic convection-diffusion problems

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Abstract

In this paper, we focus on investigating a post-processing technique designed for one-dimensional singularly perturbed parabolic convection-diffusion problems that demonstrate a regular boundary layer. We use a backward Euler numerical approach for time derivatives with uniform mesh in the temporal direction, and a simple upwind scheme is used for spatial derivatives with modified graded mesh in the spatial direction. In this study, we demonstrate the effectiveness of the Richardson extrapolation technique in enhancing the ε-uniform accuracy of simple upwinding

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within the discrete supremum norm, as evidenced by an improvement from $O(N^{-1}\ln(1/\varepsilon)+\triangle\theta)$ to $O(N^{-2}\ln^2(1/\varepsilon)+\triangle\theta^2)$. Furthermore, to validate the theoretical findings, computational experiments are conducted for two test examples by applying the proposed technique.

AMS subject classifications (2020): 65M06, 65M12, 65M15, 65M22.

Keywords: Singularly perturbed parabolic problem; Regular boundary layer; Upwind scheme; Richardson extrapolation; Modified graded mesh; Uniform convergence.

1 Introduction

We consider the following one-dimensional singularly perturbed parabolic convection-diffusion initial-boundary value problem (IBVP) posed on the domain $\aleph=\Lambda_r\times\Lambda_\theta$, where $\Lambda_r=(0,1),\Lambda_\theta=(0,T]$:

$$\begin{cases} L_\varepsilon y(r,\theta)\equiv\left(\dfrac{\partial y(r,\theta)}{\partial\theta}-\varepsilon\dfrac{\partial^2 y(r,\theta)}{\partial r^2}+\Psi_1(r)\dfrac{\partial y(r,\theta)}{\partial r}+\Psi_2(r)y(r,\theta)\right)\\ \qquad\qquad\quad =f(r,\theta),\qquad (r,\theta)\in\aleph,\\ y(r,0)=y_0(r),\qquad r\in\overline{\Lambda}_r,\\ y(0,\theta)=0,\qquad \theta\in\Lambda_\theta,\\ y(1,\theta)=0,\qquad \theta\in\Lambda_\theta,\end{cases} \tag{1}$$

where $0<\varepsilon\ll 1$ is a small parameter and the coefficients Ψ_1 and Ψ_2 are sufficiently smooth function such that

$$\Psi_1(r)>2\alpha>0,\quad \Psi_2(r)\geq\beta\geq 0\quad \text{on}\quad \overline{\Lambda}_r. \tag{2}$$

Under sufficient smoothness and compatibility conditions imposed on the functions y_0 and f, the parabolic IBVP (1)–(2), in general, admits a unique solution $y(r,\theta)$, which exhibits a regular boundary layer at $r=1$. This type of problem encompasses the linearized Navier–Stokes equation, which emerges in the modeling of convection-dominated flow issues in fluid dynamics, particularly when dealing with large Reynolds numbers. In the presence of a boundary layer, conventional numerical methods, such as standard finite difference or finite element schemes, when applied on uniform meshes, fail to produce accurate numerical solutions as the singular perturbation parameter ε approaches zero. This limitation serves as a driving force to devise ε-uniformly convergent numerical methods. Among these approaches, the fitted mesh method stands out as a satisfying and widely embraced technique for surmounting numerical challenges. By employing an especially designed layer-adapted mesh, this method successfully overcomes the shortcomings encountered with traditional approaches.

Over the past few years, numerous authors have made significant strides in the development of uniformly convergent numerical methods on Shishkin meshes, specifically tailored for singularly perturbed parabolic convection-diffusion problems. These methods address the challenges posed by such complex scenarios and have shown great promise in achieving convergence, mainly Cai and Liu [1], Clavero, Jorge, and Lisbona [3], O'Riordan, Pickett, and Shishkin [28] in the presence of regular boundary layers. However, despite these advancements, it is important to note that all of these methods exhibit first-order accuracy in both the spatial and temporal variables. Hemker [9] employed a defect correction technique to enhance the accuracy of temporal variable computations for parabolic singularly perturbed problems, specifically those lacking the convective term. Additionally, in [10], the same approach was applied to parabolic convection-diffusion problems. This technique has proved to be effective in refining the precision of the results in both cases.

Woldaregay and Duressa [35] considered the study of singularly perturbed differential-difference equations having delay and advance in the reaction term. Debela, Woldaregay, and Duressa [6] studied the singularly perturbed differential equations of convection diffusion type with nonlocal boundary conditions. Mekonnen and Duressa [15] considered the study of numerical treatment of two parametric singularly perturbed parabolic convection-diffusion problems. The scheme is developed through the Crank–Nicholson discretization method in the temporal dimension followed by fitting the B-spline collocation method in the spatial direction. Also, Hailu and Duressa [8] focused on the study of singularly perturbed parabolic convection-diffusion equations with integral boundary conditions and a large negative shift.

Furthermore, Natesan and Deb [5] have recently introduced two novel numerical schemes that exhibit uniform convergence and higher-order accuracy in time for singularly perturbed parabolic reaction-diffusion problems. These schemes are specifically designed to handle scenarios involving only parabolic layers. In their recent work, Mukherjee and Natesan [17] proposed a novel hybrid numerical scheme for singularly perturbed parabolic convection-diffusion problems. This innovative approach demonstrates uniform convergence, achieving an impressive order of convergence one in time and nearly two in space. The results of their study pave the way for more efficient and accurate solutions to such challenging problems. Several researchers, including Clavero, Gracia, and Jorge [2], Kopteva [14], and Shishkin [30], have made significant contributions to the development of uniformly convergent methods of order two in both variables for singularly perturbed parabolic convection-diffusion problems. In their study, Mukherjee and Natesan [18] employed the Richardson extrapolation technique to enhance the ε-uniform accuracy of the simple upwinding method. This improvement was observed in the discrete supremum norm and extended the accuracy from $O(N^{-1}\ln N + \triangle t)$ to $O(N^{-2}\ln^2 N + \triangle t^2)$ on the Shishkin mesh. The focus of their investigation was one-dimensional singularly perturbed parabolic convection-diffusion problems that exhibit a regular boundary layer.

Maneesh and Natesan [33] presented an advanced numerical approach designed to handle singularly perturbed systems of parabolic convection-diffusion problems, which feature overlapping exponential boundary layers. The proposed method achieves uniform convergence of higher-order accuracy. The numerical scheme integrates the backward-Euler method for the time derivative on a uniform mesh, along with the classical upwind scheme for spatial derivatives on a piecewise-uniform Shishkin mesh. This combination yields near first-order convergence in both space and time. To further enhance precision, the authors employed the Richardson extrapolation technique, elevating the accuracy of the scheme to second-order, maintaining uniform convergence in both time and space.

Negero [20, 25] considered the study of an exponentially fitted numerical scheme and analyzed it for solving singularly perturbed two-parameter parabolic problems with large temporal lag. Negero [22] studied the numerical approximation for two-parameter singularly perturbed parabolic partial differential equations with time delay. In [23], the author considered the class of time-delayed, singularly perturbed parabolic reaction-diffusion problems. Negero [21, 24, 26] studied time delay problems and parameter-uniform robust schemes for singularly perturbed parabolic convection-diffusion problems with large time-lag.

In this current study, our objective is to employ and examine a straightforward post-processing technique on the basic upwinding solution. The primary goal is to attain a convergence order greater than one concerning both the spatial and temporal variables. Richardson extrapolation, a renowned post-processing technique, offers a superior approximation to the exact solution achieved by averaging the numerical solutions computed on two embedded meshes. This method has been extensively investigated in the literature [11, 31] to enhance the accuracy of numerical solutions for singularly perturbed elliptic reaction-diffusion equations. However, the crux of their analysis relies on the direct expansion of the upwinding solution, which proves to be intricate. Shishkin and Shishkina [32] have consistently adopted this approach for quasilinear parabolic convection-diffusion problems. However, Natividad and Stynes [19] presented a comparatively simpler and distinctive analysis for applying Richardson extrapolation to one-dimensional singularly perturbed convection-diffusion boundary-value problems (BVPs). More recently, Deb and Natesan [4] conducted an analysis of the Richardson extrapolation technique for solving singularly perturbed coupled systems of convection-diffusion BVPs.

This paper represents the inaugural analysis of Richardson extrapolation on the nonuniform graded mesh applied to singularly perturbed parabolic convection-diffusion IBVPs through error decomposition after extrapolation. Initially, we address the IBVP (1)–(2) on a nonuniform graded mesh using the classical implicit upwind finite difference scheme. Subsequently, we demonstrate that the implicit upwind scheme achieves ε-uniform convergence with an almost first-order accuracy in the discrete supremum norm. Next, we incorporate the Richardson extrapolation technique to enhance the nearly first-order convergence achieved by the simple upwinding method, leading to an almost second-order convergence. During the error analysis, we derive separate estimates for the smooth component and the layer component of the error. Consequently, we attain the necessary ε-uniform convergence result.

The subsequent sections of this paper are organized as follows: In section 2, an apriori bound on the analytical solution is presented, and stronger bounds on the derivatives of the solution are derived through decomposition. Section 3 introduces the modified graded mesh and the classical implicit upwind scheme used for discretizing the continuous problem. Here, we establish the ε-uniform convergence result for the implicit upwind scheme and provide some technical results to be utilized later in this paper. Moving on, section 4 introduces the Richardson extrapolation technique, and the main theoretical result, demonstrating that the extrapolated solution of the upwind scheme achieves ε-uniform convergence with almost second-order accuracy to the exact solution of the continuous problem, is proven. In section 5, numerical experiments are conducted on two test examples to validate the theoretical results. Finally, section 6 summarizes the main conclusions of this study.

In this paper, we use the notation C to represent a universal positive constant that remains unaffected by changes in the perturbation parameter ε, the values of N and M (representing the number of mesh intervals in the spatial and temporal directions, respectively), as well as the mesh points. In the analysis, we use the standard supremum norm $\|\cdot\|_{\infty,D}$, which is defined by

$$\|z\|_{\infty,D} = \sup_{\kappa\in D} |z(\kappa)|,$$

for a function z defined on some domain D.

Before we analyze the problem, some of the compatibility conditions are necessary. Therefore, the following compatibility conditions at the corners for functions and its first-order derivatives are assumed to satisfy,

$$\begin{cases} y_0(0) = y_0(1) = 0, \\ -\varepsilon y_0''(0) + \Psi_1(0) y_0'(0) = f(0,0), \\ -\varepsilon y_0''(1) + \Psi_1(1) y_0'(1) = f(1,0). \end{cases} \tag{3}$$

Then (1) has a unique solution in the parabolic Holder space $C^{2+\alpha,1+\alpha/2}(\overline{\aleph})$; see [29, 7]. Moreover, the fulfillment of second-order corner compatibility conditions requires the Holder space to be $C^{4+\alpha,2+\alpha/2}(\overline{\aleph})$. These conditions can be written down explicitly in terms of the data of the problem in the following way: differentiating (1) with respect to θ, we get

$$f_\theta = y_{\theta\theta} + L_\varepsilon y_\theta + \Psi_{2\theta} y = y_{\theta\theta} + L_\varepsilon(f - L_\varepsilon y) + \Psi_{2\theta} y.$$

Hence, recalling (1) and (3), the second-order corner compatibility condition is

$$L_\varepsilon(L_\varepsilon y_0) = L_\varepsilon f - f_\theta \tag{4}$$

at the corners $(0,0)$ and $(1,0)$. Under these hypotheses, the solution y of (1) has an exponential layer along the boundary $r=1$ of $\aleph$.

2 Bounds on the solution and its derivatives

In this section, we introduce the conventional a-priori bound for the analytical solution and further establish enhanced bounds on the derivatives of the solution for problems (1)–(2). This is achieved through a decomposition of the solution into smooth and layer components. The utilization of these bounds becomes essential in the subsequent section as we aim to prove the ε-uniform error estimate. We first present some initial reports. Let $G = \overline{\aleph}/\aleph$. The differential operator L_ε then fulfills the following minimum principle on $\overline{\aleph}$, and [29, Theorem 2.2] provides the evidence for this.

Lemma 1. [Minimum principle] Assume that a function $z \in \mathbb{C}^0(\overline{\aleph}) \cap \mathbb{C}^2(\aleph)$ satisfies $z(r,\theta) \geq 0, (r,\theta) \in G$ and $L_\varepsilon z(r,\theta) \geq 0, (r,\theta) \in \aleph$. Then we have $z(r,\theta) \geq 0,$ for all $(r,\theta) \in \overline{\aleph}$.

The following is a direct result of the aforementioned minimum principle: ε-uniform bound of the solution of the problem (1)–(2).

Lemma 2. The solution y of the IBVP (1)–(2) satisfies

$$|y(r,\theta)| \leq C, \quad (r,\theta) \in \overline{\aleph}.$$

Proof. The proof follows from Lemma 2.3 of Roos, Stynes, and Tobiska [29]. □

Subsequently, we present a priori bounds concerning the derivatives of the analytical solution y to problem (1)–(2) in the forthcoming theorem, which will be utilized in Theorem 2.

Theorem 1. For all nonnegative integers p, q satisfying $0 \leq p+q \leq 5$, the exact solution y of the IBVP (1)–(2) satisfies the estimate

$$\left|\frac{\partial^{p+q} y}{\partial r^p \partial \theta^q}(r,\theta)\right| \leq C\left(1+\varepsilon^{-p}\exp(-\alpha(1-r)/\varepsilon)\right), \quad (r,\theta)\in\aleph. \tag{5}$$

Proof. In [27], this conclusion was verified for $0 \leq p+q \leq 2$,. Under necessary compatibility conditions and sufficient smoothness on the data, the proof of the estimate (5) for higher values of p, q follows similarly from [2]. □

Let us now decompose the solution y of the IBVP (1)–(2) into the form of $y = l + m$, where l and m represent the smooth component and the layer component, respectively. We further break down the smooth component into the sum

$$l = \sum_{j=0}^{4} \varepsilon^j l_j,$$

where the functions $l_j, \quad j = 0,1,2,3$, are solutions of the following first-order problems:

$$\begin{cases} \dfrac{\partial l_0}{\partial \theta} + \Psi_1 \dfrac{\partial l_0}{\partial r} + \Psi_2 l_0 = f \quad \text{in} \quad \aleph, \\ l_0(0,\theta) = y(0,\theta), \quad \theta \in (0,T], \quad l_0(r,0) = y(r,0), \quad r \in \overline{\Lambda}_r, \\ \dfrac{\partial l_j}{\partial \theta} + \Psi_1 \dfrac{\partial l_j}{\partial r} + \Psi_2 l_j = \dfrac{\partial^2 l_{j-1}}{\partial r^2} \quad \text{in} \quad \aleph, \\ l_j(0,\theta) = 0, \quad \theta \in (0,T], \quad l_j(r,0) = 0, \quad r \in \overline{\Lambda}_r, \quad j = 1,2,3, \end{cases} \tag{6}$$

and the lastly, the function l_4 satisfies

$$\begin{cases} L_\varepsilon l_4 = \dfrac{\partial^2 l_3}{\partial r^2} \quad \text{in} \quad \aleph, \\ l_4(0,\theta) = l_4(1,\theta) = 0, \quad \theta \in (0,T], \quad l_4(r,0) = 0, \quad r \in \overline{\Lambda}_r. \end{cases} \tag{7}$$

Hence, the smooth component l satisfies the following IBVP:

$$\begin{cases} L_\varepsilon l = f \quad \text{in} \quad \aleph, \\ l(0,\theta) = 0, \quad l(1,\theta) = \sum_{j=0}^{4} \varepsilon^j l_j(1,\theta), \quad \theta \in (0,T], \\ l(r,0) = y(r,0), \quad r \in \overline{\Lambda}_r, \end{cases} \tag{8}$$

and therefore, the layer component m must satisfy

$$\begin{cases} L_\varepsilon m = 0 \quad \text{in} \quad \aleph, \\ m(0,\theta) = 0, \quad m(1,\theta) = y(1,\theta) - l(1,\theta), \ \theta \in (0,T], \\ m(r,0) = 0, \quad r \in \overline{\Lambda}_r. \end{cases} \tag{9}$$

Theorem 2. For any nonnegative integers p and q that fulfill the condition $0 \leq p + q \leq 5$, the smooth component l and the layer component m, as defined in (8) and (9), respectively, adhere to the subsequent bounds,

$$\left\| \frac{\partial^{p+q} l}{\partial r^p \partial \theta^q} \right\|_{\infty,\aleph} \leq C(1 + \varepsilon^{4-p}),$$

and

$$\left| \frac{\partial^{p+q} m}{\partial r^p \partial \theta^q}(r,\theta) \right| \leq C\varepsilon^{-p} \exp(-\alpha(1-r)/\varepsilon), \quad (r,\theta) \in \aleph.$$

Proof. In the initial step, we will establish more stringent bounds for the smooth component l, as defined in (8), and its derivatives. The functions $l_j, j = 0,1,2,3$, serve as solutions to the problems outlined in (6). Importantly, these functions remain unaffected by the parameter ε. As a consequence, their derivatives exhibit the property of ε-uniformly bounded behavior. Hence

$$\left\| \frac{\partial^{p+q} l_j}{\partial r^p \partial \theta^q} \right\|_{\infty,\aleph} \leq C, \quad \text{j} = 0,1,2,3, \quad \text{for} \quad 0 \leq p+q \leq 5. \tag{10}$$

Similarly, the function l_4 corresponds to the solution of a problem akin to (1). By applying estimate(5) in a manner analogous to l_4, we obtain the bounds

$$\left\| \frac{\partial^{p+q} l_4}{\partial r^p \partial \theta^q} \right\|_{\infty,\aleph} \leq C(1 + \varepsilon^{-p} \exp(-\alpha(1-r)/\varepsilon)), \quad \text{for} \quad 0 \leq p+q \leq 5. \tag{11}$$

Therefore, by combining the estimates (10) and (11), for $0 \leq p + q \leq 5$, we can establish the necessary bounds on the smooth component l as follows:

$$\left\|\frac{\partial^{p+q} l}{\partial r^p \partial \theta^q}\right\|_{\infty,\aleph} \leq \sum_{j=0}^{4} \varepsilon^j \left\|\frac{\partial^{p+q} l_j}{\partial r^p \partial \theta^q}\right\|_{\infty,\aleph} \leq C(1+\varepsilon^{4-p}).$$

Alternatively, we can utilize the minimum principle (Lemma 1) on $\overline{\aleph}$ with the barrier function

$$\Psi(r,\theta) = C\exp(-\alpha(1-r)/\varepsilon), \quad (r,\theta) \in \aleph.$$

By selecting a sufficiently large C, we achieve the necessary bound on m. Moreover, the bounds on the derivatives of m can be deduced, following the arguments presented in [16, 28], thus concluding the proof. □

3 Numerical discretization

Within this section, a well-suited mesh is presented for discretizing the domain, ensuring the attainment of an ε-uniformly convergent difference scheme. Additionally, a detailed exposition of the employed difference scheme for discretizing the problem (1)–(2) is provided.

3.1 Temporal Discretization

To establish the convergence of (1)–(2) at each instance, we use the uniform time grid

$$\mathbb{W}_\theta^M = \{\theta_n = n\Delta\theta,\ n = 0,\ 1,\ \ldots,\ M,\ \ \triangle\,\theta = T/M\},$$

Here M represents the grid points.

3.2 Spatial discretization

We generate a modified graded mesh, Λ_r^N in the interval $[0,1]$ and order to resolve the boundary layer at $r = 1$ as follows:

$$\mu_j = 1 - \vartheta_{N-1} \quad \text{for} \quad j, \tag{12}$$

where μ is defined as the following form:

$$\begin{cases} \vartheta_0 = 0, & \\ \vartheta_j = 2\varepsilon \frac{j}{N}, & 1 \leq j \leq \frac{N}{2}, \\ \vartheta_{j+1} = \vartheta_j(1+\rho h), & \frac{N}{2} \leq j \leq N-2, \\ \vartheta_N = 1, & \end{cases} \tag{13}$$

where the parameter h satisfies the following nonlinear equation:

$$\ln(1/\varepsilon) = (N/2)\ln(1+\rho h). \tag{14}$$

The above section of the parameter h ensures that there are $N/2$ grid points in the interval $[0, 1-\varepsilon]$ which are distributed gradely in the interval $[0, 1-\varepsilon]$. Numerical verification stimulate that the interval $(\vartheta_{N-1}, 1)$ is not too small in comparison with the previous one $(\vartheta_{N-2}, \vartheta_{N-1})$. In the subinterval $[1-\varepsilon, 1]$, we distribute $N/2$ points with uniform step length $2\varepsilon/N$, while in the subinterval $[0, 1-\varepsilon]$, we first find h for some fix N, by means of the nonlinear equation (14), and corresponding to that h we distribute $N/2$ points in the interval $[0, 1-\varepsilon]$. The mesh length is denoted by $h_j = \vartheta_j - \vartheta_{j-1}$, for $j = 1, 2, \ldots, N$.

Remark 1. The mesh size in piecewise uniform and the modified graded region is given by

$$h_j = \begin{cases} 2\varepsilon/N & \text{for } j = 1, 2, \ldots, N/2, \\ \rho h \vartheta_{j-1} & \text{for } j = N/2+1, N/2+2, \ldots, N. \end{cases}$$

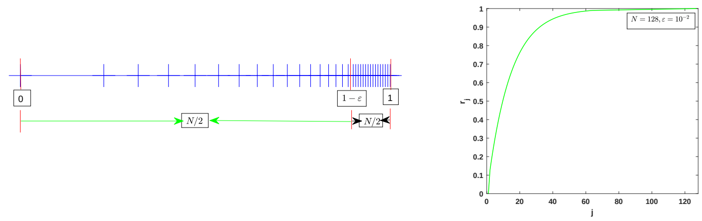
Lemma 3. The mesh defined in (13) satisfies the following estimates:

$$|h_{j+1} - h_j| \leq \begin{cases} 0 & \text{for} \quad j = 1, 2, \ldots, N/2, \\ Ch & \text{for} \quad j = N/2+1, N/2+2, \ldots, N. \end{cases}$$

Proof. Initially, we consider $j = 1, 2, \ldots, N/2$. As the mesh is uniform in this portion, so there is nothing to prove.

For $j = N/2+1, N/2+2, \ldots, N$, we have

$$|h_{j+1} - h_j| = |\rho h \vartheta_j - \rho h \vartheta_{j-1}|$$



(a) (b)

Figure 1: Modified graded mesh layer at $r = 1$ for values of N and ε (a) $N = 32, \varepsilon = 10^{-1}$ (b) $N = 128, \varepsilon = 10^{-2}$.

$$
\begin{aligned}
&= \rho h|\vartheta_i - \vartheta_{j-1}| \\
&= \rho^2 h^2 \vartheta_{j-1} \\
&\leq Ch.
\end{aligned}
$$

Here, we have taken $0 < \rho, h < 1$. □

Lemma 4. For the modified graded mesh defined in (13), the parameter h satisfies the following bound:

$$h \leq CN^{-1} \ln(1/\varepsilon).$$

Proof. Let K_1 be the number of points ϑ_j in the partition (13) such that $\vartheta_j \leq \varepsilon$, for $j = 1,\ 2,\ \ldots,\ N/2$. Clearly $K_1 \leq C/h$ and K_2 are the number of points in the partition (13) such that $\vartheta_j > \varepsilon$. Let $\vartheta_{N/2+1}$ be the smallest point such that $\vartheta_j > \varepsilon$. We have to estimate the bound for K_2. Assuming $\rho h \leq 1$, we have

$$
\begin{aligned}
K_2 = \sum_{N/2+1}^{N} 1 = \sum_{N/2+1}^{N} (\vartheta_{j+1} - \vartheta_j)^{-1} \int_{\vartheta_j}^{\vartheta_{j+1}} d\vartheta \\
= \sum_{N/2+1}^{N} (h_{j+1})^{-1} \int_{\vartheta_j}^{\vartheta_{j+1}} d\vartheta \\
= \sum_{N/2+1}^{N} (\rho h \vartheta_j)^{-1} \int_{\vartheta_j}^{\vartheta_{j+1}} d\vartheta
\end{aligned}
$$

$$\leq \sum_{N/2+1}^{N} (2/\rho h \vartheta_{j+1})^{-1} \int_{\vartheta_j}^{\vartheta_{j+1}} d\vartheta,$$

because $\vartheta_{j+1} < 2\vartheta_j$. For any $\vartheta \in [\vartheta_j, \vartheta_{+1}]$, we have

$$\begin{aligned} K_2 &\leq \sum_{N/2+1}^{N} 2(\rho h)^{-1} \int_{\vartheta_j}^{\vartheta_{j+1}} \frac{1}{\vartheta} d\vartheta \\ &\leq 2(\rho h)^{-1} \int_{\varepsilon}^{1} \frac{1}{\vartheta} d\vartheta \\ &\leq 2(\rho h)^{-1} \ln(1/\varepsilon). \end{aligned}$$

Recalling $N = K_1 + K_2$, we have

$$\begin{aligned} N &\leq C/\rho h + 2(\rho h)^{-1} \ln(1/\varepsilon) \\ N &\leq 1/h(C\rho + 2(\rho)^{-1} \ln(1/\varepsilon)) \\ N &\leq 1/h(C \ln(1/\varepsilon)). \end{aligned}$$

Finally, we get

$$h \leq CN^{-1} \ln(1/\varepsilon),$$

where N is the number of grid points in the r-direction. □

3.3 The classical implicit upwind scheme

Prior to explaining the scheme, we establish necessary operators for a given mesh function $z(r_j, \theta_n)$, in space and time. First, we define the forward difference operator D_r^+, the backward difference operator D_r^-, and the central difference operator D_r^0 in the spatial domain. These operators allow us to approximate derivatives with respect to the spatial variable, enabling us to capture the behavior of the function at different points. Additionally, we introduce the backward difference operator D_θ^- in the temporal domain represented by

$$\begin{cases} D_r^+ z(r_j,\theta_n) = \dfrac{z(r_{j+1},\theta_n) - z(r_j,\theta_n)}{h_{j+1}}, \\ D_r^- z(r_j,\theta_n) = \dfrac{z(r_j,\theta_n) - z(r_{j-1},\theta_n)}{h_j}, \\ D_r^0 z(r_j,\theta_n) = \dfrac{z(r_{j+1},\theta_n) - z(r_{j-1},\theta_n)}{\hat{h_j}}, \\ D_\theta^- z(r_j,\theta_n) = \dfrac{z(r_j,\theta_n) - z(r_j,\theta_{n-1})}{\triangle\theta}, \end{cases} \tag{15}$$

respectively, and define the second-order finite difference operator δ_r^2 in space by

$$\delta_r^2 z(r_j,\theta_n) = \frac{2\left(D_r^+ z(r_j,\theta_n) - D_r^- z(r_j,\theta_n)\right)}{\hat{h_j}},$$

where $\hat{h_j} = h_j + h_{j+1} \quad j = 1,\dots,N-1$ and $\quad h_j = r_j - r_{j-1}, \quad j = 1,\dots,N.$

Let $\aleph_\varepsilon^{N,M} = \overline{\aleph}_\varepsilon^{N,M} \bigcap \aleph$ and let $\Upsilon_\varepsilon^{N,M} = \overline{\aleph}_\varepsilon^{N,M} \setminus \aleph_\varepsilon^{N,M}$. For the discretization of the continuous problem (1), the following classical implicit upwind finite difference scheme is used on the mesh $\overline{\aleph}_\varepsilon^{N,M}$:

$$\begin{cases} L_\varepsilon^{N,M} Y^{N,\triangle\theta} \equiv (D_\theta^- - \varepsilon\delta_r^2 + \Psi_1 D_r^- + \Psi_2) Y^{N,\triangle\theta} = f \quad \text{in} \quad \aleph_\varepsilon^{N,M}, \\ Y^{N,\triangle\theta} = y, \quad \text{on} \quad \mathrm{G}_\varepsilon^{N,M}. \end{cases} \tag{16}$$

Demonstrating the discrete minimum principle of the finite difference operator $L_\varepsilon^{N,M}$ establishes its well-known property, ultimately leading to the ε-uniform stability of the difference operator $L_\varepsilon^{N,M}$.

Lemma 5. [Discrete Minimum Principle] Assume that a mesh function Usatisfies $U \geq 0$ on $G_\varepsilon^{N,M}$. Then $L_\varepsilon^{N,M} U \geq 0$ in $\aleph_\varepsilon^{N,M}$ implies that $U \geq 0$ at each point of $\overline{\aleph}_\varepsilon^{N,M}$.

Proof. Let s and l be indices such that $U_s^l = \min\limits_{(j,\theta)} U_j^\theta$, for $U_j^\theta \in \overline{\aleph}_\varepsilon^{N,M}$. Assume that $U_s^l < 0$. It is easy to see that $(s,l) \in \{1,2,\dots,N\} \times \{1,2,\dots,M\}$, because otherwise $U_s^l \geq 0$. It follows that $U_{s+1}^l - U_s^l > 0$ and $U_{s-1}^l - U_s^l > 0$. Thus $L_\varepsilon^{N,M} U_s^l < 0$, which is a contradiction. Therefore $U_s^l \geq 0$. The indices s and l being arbitrary, we obtain $U_j^\theta \geq 0$ in $\overline{\aleph}_j^\theta$. □

The subsequent theorem asserts the ε-uniform convergence of the numerical scheme (16) on the modified graded mesh $\overline{\aleph}_\varepsilon^{N,M}$, demonstrating nearly first-order accuracy.

Theorem 3. [Error due to up-winding] Consider the solution y to the continuous problem defined by (1)–(2), and let $Y^{N,\triangle\theta}$ represent the solution corresponding to the discrete problem outlined in (16). We can then examine the error linked to the discrete solution $Y^{N,\triangle\theta}$ at the time instance θ_n that satisfies

$$\left| y(r_j,\theta_n) - Y^{N,\triangle\theta}(r_j,\theta_n) \right| \leq C\big(N^{-1}\ln(1/\varepsilon) + \triangle\theta\big), \quad \text{for } 1 \leq j \leq N-1.$$

Proof. We obtained this result with the help of article Mukherjee and Natesan; see [18, Appendix A]. □

The primary objective of this article is to enhance the accuracy of the discrete solution $Y^{N,\triangle\theta}$ for the problem (16) through a post-processing technique. The aim is to achieve an ε-uniform order of accuracy greater than one, considering both the spatial and temporal variables for the IBVP (1)–(2). To accomplish this, we will employ the Richardson extrapolation technique. Prior to introducing this technique, we present some essential technical lemmas that will be utilized in section 4.

Lemma 6. On $\overline{\Lambda}_r^{N,\varepsilon} = \{r_j\}_0^N$, define the mesh function

$$R_j = \prod_{k=1}^{j}\left(1 + \frac{\alpha h_k}{\varepsilon}\right),$$

(with the usual convention that $R_0 = 1$). Then for $1 \leq j \leq N-1$,

$$L_\varepsilon^{N,M} R_j \geq \left(\frac{C_1}{\varepsilon + \alpha h_j}\right) R_j, \tag{17}$$

for some constant C_1. Moreover, for $N/2 + 1 \leq j \leq N-1$, we have

$$L_\varepsilon^{N,M} R_j \geq C_2 \varepsilon^{-1} R_j, \tag{18}$$

for some constant C_2.

Proof. Firstly, $R_j - R_{j-1} = \dfrac{\alpha h_j}{\varepsilon} R_{j-1}$ we have

$$\begin{aligned} L_\varepsilon^{N,M} R_j &= -\frac{2\alpha}{h_j + h_{j+1}}(R_j - R_{j-1}) + \Psi_j \frac{\alpha}{\varepsilon} R_{j-1} + \Psi_j R_i \\ &\geq \frac{\alpha}{\varepsilon} R_{j-1}\left[\Psi_j - \frac{2\alpha h_j}{h_j + h_{j+1}}\right] \end{aligned}$$

$$\geq \frac{C\alpha}{\varepsilon + \alpha h_j} R_i \quad for\ 1 \leq j \leq N-1.$$

Therefore, (17) is proved and (18) is an easy consequence of it, since $h/\varepsilon < 4/\alpha$. □

Lemma 7. For the modified graded mesh $\overline{\Lambda}_r^{N,\varepsilon} = \{r_j\}_0^N$, the following inequalities hold true:

(i)
$$\exp(-\alpha(1-r_j)/\varepsilon) \leq \prod_{k=j+1}^{N} \left(1 + \frac{\alpha h_k}{\varepsilon}\right)^{-1}, \quad \text{for all } j. \tag{19}$$

(ii) There exists a constant C such that
$$\prod_{k=j+1}^{N} \left(1 + \frac{\alpha h_k}{\varepsilon}\right)^{-1} \leq C N^{-4(1-j/N)}, \quad \text{for } N/2 \leq j \leq N-1. \tag{20}$$

Proof. The proof of (i) and (ii) follows from [34]. □

4 Extrapolation of $Y^{N,\triangle\theta}$

Within this section, we introduce the extrapolation technique and concurrently present a thorough error analysis. This analysis involves the decomposition of the solution pertaining to the discrete problem outlined in (16). Ultimately, we establish an ε-uniform error estimate that is closely linked with the extrapolated solution.

4.1 Extrapolation technique

To improve the accuracy of the numerical solution $Y^{N,\triangle\theta}$ by extrapolation, we shall solve the discrete problem (16) on the fine mesh $\overline{\aleph}_\varepsilon^{2N,2M} = \overline{\Lambda}_r^{2N,\varepsilon} \times \mathbb{W}_\theta^{2M}$, where $\Lambda_r = (0,1)$ with $2N$ mesh interval in the spatial direction and $2M$ mesh interval in the θ-direction, where $\overline{\Lambda}_r^{2N,\varepsilon}$ is a Modified graded mesh having the same transition point $(1-\varepsilon)$ as $\overline{\Lambda}_r^{N,\varepsilon}$ and is obtained by bisecting each mesh interval of $\overline{\Lambda}_r^{N,\varepsilon}$. Clearly, $\overline{\aleph}_\varepsilon^{N,M} = \{(r_j, \theta_n)\} \subset \overline{\aleph}_\varepsilon^{2N,2M} = \{(\tilde{r}_j, \tilde{\theta}_n)\}$

and the following are the corresponding mesh widths in $\overline{\aleph}_{\varepsilon}^{2N,2M}$:

$$\tilde{r}_j - \tilde{r}_{j-1} = \begin{cases} \overline{H}/2, & \text{for } \tilde{r}_j \in \overline{\Lambda}^{2N,\varepsilon} \bigcap [0, 1-\varepsilon], \quad \text{where } \overline{H} = 2\varepsilon/N, \\ \overline{h}/2, & \text{for } \tilde{r}_j \in \overline{\Lambda}^{2N,\varepsilon} \bigcap [1-\varepsilon, 1], \quad \text{where } \overline{h} = \rho h \vartheta_{j-1}, \\ \tilde{\theta}_n - \tilde{\theta}_{n-1} = \triangle\theta/2 & \text{for } \tilde{\theta}_n \in \mathbb{W}_{\theta}^{2M}. \end{cases}$$

Let $\tilde{Y}^{2N,\triangle\theta/2}$ be the solution of the discrete problem (16) on the mesh $\overline{\aleph}_{\varepsilon}^{2N,2M}$. Now, it follows from Theorem 3 that

$$\begin{aligned} Y^{N,\triangle\theta}(r_j,\theta_n) - y(r_j,\theta_n) &= C_3\left(N^{-1}\ln(1/\varepsilon) + \triangle\theta\right) + \mathbb{R}_{N,\triangle\theta}(x_j,\theta_n) \\ &= C_3\left(N^{-1}(\alpha\varepsilon/2\varepsilon) + \triangle\theta\right) + \mathbb{R}_{N,\triangle\theta}(r_j,\theta_n), \\ &\qquad (r_j,\theta_n) \in \overline{\aleph}_{\varepsilon}^{N,M}, \end{aligned} \tag{21}$$

where C_3 is some fixed constant and the remainder term $\mathbb{R}_{N,\triangle\theta}$ is $O\left(N^{-1}\ln(1/\varepsilon)+\triangle\theta\right)$. keeping in the mind that $\tilde{Y}^{2N,\triangle\theta/2}$ is obtaining using the same transition point $1-\varepsilon$, we have

$$\begin{aligned} \tilde{Y}^{2N,\triangle\theta/2}(\tilde{r}_j,\tilde{\theta}_n) - y(\tilde{r}_j,\tilde{\theta}_n) &= C_3\left((2N)^{-1}(\alpha\varepsilon/2\varepsilon) + (\triangle\theta/2)\right) + \tilde{\mathbb{R}}_{2N,\triangle\theta/2}(\tilde{r}_j,\tilde{\theta}_n), \\ &\qquad (\tilde{r}_j,\tilde{\theta}_n) \in \overline{\aleph}_{\varepsilon}^{2N,2M}, \end{aligned} \tag{22}$$

where the remainder term $\mathbb{R}_{2N,\triangle\theta/2}$ is $O\left(N^{-1}\ln(1/\varepsilon) + \triangle\theta\right)$. Now, the elimination of the $O(N^{-1})$ and $O(\triangle\theta)$ terms from (21) and (22) lead to the following approximation:

$$\begin{aligned} &y(r_j,\theta_n) - \left(2\tilde{Y}^{2N,\triangle\theta/2}(r_j,\theta_n) - Y^{N,\triangle\theta}(r_j,\theta_n)\right) \\ &\quad = -2\tilde{\mathbb{R}}_{2N,\triangle\theta/2}(r_j,\theta_n) + \mathbb{R}_{N,\triangle\theta}(r_j,\theta_n) \\ &\quad = O\left(N^{-1}\ln(1/\varepsilon) + \triangle\theta\right), \qquad (r_j,\theta_n) \in \overline{\aleph}_{\varepsilon}^{N,M}. \end{aligned}$$

Hence, we will employ the subsequent straightforward extrapolation formula,

$$Y_{extp}^{N,\triangle\theta}(r_j,\theta_n) = 2\tilde{Y}^{2N,\triangle\theta/2}(r_j,\theta_n) - Y^{N,\triangle\theta}(r_j,\theta_n), \quad (r_j,\theta_n) \in \overline{\aleph}_{\varepsilon}^{N,M}, \tag{23}$$

which is expected to yield an approximation of $y(r_j,\theta_n)$ is more accurate than either $Y^{N,\triangle\theta}(r_j,\theta_n)$ or $\tilde{Y}^{2N,\triangle\theta/2}(r_j,\theta_n)$.

4.2 Solution decomposition

To obtain the estimate of the nodal error $\left|y(r_j,\theta_n) - Y^{N,\triangle\theta}_{extp}(r_j,\theta_n)\right|$ after extrapolation of $Y^{N,\triangle\theta}$, we decompose the solution $Y^{N,\triangle\theta}$ on the mesh $\overline{\aleph}^{N,M}$ into the sum

$$Y^{N,\triangle\theta} = R^{N,\triangle\theta} + S^{N,\triangle\theta},$$

where the smooth component $R^{N,\triangle\theta}$ and the layer component $S^{N,\triangle\theta}$ are, respectively, the solutions of the following discrete problems:

$$\begin{cases} L^{N,M}_{\varepsilon} R^{N,\triangle\theta} = f \quad \text{in} \quad \aleph^{N,M}_{\varepsilon}, R^{N,\triangle\theta} = l \quad \text{on} \quad G^{N,M}_{\varepsilon}, \\ L^{N,M}_{\varepsilon} S^{N,\triangle\theta} = 0 \quad \text{in} \quad \aleph^{N,M}_{\varepsilon}, S^{N,\triangle\theta} = m \quad \text{on} \quad G^{N,M}_{\varepsilon}, \end{cases} \tag{24}$$

Similarly, we decompose the solution $\tilde{Y}^{2N,\triangle\theta/2}$ into the smooth component $\tilde{R}^{2N,\triangle\theta/2}$ and the layer component $\tilde{S}^{2N,\triangle\theta/2}$ on the fine mesh $\overline{\aleph}^{2N,2M}$ as

$$\tilde{Y}^{2N,\triangle\theta/2} = \tilde{R}^{2N,\triangle\theta/2} + \tilde{S}^{2N,\triangle\theta/2}.$$

Then, the errors can be written in the following form:

$$Y^{N,\triangle\theta} - y = \left(R^{N,\triangle\theta} - l\right) + \left(S^{N,\triangle\theta} - m\right).$$

4.3 Extrapolation of $R^{N,\triangle\theta}$

Let

$$\eta_1(r,\theta) = \frac{1}{2}\Psi_1(r)\frac{\partial^2 l}{\partial r^2(r,\theta)} \quad \text{and} \quad \eta_2(r,\theta) = \frac{1}{2}\frac{\partial^2 l}{\partial r^2(r,\theta)}, \quad (r,\theta)\in\aleph.$$

Lemma 8. Assume that $\varepsilon \le N^{-1}$. Then, the local truncation error related to the smooth component fulfills the following condition:

$$L_\varepsilon^{N,M}\left(R^{N,\triangle\theta}-l\right)(r_j,\theta_{n+1})$$
$$=h_j\eta_1(r_j,\theta_{n+1})+\triangle\theta\eta_2(r_j,\theta_{n+1})+O\left(\overline{H}^2+\triangle\theta^2\right),\quad \text{for } 1\leq j\leq N-1.$$

Proof. By utilizing the bounds on the derivatives of l as provided in Theorem 2 and $\varepsilon \leq N^{-1} \leq \overline{H}$, we can readily derive the following outcome from Taylor's expansion:

$$L_\varepsilon^{N,M}\left(R^{N,\triangle\theta}-l\right)(r_j,\theta_{n+1})$$
$$=\frac{\varepsilon}{3\hat{h}_j}\left[h_{j+1}^2\frac{\partial^3 l}{\partial r^3}(\omega_1,\theta_{n+1})-h_j^2\frac{\partial^3 l}{\partial r^3}(\omega_2,\theta_{n+1})\right]+\frac{h_j}{2}\Psi_1(r_j)\frac{\partial^2 l}{\partial r^2}(r_j,\theta_{n+1})$$
$$-\frac{h_j^2}{3!}\Psi_1(r_j)\frac{\partial^3 l}{\partial r^3}(\omega_3,\theta_{n+1})-\frac{\triangle\theta}{2}\frac{\partial^2 l}{\partial\theta^2}(r_j,\theta_{n+1})-\frac{\triangle\theta}{3!}\frac{\partial^3 l}{\partial\theta^3}(r_j,\tilde{\theta}),$$

for some $\omega_1\in(r_j,r_{j+1}),\omega_2,\omega_3\in(r_{j-1},r_j)$ and $\tilde{\theta}\in(\theta_n,\theta_{n+1})$. □

Now, following the classical approach of Keller [12], we define the functions Q_i, where $i=1,2$, as the solutions of the following IBVPs:

$$\begin{cases} L_\varepsilon Q_i=\eta_i \quad \text{in} \quad \aleph,\\ Q_i(r,0)=0,\quad r\in\overline{\Lambda}_r,\\ Q_i(0,\theta)=Q_i(1,\theta)=0,\quad \theta\in(0,T],\quad i=1,2.\end{cases} \tag{25}$$

Next, decompose Q_i as $Q_i=\Theta_i+\Phi_i,\ \ i=1,2$, where the smooth component Θ_i and the layer component Φ_i satisfy the following IBVPs:

$$\begin{cases} L_\varepsilon\Theta_i=\eta_i,\quad L_\varepsilon\Phi_i=0 \quad \text{in} \quad \aleph,\\ \Theta_i(r,0)=\Phi_i(r,0)=0,\quad \text{r}\in\overline{\Lambda}_r,\\ \Theta_i(0,\theta)=\Phi_i(0,\theta)=0,\quad \Theta_i(1,\theta)=-\Phi_i(1,\theta)\quad \theta\in(0,T],\quad i=1,2.\end{cases} \tag{26}$$

Theorem 4. For all nonnegative integers p,q satisfying $0\leq p+q\leq 2$, the smooth components $\Theta_i, i=1,2$, defined in (26) satisfy the following bounds:

$$\left|\left|\frac{\partial^{p+q}\Theta_i}{\partial r^p\partial\theta^q}\right|\right|_{\infty,\aleph}\leq C,\quad \left|\left|\frac{\partial^3\Theta_i}{\partial r^3}\right|\right|_{\infty,\aleph}\leq C\varepsilon^{-1}.$$

Proof. firstly, following the approach of [27] and using the bounds on the derivatives of l given in Theorem 2, we obtain that the solution Q_i of the IBVP (25) satisfies the estimate

$$\left|\frac{\partial^{p+q}Q_i}{\partial r^p \partial \theta^q}(r,\theta)\right| \leq C\left(1+\varepsilon^{-p}\exp(-\alpha(1-r)/\varepsilon\right), \quad (r,\theta)\in\aleph, \text{ for } 0\leq p+q\leq 2. \tag{27}$$

This shows that away from the side $r=1$, the derivatives of Q_i are bounded independent of ε; see, for example, [33]. □

Remark 2. It is to be noted that the bounds obtained in Theorem 2 on $\dfrac{\partial^{p+q}l}{\partial r^3}$, for $0\leq p+q\leq 4$, are sufficient to derive the estimate (27) for the solution Q_i of the IBVP (25).

Lemma 9. Assume that $\varepsilon \leq N^{-1}$. Then,

$$\left(R^{N,\triangle\theta}-l\right)(r_j,\theta_{n+1})$$
$$= h_j\Theta_1(r_j,\theta_{n+1})+\triangle\theta\Theta_2(r_j,\theta_{n+1})+O\left(N^{-2}+\triangle\theta^2\right), \quad \text{for } 1\leq j\leq N-1.$$

Proof. Assume that $1\leq j\leq N-1$. Then, applying Lemma 8 and (26), we get

$$\left(R^{N,\triangle\theta}-l\right)(r_j,\theta_{n+1} = h_jL_\varepsilon\Theta_1(r_j,\theta_{n+1})+\triangle\theta L_\varepsilon\Theta_2(r_j,\theta_{n+1})+O\left(\overline{H}+\triangle\theta^2\right). \tag{28}$$

Now, following the Taylor's expansion, it can be deduced that for $i=1,2$,

$$\left|\left(L_\varepsilon - L_\varepsilon^{N,M}\right)\Theta_i(r_i,\theta_{n+1})\right| \leq \left[\frac{\varepsilon}{3}(h_j+h_{j+1})\left\|\frac{\partial^3\Theta_i}{\partial r^3}\right\|_\infty + \frac{h_j}{2}\Psi_1(r_j)\left\|\frac{\partial^2\Theta_i}{\partial r^2}\right\|_\infty + \frac{\triangle\theta}{2}\left\|\frac{\partial^2\Theta_i}{\partial\theta^2}\right\|_\infty\right].$$

In accordance with Theorem 4 and the inequality $h_j\leq\overline{H}$ for all j, where $\overline{H}=2\varepsilon/N$, it follows that

$$\left|h_j\left(L_\varepsilon-L_\varepsilon^{N,M}\right)\Theta_1(r_{j,\theta_{n+1}})+\triangle\theta\left(L_\varepsilon-L_\varepsilon^{N,M}\right)\Theta_2(r_{j,\theta_{n+1}})\right|$$
$$\leq C\left(\overline{H}^2+\overline{H}\triangle\theta+\triangle\theta^2\right)\leq C\left(\overline{H}^2+\triangle\theta^2\right). \tag{29}$$

Hence, from the combination of (28) and (29), we get

$$L_{\varepsilon}^{N,M}\left[R^{N,\triangle\theta}-l-h_j\Theta_1-\triangle\theta\Theta_2\right](r_j,\theta_{n+1})=O\left(\overline{H}^2+\triangle\theta^2\right),\quad 1\leq j\leq N-1. \tag{30}$$

Provide the definitions for the following pair of discrete functions:

$$\upsilon^{\pm}(r_j,\theta_n)=C_4\left(N^{-2}+\triangle\theta^2\right)(1+r_j)\pm\xi(r_j,(\theta_n),\quad \text{for } 0\leq j\leq N \text{ and } n\geq 0,$$

where

$$\xi(r_j,\theta_n)=\begin{cases}\left[R^{N,\triangle\theta}-l-h_j\Theta_1-\triangle\theta\Theta_2\right](r_j,\theta_n), & \text{for } 0\leq j\leq N-1,\\ 0, & \text{for } j=0,N.\end{cases} \tag{31}$$

Now, for $1\leq j\leq N-1$, using the inequality $\overline{H}\leq 2N^{-1}$, where $\overline{H}=2\varepsilon/N$, we have

$$L_{\varepsilon}^{N,M}\left(\left(N^{-2}+\triangle\theta^2\right)(1+r_j)\right)\geq\alpha\left(N^{-2}+\triangle\theta^2\right)\geq\left(\overline{H}^2+\triangle\theta^2\right)/4. \tag{32}$$

Therefore, by utilizing (30)and (32), one can select the constant C_4 to be suitably large, ensuring that $L_{\varepsilon}^{N,M}\upsilon^{\pm}\geq 0$ within $\aleph_{\varepsilon}^{N,M}$. Additionally, in accordance with (24), (26), and (31), it is established that $\upsilon^{\pm}\geq 0$ over the domain $G_{\varepsilon}^{N,M}$. Consequently, by employing Lemma 5 of the discrete minimum principle across the region $\aleph_{\varepsilon}^{N,M}$, we achieve

$$\left|\left[R^{N,\triangle\theta}-l-h_j\Theta_1-\triangle\theta\Theta_2\right](r_j,\theta_{n+1})\right|\leq C\left(N^{-2}+\triangle\theta^2\right),\ \text{for } 1\leq j\leq N-1,$$

and hereby, the desired result follows. □

We deduce the estimate for the smooth part of the error after extrapolation, in the following lemma.

Lemma 10. Assume that $\varepsilon\leq N^{-1}$. Then, the error after extrapolation associated to the smooth component $R^{N,\triangle\theta}$ satisfies

$$\left|l(r_j,\theta_{n+1})-R_{extp}^{N,\triangle\theta}(r_j,\theta_{n+1})\right|\leq C\left(N^{-2}+\triangle\theta^2\right),\quad \text{for }\ 1\leq j\leq N-1.$$

Proof. Since the mesh widths of $\overline{\aleph}_{\varepsilon}^{2N,2M}$ are half of those of $\overline{\aleph}_{\varepsilon}^{N,M}$, applying Lemma 9 on the fine mesh $\overline{\aleph}_{\varepsilon}^{2N,2M}$ we have

$$\left(\tilde{R}^{2N,\triangle\theta/2}(r_j,\theta_{n+1}-l\right)(r_j,\theta_{n+1})$$
$$=\begin{cases}(\overline{H}/2)\Theta_1(r_j,\theta_{n+1})+(\triangle\theta/2)\Theta_2(r_j,\theta_{n+1})+O\left(N^{-2}+\triangle\theta^2\right),\\ \qquad\qquad \text{for}\ \ 1\le j\le N/2,\\ (\overline{h}/2)\Theta_1(r_j,\theta_{n+1})+(\triangle\theta/2)\theta_2(r_j,\theta_{n+1})+O\left(N^{-2}+\triangle\theta^2\right),\\ \qquad\qquad \text{for}\ \ N/2+1\le j\le N-1,\end{cases} \tag{33}$$

where $\overline{H}=2\varepsilon/N$ and $\overline{h}=\rho h\vartheta_{j-1}$. Therefore, according to the extrapolation formula (23), from Lemma 9 and (33), it immediately follows that

$$\begin{aligned}&l(r_j,\theta_{n+1})-R^{N,\triangle\theta}_{extp}(r_j,\theta_{n+1})\\ &\quad=l(r_j,\theta_{n+1})-\left(2\tilde{R}^{2N,\triangle\theta/2}(r_j,\theta_{n+1})-R^{N,\triangle\theta}(r_j,\theta_{n+1})\right)\\ &\quad=-2\left(\tilde{R}^{2N,\triangle\theta/2}-l\right)(r_j,\theta_{n+1})+\left(R^{N,\triangle\theta}-l\right)(r_j,\theta_{n+1})\\ &\quad=O\left(N^{-2}+\triangle\theta^2\right),\quad \text{for}\ \ 1\le j\le N-1.\end{aligned}$$

□

4.4 Extrapolation of $S^{N,\triangle\theta}$

Lemma 11. The error after extrapolation associated to the layer component $S^{N,\triangle t}$ satisfies

$$\left|m(r_j,\theta_{n+1})-S^{N,\triangle\theta}_{extp}(r_j,\theta_{n+1})\right|\le CN^{-2},\quad \text{for}\ \ 1\le j\le N/2.$$

Proof. Let $1\le j\le N/2$. Now, following the argument resulting to [34] over $\overline{\aleph}^{N,M}$ and using (20), we can show that

$$\left|S^{N,\triangle\theta}(r_j,\theta_{n+1})\right|\le C\prod_{k=j+1}^{N}\left(1+\frac{\alpha h_k}{\varepsilon}\right)^{-1}\le CN^{-2}.$$

Next, from (19) and Theorem 2 we obtain $|m(r_j,\theta_{n+1}|\le CN^{-2}$. Hence, we have

$$\left|\left(S^{N,\triangle\theta} - m\right)(r_j, \theta_{n+1})\right| \le CN^{-2}.$$

Similarly, $\left|\left(\tilde{S}^{2N,\triangle\theta/2} - m\right)(r_j, \theta_{n+1})\right| \le CN^{-2}$ and hereby, the required extrapolated error bound is obtained from the extrapolation formula (23). □

Let $\varrho_1(r,\theta) = \dfrac{2\varepsilon}{\alpha}\Psi_1(r)\dfrac{\partial^2 m}{\partial r^2}(r,\theta)$, and let $\varrho_2(r,\theta) = \dfrac{1}{2}\dfrac{\partial^2 m}{\partial \theta^2}(r,\theta)$, $(r,\theta) \in \aleph$.

Lemma 12. For $N/2+1 \le j \le N-1$, the local truncation error associated to the layer component satisfies

$$\begin{aligned} L_\varepsilon^{N,M}\left(S^{N,\triangle\theta} - m\right)(r_j, \theta_{n+1}) = &\left(N^{-1}\ln(1/\varepsilon)\right)\varrho_1(r_j,\theta_{n+1}) + \triangle\theta\varrho_2(r_j,\theta_{n+1}) \\ &+ O\Big(\varepsilon^{-1}\exp(-\alpha(1-r_{j+1})/\varepsilon)N^{-2}\ln^2(1/\varepsilon) \\ &+ \exp(-\alpha(1-r_j)/\varepsilon) \triangle \theta^2\Big). \end{aligned}$$

Proof. Let $N/2+1 \le j \le N-1$, Then, from the Taylor's expansion and using Theorem 2, for some $\omega_1 \in (r_j, r_{j+1}), \omega_2, \omega_3 \in (r_{j-1}, r_j)$, and $\tilde{\theta} \in (\theta_n, \theta_{n+1})$, we have

$$\begin{aligned} &L_\varepsilon^{N,M}\left(S^{N,\triangle\theta} - m\right)(r_j, \theta_{n+1}) \\ &= \frac{\varepsilon h^2}{4!}\left[\frac{\partial^4 m}{\partial r^4}(\omega_1,\theta_{n+1}) + \frac{\partial^4 m}{\partial r^4}(\omega_2,\theta_{n+1})\right] + \frac{h}{2}\Psi_1(r_j)\frac{\partial^2 m}{\partial r^2}(r_j,\theta_{n+1}) \\ &\quad - \frac{h^2}{3!}\Psi_1(r_j)\frac{\partial^3 m}{\partial r^3}(\omega_3,\theta_{n+1}) + \frac{\triangle\theta}{2}\frac{\partial^2 m}{\partial \theta^2}(r_j,\theta_{n+1}) - \frac{\triangle\theta^2}{3!}\frac{\partial^3 m}{\partial \theta^3}(r_j,\tilde{\theta}) \\ &= \frac{2\varepsilon}{\alpha}\left(N^{-1}\ln(1/\varepsilon)\right)\Psi_1(r_j)\frac{\partial^2 m}{\partial r^2}(r_j,\theta_{n+1}) + \frac{\triangle\theta}{2}\frac{\partial^2 m}{\partial \theta^2}(x_j,t_{n+1}) \\ &\quad + O\left(\varepsilon^{-1}\exp(-\alpha(1-r_{j+1})/\varepsilon)N^{-2}\ln^2(1/\varepsilon) + \exp(-\alpha(1-r_j)/\varepsilon) \triangle \theta^2\right). \end{aligned}$$

□

Let the functions $\mathbb{Q}_i, i = 1,2,$ represent the solutions of the following IBVPs:

$$
\begin{cases}
L_\varepsilon \mathbb{Q}_i = \varrho_i \ \text{ in } \ \Omega = (1-\varepsilon, 1) \times (0, T], \\
\mathbb{Q}_i(r, 0) = 0, \quad r \in [1-\varepsilon, 1], \\
\mathbb{Q}_i(1-\varepsilon, \theta) = \mathbb{Q}_i(1, \theta) = 0, \quad \theta \in (0, T], \quad i = 1, 2.
\end{cases} \tag{34}
$$

The bounds on the derivatives of $\mathbb{Q}_i, i = 1, 2$, are obtained by using Lemmas 13–14. We provide a detailed proof for the function $\mathbb{Q}_1$ and show that the proof for $\mathbb{Q}_2$ follows in a similar manner.

Lemma 13. The bounds for the temporal derivatives of $\mathbb{Q}_1$ are as follows:

$$
\left| \frac{\partial^q \mathbb{Q}_1}{\partial \theta^q}(r, \theta) \right| \leq C \exp(-\alpha(1-r)/\varepsilon), \quad (r, \theta) \in \Omega, \quad \text{for} \quad 0 \leq q \leq 3.
$$

Proof. From (34), we have $\mathbb{Q}_1 \equiv 0$ on $\Upsilon_\Omega = \overline{\Omega} \setminus \Omega$. Also, from Theorem 2, we obtain

$$
|L_\varepsilon \mathbb{Q}_1(r, \theta)| = |\varrho_1| \leq C\varepsilon^{-1} \exp(-\alpha(1-r)/\varepsilon), \quad (r, \theta) \in \Omega.
$$

Now, consider the barrier function

$$
\psi(r, \theta) = C \exp(-\alpha(1-r)/\varepsilon), \quad (r, \theta) \in \Omega, \tag{35}
$$

where C is sufficiently large. It follows that $L_\varepsilon \psi(r, \theta) \geq |L_\varepsilon \mathbb{Q}_1(r, \theta)|$ in Ω and $\psi \geq \mathbb{Q}_1$ on Υ_Ω. Since L_ε satisfies the minimum principle Lemma 1 on $\overline{\Omega}$, there we have $\mathbb{Q}_1 \leq C \exp(-\alpha(1-r)/\varepsilon)$, in Ω.

Next, we shall obtain the bound on $\dfrac{\partial \mathbb{Q}_1}{\partial \theta}$. On the sides $r = 1-\varepsilon$ and $r = 1$ of $\overline{\Omega}$, we have $\mathbb{Q}_1 \equiv 0$ and so $\dfrac{\partial \mathbb{Q}_1}{\partial \theta} \equiv 0$. On the side $\theta = 0$, we have $\mathbb{Q}_1 \equiv 0$, and so $\dfrac{\partial \mathbb{Q}_1}{\partial r} \equiv \dfrac{\partial^2 \mathbb{Q}_1}{\partial r^2} \equiv 0$. Hence, from (34), we obtain

$$
\frac{\partial \mathbb{Q}_1}{\partial \theta}(r, 0) = \frac{2\varepsilon}{\alpha} \Psi_1(r) \frac{\partial^2 m}{\partial r^2}(r, 0), \quad r \in [1-\varepsilon, 1]. \tag{36}
$$

Since on the side $\theta = 0$, we have $m \equiv 0$, which implies that $\dfrac{\partial^2 m}{\partial r^2} \equiv 0$, and hence from (36), we obtain $\dfrac{\partial \mathbb{Q}_1}{\partial \theta}(r, 0) = 0$, r $\in [1-\varepsilon, 1]$. Therefore, it is clear that

$$
\frac{\partial \mathbb{Q}_1}{\partial \theta}(r, 0) \equiv 0, \quad \text{on} \quad \Upsilon_\Omega.
$$

Now, differentiating the equation given in (34) with respect to θ, we get

$$\frac{\partial^2 \mathbb{Q}_1}{\partial \theta^2} - \varepsilon \frac{\partial^3 \mathbb{Q}_1}{\partial \theta \partial r^2} + \Psi_1(r) \frac{\partial^2 \mathbb{Q}_1}{\partial \theta \partial r} + \Psi_2(r) \frac{\partial \mathbb{Q}_1}{\partial \theta} = \frac{2\varepsilon}{\alpha} \Psi_1(r) \frac{\partial^3 m}{\partial \theta \partial r^2} \quad \text{in} \quad \Omega, \quad (37)$$

and employing Theorem 2 in (37) yields that

$$\left| L_\varepsilon \left(\frac{\partial \mathbb{Q}_1}{\partial \theta} \right)(r, \theta) \right| \leq C\varepsilon^{-1} \exp(-\alpha(1-r)/\varepsilon), \quad (r, \theta) \in \Omega.$$

Therefore, choosing the barrier function $\psi(r, \theta)$ given in (20), we obtain that

$$L_\varepsilon \psi(r, \theta) \geq \left| L_\varepsilon \left(\frac{\partial \mathbb{Q}_1}{\partial \theta} \right)(r, \theta) \right| \text{ in } \Omega \text{ and } \psi \geq \left| \frac{\partial \mathbb{Q}_1}{\partial \theta} \right| \text{ on } \Upsilon_\Omega.$$

Thus, applying the minimum principle on $\overline{\Omega}$, we have

$$\left| \frac{\partial \mathbb{Q}_1}{\partial \theta} \right| \leq C \exp(-\alpha(1-r)/\varepsilon) \quad \text{in} \quad \Omega.$$

Similarly, we shall obtain the bound on $\frac{\partial^2 \mathbb{Q}_1}{\partial \theta^2}$. On the side $r = 1 - \varepsilon$ and $r = 0$ of $\overline{\Omega}$ Hence,

$$\left| \frac{\partial^2 \mathbb{Q}_1}{\partial \theta^2}(r, \theta) \right| \leq C \exp(-\alpha(1-r)/\varepsilon), \quad (r, \theta) \in \Omega.$$

Finally, by employing a similar approach as for the second-order derivative $\frac{\partial^2 \mathbb{Q}_1}{\partial \theta^2}$, one can derive the bound for $\frac{\partial^3 \mathbb{Q}_1}{\partial \theta^3}$, thereby completing the proof. □

Lemma 14. The following mixed derivative of $\mathbb{Q}_1$, bound on the derivative of $\frac{\partial \mathbb{Q}_1}{\partial \theta}$ and spatial derivative of $\mathbb{Q}_1$ satisfies the following bound:

$$\begin{cases} (A). \quad \left| \frac{\partial^3 \mathbb{Q}_1}{\partial r \theta^2}(r, \theta) \right| \leq C\varepsilon^{-1} \exp(-\alpha(1-r)/\varepsilon), \quad (r, \theta) \in \Omega \\ (B). \quad \left| \frac{\partial^p}{\partial r^p} \left(\frac{\partial \mathbb{Q}_1}{\partial \theta} \right)(r, \theta) \right| \leq C\varepsilon^{-p} \exp(-\alpha(1-r)/\varepsilon), \quad (r, \theta) \in \Omega \text{ for p} = 1, 2. \\ (C). \quad \left| \frac{\partial^p \mathbb{Q}_1}{\partial r^p}(r, \theta) \right| \leq C\varepsilon^{-p} \exp(-\alpha(1-r)/\varepsilon), \quad (r, \theta) \in \Omega \text{ for } 1 \leq p \leq 3. \end{cases}$$

Proof. For (A) from Lemma 13, we have

$$-\varepsilon \frac{\partial}{\partial r} \left(\frac{\partial^3 \mathbb{Q}_1}{\partial r \partial \theta^2} \right) + \Psi_1(r) \frac{\partial^3 \mathbb{Q}_1}{\partial r \partial \theta^2} + \Psi_2(r) \frac{\partial^2 \mathbb{Q}_1}{\partial r^2} = J_1 \quad \in \Omega, \qquad (38)$$

where $J_1(r,\theta) = -\varepsilon \dfrac{\partial^3 \mathbb{Q}_1}{\partial \theta^3} + \dfrac{2\varepsilon}{\alpha}\Psi_1(r)\dfrac{\partial^4 m}{\partial \theta^2 \partial r^2}$. From Lemma 13 and Theorem 2, we have

$$|J_1(r,\theta)| \leq C\varepsilon^{-1}\exp(-\alpha(1-r)/\varepsilon), \quad (r,\theta) \in \Omega. \tag{39}$$

Let $\theta \in [0,T]$. be fixed. To obtain the desired bound on $\dfrac{\partial^3 \mathbb{Q}_1}{\partial r \partial \theta^2}$, we apply the argument presented by Kellogg and Tsan [13] on the line segment $\{(r,\theta), r \in [1-\varepsilon, 1]\}$. This requires to use (38), inequality (39), and the previously established bound on $\dfrac{\partial^2 \mathbb{Q}_1}{\partial \theta^2}$.

For (B) from Lemma 13, rewriting (37), we get the following form:

$$-\varepsilon \frac{\partial}{\partial r}\left(\frac{\partial^2 \mathbb{Q}_1}{\partial r \partial \theta}\right) + \Psi_1(r)\frac{\partial^2 \mathbb{Q}_1}{\partial r \partial \theta} + \Psi_2(r)\frac{\partial \mathbb{Q}_1}{\partial} = J_2 \quad \in \Omega, \tag{40}$$

$J_2(r,\theta) = -\varepsilon \dfrac{\partial^2 \mathbb{Q}_1}{\partial \theta^2} + \dfrac{2\varepsilon}{\alpha}\Psi_1(r)\dfrac{\partial^3 m}{\partial \theta \partial r^2}$. From Lemmas 13 and 14 (A) and Theorem 2, then we get

$$\left|\frac{\partial^p J_2}{\partial r^p}(r,\theta)\right| \leq C\varepsilon^{-1}\exp(-\alpha(1-r)/\varepsilon), \quad (r,\theta) \in \Omega. \tag{41}$$

Now, let us consider the fixed time $\theta \in [0,T]$. Applying the argument presented by Kellogg and Tsan [13] to the line segment $\{(r,\theta), r \in [1-\varepsilon, 1]\}$, using (40) inequality (40) and the bound on $\dfrac{\partial \mathbb{Q}_1}{\partial \theta}$, we can obtain the desired bound for $\dfrac{\partial^2 \mathbb{Q}_1}{\partial r \partial \theta}$. Similarly, differentiating (40) with respect to r and use of the inequality (41), the obtained bound on $\dfrac{\partial \mathbb{Q}_1}{\partial \theta}$, $\dfrac{\partial^2 \mathbb{Q}_1}{\partial r \partial \theta}$ directly implies the desired bound on $\dfrac{\partial^3 \mathbb{Q}_1}{\partial r^2 \partial \theta}$.

For (C) from Lemma 12, rewriting (34) we get the following form:

$$-\varepsilon \frac{\partial}{\partial r}\left(\frac{\partial \mathbb{Q}_1}{\partial r}\right) + \Psi_1(r)\frac{\partial \mathbb{Q}_1}{\partial r} + \Psi_2 \mathbb{Q}_1 = J_3 \in \Omega, \tag{42}$$

where $J_3(r,\theta) = -\dfrac{\partial \mathbb{Q}_1}{\partial \theta} + \dfrac{2\varepsilon}{\alpha}\Psi_1 \dfrac{\partial^2 m}{\partial r^2}$. From Lemmas 13 and 14 (B), we have

$$\left|\frac{\partial^p J_3}{\partial r^p}(r,\theta)\right| \leq C\varepsilon^{-1}\exp(-\alpha(1-r)/\varepsilon), \quad (r,\theta) \in \Omega \quad \text{for} \quad 0 \leq p \leq 2. \tag{43}$$

Let us now fix $\theta \in [0,T]$ and proceed to apply the argument presented by Kellogg and Tsan [13] to the line segment $\{(r,\theta), r \in [1-\varepsilon, 1]\}$. By utilizing (42), inequality (43), and the given bound on $\mathbb{Q}_1$, we can derive the desired bound for $\dfrac{\partial \mathbb{Q}_1}{\partial r}$.

The bound on $\dfrac{\partial^2 \mathbb{Q}_1}{\partial r^2}$ can be deduced in a similar manner as in the previous cases. By employing (43), along with the bounds on $\mathbb{Q}_1$ and $\dfrac{\partial \mathbb{Q}_1}{\partial r}$, we can establish the desired bound for $\dfrac{\partial^2 \mathbb{Q}_1}{\partial r^2}$. Lastly, to obtain the bound on $\dfrac{\partial^3 \mathbb{Q}_1}{\partial r^3}$, one can differentiate with respect to r and employ (42) in a similar fashion. □

Subsequently, we consolidate the preceding lemmas into the following theorem.

Theorem 5. For all nonnegative integers p, q, satisfying $0 \le p+q \le 3$, the solution $\mathbb{Q}_i, i = 1, 2$, of (34) satisfies the following bounds:

$$\left| \frac{\partial^{p+q} \mathbb{Q}_i}{\partial r^p \theta^q}(r,\theta) \right| \le C\varepsilon^{-p} \exp(-\alpha(1-r)/\varepsilon), \quad (r,\theta) \in \Omega.$$

Lemma 15. For $N/2 + 1 \le j \le N-1$, we obtain

$$\begin{aligned} \left(S^{N,\triangle\theta} - m\right)(r_j, \theta_{n+1}) &= \left(N^{-1} \ln(1/\varepsilon)\right) \mathbb{Q}_1(r_j, \theta_{n+1}) + \triangle\theta \mathbb{Q}_2(r_j, \theta_{n+1}) \\ &\quad + O\left(N^{-2} \ln^2(1/\varepsilon) + \triangle\theta^2\right). \end{aligned}$$

Proof. Let $N/2 + 1 \le j \le N-1$.
Then, Lemma 12 and (34) imply that

$$\begin{cases} L_\varepsilon^{N,M}\left(S^{N,\triangle\theta} - m\right)(r_j, \theta_{n+1}) \\ = \left(N^{-1} \ln(1/\varepsilon)\right) L_\varepsilon \mathbb{Q}_1(r_j, \theta_{n+1}) + \triangle\theta L_\varepsilon \mathbb{Q}_2(r_j, \theta_{n+1}) \\ \quad + O\left(\varepsilon^{-1} \exp(-\alpha(1-r)/\varepsilon) N^{-2} \ln^2(1/\varepsilon) + \exp(-\alpha(1-r_j)/\varepsilon) \triangle \theta^2\right). \end{cases} \tag{44}$$

Again, from the Taylor's expansion and Theorem 5 it follows that for $l = 1, 2$,

$$\left(L_\varepsilon - L_\varepsilon^{N,M}\right) \mathbb{Q}_i(r_j, \theta_{n+1})$$

$$= O\left(\varepsilon^{-1}\exp(-\alpha(1-r_{j+1})/\varepsilon)N^{-1}\ln(1/\varepsilon) + \exp(-\alpha(1-r_j)/\varepsilon)\triangle\theta\right),$$

and so

$$\begin{aligned}&\left|\left(N^{-1}\ln(1/\varepsilon)\right)\left(L_\varepsilon - L_\varepsilon^{N,M}\right)\mathbb{Q}_l(r_j,\theta_{n+1}) + \triangle\theta\left(L_\varepsilon - L_\varepsilon^{N,M}\right)\mathbb{Q}_2(r_j,\theta_{n+1})\right|\\ &\quad\le C\left[\varepsilon^{-1}\exp(-\alpha(1-r_{j+1})/\varepsilon)N^{-2}\ln^{-2}(1/\varepsilon) + \exp(-\alpha(1-r_j)/\varepsilon)\triangle\theta^2\right.\\ &\qquad\left. + \left(\varepsilon^{-1}\exp(-\alpha(1-r_{j+1})/\varepsilon) + \exp(-\alpha(1-r_j)/\varepsilon)\right)\left(N^{-1}\ln(1/\varepsilon)\right)\triangle\theta\right]\\ &\quad\le C\varepsilon^{-1}\exp(-\alpha(1-r_{j+1})/\varepsilon)\left(N^{-2}\ln^2(1/\varepsilon) + \triangle\theta^2\right).\end{aligned} \tag{45}$$

Therefore, combining (44) and (45), we have

$$\begin{aligned}&\left|L_\varepsilon^{N,M}\left[S^{N,\triangle\theta} - m - \left(N^{-1}\ln(1/\varepsilon)\right)\mathbb{Q}_1 - \triangle\theta\mathbb{Q}_2\right](r_i,\theta_{n+1})\right|\\ &\quad\le C_5\left(\varepsilon^{-1}\exp(-\alpha(1-r_{j+1})/\varepsilon)\left(N^{-2}\ln^2(1/\varepsilon) + \triangle\theta^2\right)\right),\\ &\qquad\text{for}\quad N/2+1\le j\le N-1,\end{aligned} \tag{46}$$

for some constant C_5. Now, consider the barrier function

$$\chi_j^n = C_6\left[N^{-2}(1+r_j) + \left(N^{-2}\ln^2(1/\varepsilon) + \triangle\theta^2\right)R_j\prod_{k=1}^{N}\left(1+\frac{\alpha h_k}{\varepsilon}\right)^{-1}\right],$$
$$\text{for } N/2\le j\le N.$$

For $N/2+1\le j\le N-1$, employing (18), we get

$$L_\varepsilon^{N,M}\chi_j^{n+1} \ge C_2C_6\varepsilon^{-1}\left(N^{-2}\ln^2(1/\varepsilon) + \triangle\theta^2\right)\prod_{k=j+1}^{N}\left(1+\frac{\alpha h_k}{\varepsilon}\right)^{-1}, \tag{47}$$

Also, for $j\ge N/2$, from (19), we get

$$\begin{aligned}\exp(-\alpha(1-r_{j+1})/\varepsilon) &\le \exp(\lambda h/\varepsilon)\prod_{k=j+1}^{N}\left(1+\frac{\alpha h_k}{\varepsilon}\right)^{-1}\\ &\le \exp(\lambda_1)\prod_{k=j+1}^{N}\left(1+\frac{\alpha h_k}{\varepsilon}\right)^{-1}.\end{aligned} \tag{48}$$

Hence, for $N/2+1 \leq j \leq N-1$, it is easy to show from (46) – (48) that

$$L_\varepsilon^{N,M}\chi_j^{n+1} \geq \left|L_\varepsilon^{N,M}\left[S^{N,\triangle\theta} - m - \left(N^{-1}\ln(1/\varepsilon)\right)\mathbb{Q}_1 - \triangle\theta\mathbb{Q}_2\right](r_j,\theta_{n+1})\right|, \tag{49}$$

provided $C_6 \geq \exp(\lambda_1)C_5/C_2$. Again, from the proof of Lemma 11, we get

$$\left|\left(S^{n,\triangle\theta} - m\right)(1-\varepsilon,\theta_{n+1})\right| \leq C_7N^{-2},$$

for some constant C_7. Hence, by (34),

$$\chi_{N/2}^{n+1} \geq C_6N^{-2} \geq \left|\left[S^{N,\triangle\theta} - m - \left(N^{-1}\ln(1/\varepsilon)\right)\mathbb{Q}_1 - \triangle\theta\mathbb{Q}_2\right](1-\varepsilon,\theta_{n+1})\right|, \tag{50}$$

provided $C_6 \geq C_7$. Also, recalling (24) and (34), we have

$$\chi_N^{n+1} \geq 0 = \left|\left[S^{N,\triangle\theta} - m - \left(N^{-1}\ln(1/\varepsilon)\right)\mathbb{Q}_1 - \triangle\theta\mathbb{Q}_2\right](1,\theta_{n+1})\right|. \tag{51}$$

Therefore, it follows from (49)–(51) that one can choose

$$C_6 = \max\{\exp(\lambda_1)C_5/C_2, C_7\}$$

so that χ_j^n is a barrier function for

$$\pm\left[S^{N,\triangle\theta} - m - \left(N^{-1}\ln(1/\varepsilon)\right)\mathbb{Q}_1 - \triangle\theta\mathbb{Q}_2\right](r_j,\theta_n).$$

Thus, by the discrete minimum principle over $\overline{\aleph}_\varepsilon^{N,M} \bigcap \left(\left[1-\varepsilon,1\right] \times \left[0,T\right]\right)$, we have

$$\begin{aligned}&\left|\left[S^{N,\triangle\theta} - m - \left(N^{-1}\ln(1/\varepsilon)\right)\mathbb{Q}_1 - \triangle\theta\mathbb{Q}_2\right](r_j,\theta_{n+1})\right|\\ &\quad\leq \chi_j^{n+1}\\ &\quad\leq C\left(N^{-2}\ln^2(1/\varepsilon) + \triangle\theta^2\right),\end{aligned}$$

for $N/2+1 \leq j \leq N-1$, and this completes the proof. □

Lemma 16. The error after extrapolation associated to the layer component $S^{N,\triangle\theta}$ satisfies

$$\left|m(r_j,\theta_{n+1}) - S_{\exp}^{N,\triangle\theta}(r_j,\theta_{n+1})\right| \leq C\left(N^{-2}\ln^2(1/\varepsilon) + \triangle\theta^2\right),$$

for $N/2+1 \leq j \leq N-1$.

Proof. Let $N/2+1 \leq j \leq N-1$. from Lemma 15, we have

$$\left(S^{N,\triangle\theta} - m\right)(r_j,\theta_{n+1}) = N^{-1}\mathbb{Q}_1(r_j,\theta_{n+1}) + \triangle\theta\mathbb{Q}_2(r_j,\theta_{n+1}) + O\left(N^{-2} + \triangle\theta^2\right). \tag{52}$$

Next, since the fine mesh $\overline{\aleph}_\varepsilon^{2N,2M}$ has the same transition point $1-\varepsilon$ as that of $\overline{\aleph}_\varepsilon^{N,M}$, Lemma 15 implies that

$$\begin{aligned}&\left(\tilde{S}^{2N,\triangle\theta/2} - m\right)(r_j,\theta_{n+1})\\ &\quad = (2N)^{-1}\mathbb{Q}_1(r_j,\theta_{n+1}) + \triangle\theta/2\mathbb{Q}_2(r_j,\theta_{n+1}) + O\left(N^{-2} + \triangle\theta^2\right).\end{aligned} \tag{53}$$

Henceforth, eliminating $O(N^{-1})$ and $O(\triangle\theta)$ terms from (52) and (53), the required extrapolated error bound is obtained. □

4.5 Convergence result of the solution $Y_{extp}^{N,\triangle\theta}$

The main result of this paper is presented in the following theorem.

Theorem 6. [Error after extrapolation] Assume that $\varepsilon \leq N^{-1}$. Let y be the solution of the continuous problem (1) – (2) and let $Y_{extp}^{N,\triangle\theta}$ be the solution obtained via the Richardson extrapolation technique by solving the discrete problem (16) on two nested meshes $\overline{\aleph}_\varepsilon^{N,M}$ and $\overline{\aleph}_\varepsilon^{2N,2M}$. Then, the error associated with the solution $Y_{extp}^{N,\triangle\theta}$ at time level θ_n satisfies

$$\left|y(r_j,\theta_n) - Y_{extp}^{N,\triangle\theta}(r_j,\theta_n)\right| \leq C\left(N^{-2}\ln^2(1/\varepsilon) + \triangle\theta^2\right), \quad \text{for} \quad 1 \leq j \leq N-1. \tag{54}$$

Proof. For each $(r_j,\theta_n) \in \overline{\aleph}_\varepsilon^{N,M}$, we have

$$y(r_j,\theta_n) - Y_{extp}^{N,\triangle\theta} = \left(l(r_j,\theta_n) - R_{extp}^{N,\triangle\theta}\right) + \left(m(r_j,\theta_n) - S_{extp}^{N,\triangle\theta}\right).$$

Therefore, the result (54) follows immediately after combining Lemma 10 for the smooth component and Lemmas 11 and (16 for the layer component. □

5 Numerical results

In this section, we showcase the numerical findings acquired through the utilization of the Richardson extrapolation technique, aimed at corroborating the theoretical outcomes asserted in the preceding section. To accomplish this objective, we conduct extensive numerical experiments on a modified graded mesh $\overline{\aleph}_{\varepsilon}^{N,M}$. For all the test examples, we carefully select the transition parameter $\rho = 0.9$, along with $\alpha = 1$ and a time step size of $\triangle\theta = 1/N$. These parameters are consistently applied throughout the numerical investigations.

Example 1. Let us examine the parabolic IBVP presented below:

$$\begin{cases} \dfrac{\partial y}{\partial \theta} - \varepsilon \dfrac{\partial^2 y}{\partial r^2} + \big(1 + r(1-r)\big)\dfrac{\partial y}{\partial r} = f(r,\theta), 7(r,\theta) \in (0,1) \times (0,1\big], \\ y(r,0) = y_0(r), \quad 0 \le r \le 1, \\ y(0,\theta) = 0, \quad y(1,\theta) = 0, \quad 0 < \theta \le 1. \end{cases} \tag{55}$$

Example 2. Let us examine the parabolic IBVP presented below:

$$\begin{cases} \dfrac{\partial y}{\partial \theta} - \varepsilon \dfrac{\partial^2 y}{\partial r^2} + \dfrac{\partial y}{\partial r} = f(r,\theta), & (r,\theta) \in (0,1) \times (0,1\big], \\ y(r,0) = y_0(r), & 0 \le r \le 1, \\ y(0,\theta) = 0, \quad y(1,\theta) = 0, & 0 < \theta \le 1, \end{cases} \tag{56}$$

where the initial data $y_0(r)$ and the source function $f(r,\theta)$ have been chosen to fit the exact solution for the IBVP (55) and (56), which is given by the expression

$$y(r,\theta) = \exp(-\theta)\left(\overline{m}1 + \overline{m}2r - \exp\left(-\frac{1-r}{\varepsilon}\right)\right),$$

where the constants are defined as $\overline{m}1 = \exp\left(-\frac{1}{\varepsilon}\right)$ and $\overline{m}2 = 1 - \exp\left(-\frac{1}{\varepsilon}\right)$. Once the exact solution is known for each ε, the maximum point-wise error $\mathbb{E}_{\varepsilon}^{N,\triangle\theta}$ before and after extrapolation is calculated using

$$\max_{(r_j,\theta_n)\in\overline{\aleph}_\varepsilon^{N,M}} \left| y(r_j,\theta_n) - Y^{N,\triangle\theta}(r_j,\theta_n) \right|$$

and

$$\max_{(r_j,\theta_n)\in\overline{\aleph}_\varepsilon^{N,M}} \left| y(r_j,\theta_n) - Y_{extp}^{N,\triangle\theta}(r_j,\theta_n) \right|,$$

respectively, where $y(r_j,\theta_n), Y^{N,\triangle\theta}(r_j,\theta_n)$ and $Y_{extp}^{N,\triangle\theta}(r_j,\theta_n)$, respectively, denote the exact solution, the upwind numerical solution and the extrapolated solution obtained on the mesh $\overline{\aleph}_\varepsilon^{N,M}$ with N mesh-points in the spatial direction and M mesh-points in the θ-direction such that $\triangle\theta = T/M$ is the uniform time step. In addition, the corresponding order of convergence is determined by

$$\mathbb{P}_\varepsilon^{N,\triangle\theta} = \log_2\left(\frac{\mathbb{E}_\varepsilon^{N,\triangle\theta}}{\mathbb{E}_\varepsilon^{2N,\triangle\theta/2}}\right).$$

Now, for each N and $\triangle\theta$, we define $\mathbb{E}_\varepsilon^{N,\triangle\theta} = \max_\varepsilon \mathbb{E}_\varepsilon^{N,\triangle\theta}$ as the ε-uniform maximum point-wise error and the corresponding local ε-uniform order of convergence is defined by

$$\mathbb{P}^{N,\triangle\theta} = \log_2\left(\frac{\mathbb{E}^{N,\triangle\theta}}{\mathbb{E}^{2N,\triangle\theta/2}}\right).$$

Tables 1 and 3 present the calculated maximum point-wise errors $\mathbb{E}_\varepsilon^{N,\triangle\theta}$, as well as the corresponding order of convergence $\mathbb{P}^{N,\triangle\theta}$ for various values of ε and N before and after extrapolation, respectively, for Example 1 and (55). Similarly, Tables 2 and 4 showcase the corresponding results for Example 2 and (56). The results presented in Tables 1-4 demonstrate a consistent decrease in the computed ε-uniform errors $\mathbb{E}^{N,\triangle\theta}$ for Examples 1 and 2 as the number of grid points, N, increases. Besides, the comparison of numerical results obtained by the proposed scheme and results in [18] are tabulated in Tables 5 and 6 for Example 1. From these tables, one can conclude that the proposed scheme gives better results than the scheme considered in [18]. Figures 2–5 exhibit the solution profile for Examples 1 and 2 for various values of ε and N. Figures 2 and 3 provide a solution profile before Richardson extrapolation whereas Figures 4 and 5 provide a solution profile after Richardson extrapolation. This observation indicates that the implicit upwind scheme (16) exhibits ε-uniform convergence before and after extrapolation. Additionally, Figure 6 depicts the maximum point-wise errors for Examples 1 and 2. The plots clearly illustrate that Richardson extrapolation enhances the order of convergence of the upwind scheme from $O\big(N^{-1}\ln(1/\varepsilon)\big)$ to $O\big(N^{-2}\ln^2(1/\varepsilon)\big)$ for $\triangle\theta = 1/N$. This finding validates the theoretical bounds established in (16) and (33).

6 Conclusion

In this article, our focus revolved around devising an efficient numerical approach for tackling one-dimensional singularly perturbed parabolic problems. Specifically, we leveraged the Richardson extrapolation technique to enhance the performance of the upwind scheme in addressing the one-dimensional singularly perturbed parabolic convection-diffusion IBVP presented in (1) − (2). The methodology began by discretizing the domain using a modified graded mesh. This mesh structure was defined by $N \times M$ points. Subsequently, we applied the classical implicit upwind finite difference scheme to solve the IBVP effectively. To validate the efficacy of our approach, we furnished rigorous proof detailing the ε-uniform error estimation. This proof substantiated the convergence of the upwind scheme in an ε-uniform manner, displaying nearly first-order accuracy. Furthermore, by maintaining the transition parameter at a constant value of $\rho = 0.9$ within the modified graded mesh framework, we extended our analysis to the fine mesh configuration, which is characterized by $2N \times 2M$ mesh-points. Finally, we amalgamated the outcomes obtained from both coarse and fine mesh computations. This amalgamation served as a foundation for the implementation of the Richardson extrapolation technique. Both theoretical and computational examinations affirmed that this extrapolation technique propels the convergence of the basic upwinding scheme from almost first-order accuracy to a nearly second-order accuracy, all within the confines of the discrete supremum norm.

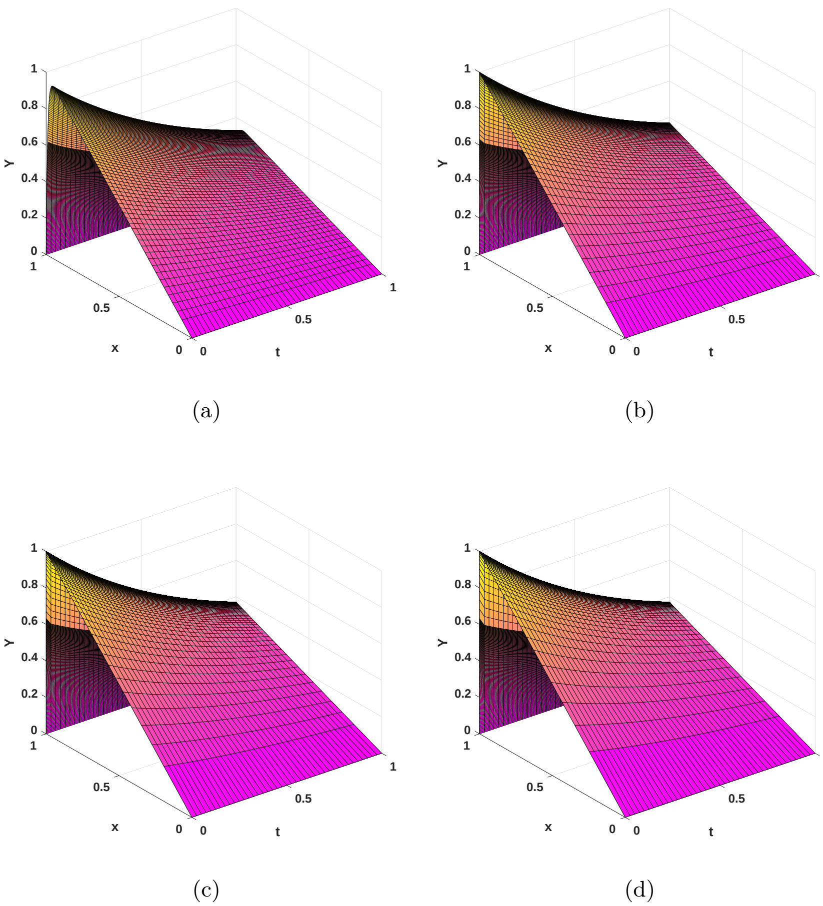
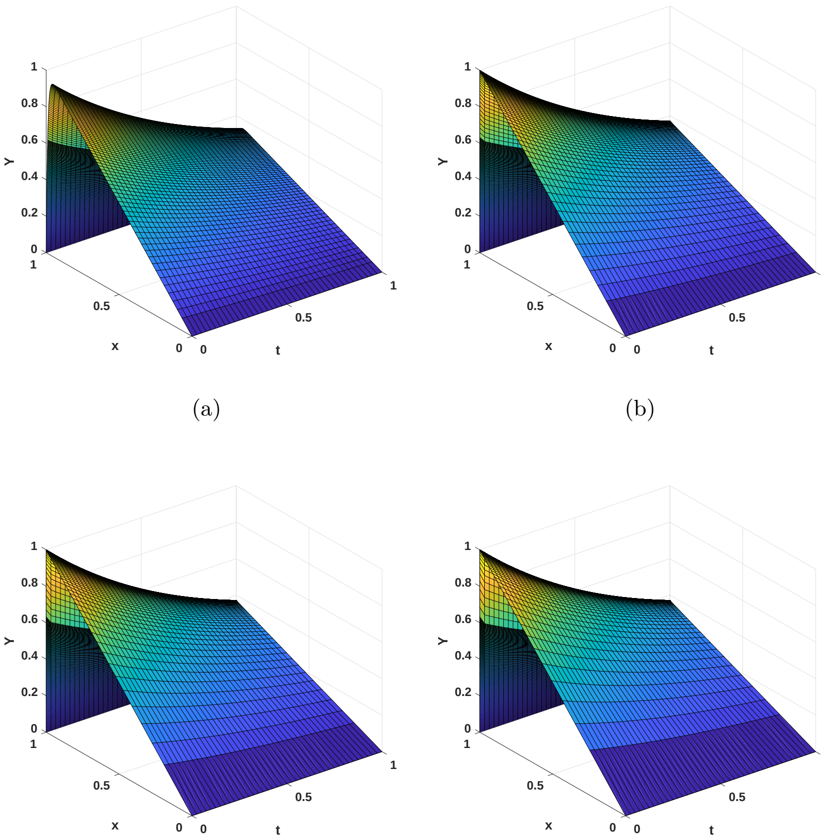
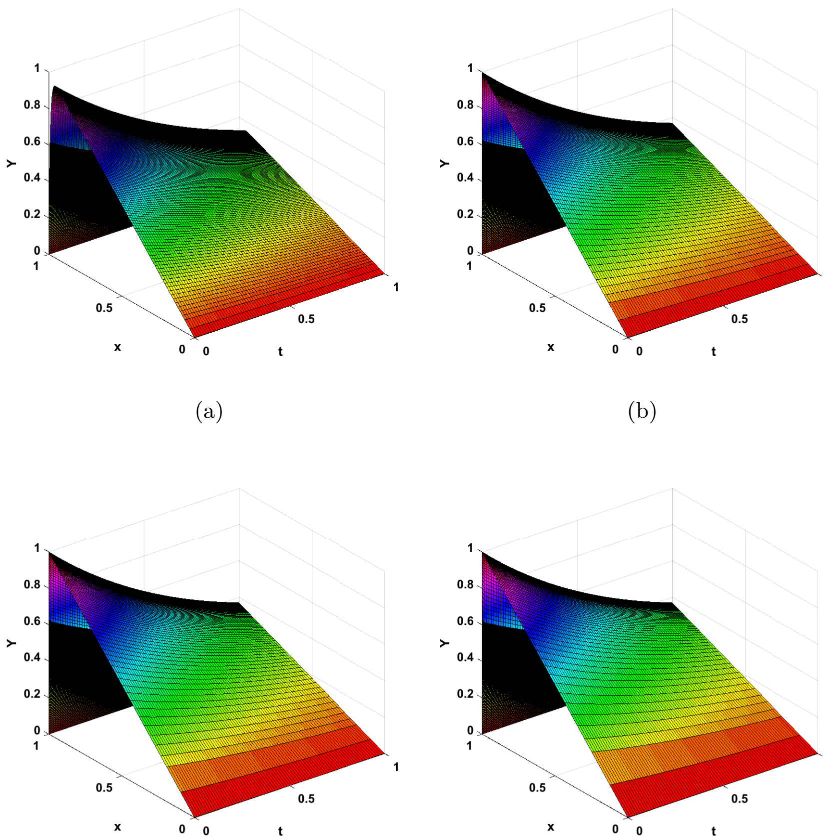


Figure 2: Solution profile for Example 1 using simple upwind scheme before Richardson extrapolation technique with different value of ε and N. (a) Solution of $N = 128, \varepsilon = 10^{-2}$ (b) Solution of $N = 128, \varepsilon = 10^{-4}$ (c) Solution of $N = 128, \varepsilon = 10^{-6}$ (d) Solution of $N = 128, \varepsilon = 10^{-8}$.



(a) (b) (c) (d)

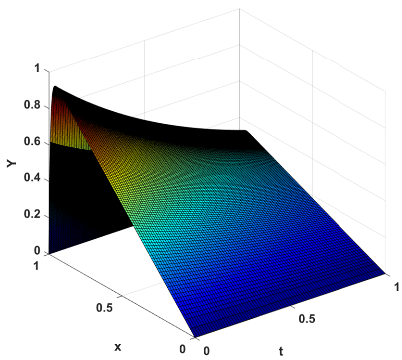
Figure 3: Solution profile for Example 2 using simple upwind scheme before Richardson extrapolation technique with different value of ε and N. (a) Solution of $N = 128, \varepsilon = 10^{-2}$ (b) Solution of $N = 128, \varepsilon = 10^{-4}$ (c) Solution of $N = 128, \varepsilon = 10^{-6}$ (d) Solution of $N = 128, \varepsilon = 10^{-8}$.



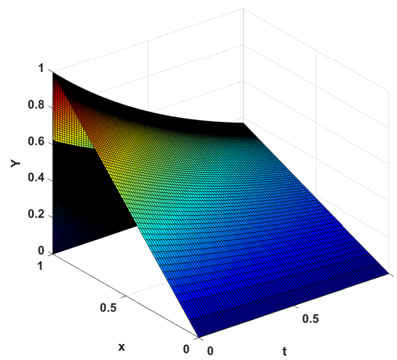
(a) (b)

(c) (d)

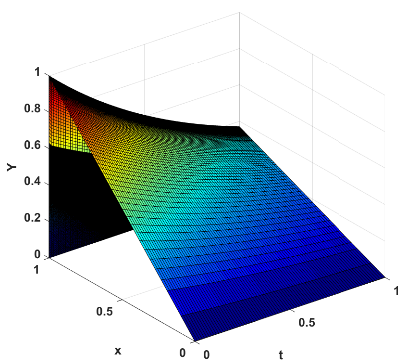
Figure 4: Solution profile for Example 1 using simple upwind scheme and after Richardson extrapolation technique with different value of ε and N. (a) Solution of $N = 128, \varepsilon = 10^{-2}$ (b) Solution of $N = 128, \varepsilon = 10^{-4}$ (c) Solution of $N = 128, \varepsilon = 10^{-6}$ (d) Solution of $N = 128, \varepsilon = 10^{-8}$.



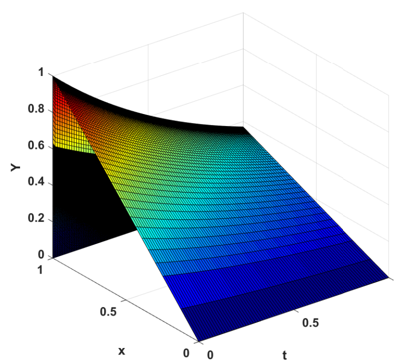
(a)



(b)



(c)



(d)

Figure 5: Solution profile for Example 2 using simple upwind scheme and after Richardson extrapolation technique with different value of ε and N. (a) Solution of $N = 128, \varepsilon = 10^{-2}$ (b) Solution of $N = 128, \varepsilon = 10^{-4}$ (c) Solution of $N = 128, \varepsilon = 10^{-6}$ (d) Solution of $N = 128, \varepsilon = 10^{-8}$.

Table 1: Maximum point-wise errors and the corresponding order of convergence before extrapolation for Example 1 on Modified graded mesh

	Number of Intervals $N/\Delta\theta$					
ε	$128/\frac{1}{10}$	$256/\frac{1}{20}$	$512/\frac{1}{40}$	$1024/\frac{1}{80}$	$2048/\frac{1}{160}$	$4096/\frac{1}{320}$
10^{-2}	$1.3707e-02$	$7.5039e-03$	$3.9359e-03$	$2.0188e-03$	$1.0231e-03$	$5.1505e-04$
	0.8692	0.9310	0.9632	0.9805	0.9902	
10^{-4}	$3.2822e-02$	$1.7760e-02$	$9.1788e-03$	$4.7257e-03$	$2.3904e-03$	$1.2045e-03$
	0.8860	0.9523	0.9578	0.9832	0.9889	
10^{-6}	$4.8001e-02$	$2.6420e-02$	$1.3866e-02$	$7.1029e-03$	$3.5934e-03$	$1.8063e-03$
	0.8614	0.9302	0.9650	0.9831	0.9923	
10^{-8}	$6.2334e-02$	$3.4617e-02$	$1.8304e-02$	$9.4150e-03$	$4.7736e-03$	$2.4034e-03$
	0.8485	0.9193	0.9591	0.9799	0.9900	
10^{-10}	$7.4394e-02$	$4.2641e-02$	$2.2683e-02$	$1.1706e-02$	$5.9463e-03$	$2.9968e-03$
	0.8030	0.9106	0.9544	0.9772	0.9886	
$\mathbb{E}^{N,\triangle\theta}$	$7.4394e-02$	$4.2641e-02$	$2.2683e-02$	$1.1706e-02$	$5.9463e-03$	$2.9968e-03$
$\mathbb{P}^{N,\triangle\theta}$	0.8030	0.9106	0.9544	0.9772	0.9886	

Table 2: Maximum point-wise errors and the corresponding order of convergence before extrapolation for Example 2 on Modified graded mesh

	Number of Intervals $N/\Delta\theta$					
ε	$128/\frac{1}{10}$	$256/\frac{1}{20}$	$512/\frac{1}{40}$	$1024/\frac{1}{80}$	$2048/\frac{1}{160}$	$4096/\frac{1}{320}$
10^{-2}	$1.3635e-02$	$7.4761e-03$	$3.9204e-03$	$2.0104e-03$	$1.0185e-03$	$5.1265e-04$
	0.8669	0.9313	0.9636	0.9811	0.9903	
10^{-4}	$3.2821e-02$	$1.7761e-02$	$9.1790e-03$	$4.7258e-03$	$2.3905e-03$	$1.2045e-03$
	0.8859	0.9523	0.9578	0.9832	0.9889	
10^{-6}	$4.7999e-02$	$2.6420e-02$	$1.3865e-02$	$7.1029e-03$	$3.5934e-03$	$1.8063e-03$
	0.8613	0.9302	0.9650	0.9831	0.9923	
10^{-8}	$6.2332e-02$	$3.4617e-02$	$1.8304e-02$	$9.4150e-03$	$4.7736e-03$	$2.4034e-03$
	0.8485	0.9193	0.9591	0.9799	0.9900	
10^{-10}	$7.4392e-02$	$4.2641e-02$	$2.2683e-02$	$1.1706e-02$	$5.9463e-03$	$2.9968e-03$
	0.8029	0.9106	0.9544	0.9772	0.9886	
$\mathbb{E}^{N,\triangle\theta}$	$7.4392e-02$	$4.2641e-02$	$2.2683e-02$	$1.1706e-02$	$5.9463e-03$	$2.9968e-03$
$\mathbb{P}^{N,\triangle\theta}$	0.8029	0.9106	0.9544	0.9772	0.9886	

Table 3: Maximum point-wise errors and the corresponding order of convergence after extrapolation for Example 1 on Modified graded mesh

ε	Number of Intervals $N/\Delta\theta$					
	$128/\frac{1}{10}$	$256/\frac{1}{20}$	$512/\frac{1}{40}$	$1024/\frac{1}{80}$	$2048/\frac{1}{160}$	$4096/\frac{1}{320}$
10^{-2}	$1.8300e-03$	$6.7491e-04$	$2.1138e-04$	$6.1563e-05$	$1.6744e-05$	$4.3943e-06$
	1.4390	1.6748	1.7797	1.8784	1.9299	
10^{-4}	$1.3132e-03$	$4.8091e-04$	$2.4003e-04$	$1.6554e-04$	$1.2649e-04$	$8.9042e-05$
	1.4492	1.0025	1.5360	1.3881	1.5065	
10^{-6}	$2.5422e-03$	$7.2236e-04$	$1.9374e-04$	$5.1776e-05$	$1.4992e-05$	$5.6506e-06$
	1.8153	1.8986	1.9038	1.7881	1.4077	
10^{-8}	$4.3611e-03$	$1.2692e-03$	$3.3970e-04$	$8.8050e-05$	$2.2424e-05$	$5.6808e-06$
	1.7808	1.9015	1.9479	1.9733	1.9809	
10^{-10}	$6.5786e-03$	$1.9550e-03$	$5.2916e-04$	$1.3762e-04$	$3.5092e-05$	$8.8610e-06$
	1.7506	1.8854	1.9430	1.9715	1.9856	
$\mathbb{E}^{N,\triangle\theta}$	$6.5786e-03$	$1.9550e-03$	$5.2916e-04$	$1.3762e-04$	$3.5092e-05$	$8.8610e-06$
$\mathbb{P}^{N,\triangle\theta}$	1.7506	1.8854	1.9430	1.9715	1.9856	

Table 4: Maximum point-wise errors and the corresponding order of convergence after extrapolation for Example 2 on Modified graded mesh

ε	Number of Intervals $N/\Delta\theta$					
	$128/\frac{1}{10}$	$256/\frac{1}{20}$	$512/\frac{1}{40}$	$1024/\frac{1}{80}$	$2048/\frac{1}{160}$	$4096/\frac{1}{320}$
10^{-2}	$1.8144e-03$	$6.6235e-04$	$2.0693e-04$	$6.0125e-05$	$1.6335e-05$	$4.2857e-06$
	1.4538	1.6784	1.7831	1.8800	1.9304	
10^{-4}	$1.3142e-03$	$4.8110e-04$	$2.4012e-04$	$1.6561e-04$	$1.2653e-04$	$8.9056e-05$
	1.4498	1.0026	1.5360	1.3882	1.5067	
10^{-6}	$2.5430e-03$	$7.2242e-04$	$1.9375e-04$	$5.1777e-05$	$1.4991e-05$	$5.6505e-06$
	1.8156	1.8987	1.9038	1.7882	1.4077	
10^{-8}	$4.3619e-03$	$1.2692e-03$	$3.3971e-04$	$8.8050e-05$	$2.2424e-05$	$5.6808e-06$
	1.7810	1.9016	1.9479	1.9733	1.9809	
10^{-10}	$6.5793e-03$	$1.9550e-03$	$5.2917e-04$	$1.3762e-04$	$3.5092e-05$	$8.8617e-06$
	1.7507	1.8854	1.9430	1.9715	1.9855	
$\mathbb{E}^{N,\triangle\theta}$	$6.5793e-03$	$1.9550e-03$	$5.2917e-04$	$1.3762e-04$	$3.5092e-05$	$8.8617e-06$
$\mathbb{P}^{N,\triangle\theta}$	1.7507	1.8854	1.9430	1.9715	1.9855	

Table 5: Comparison of maximum point-wise errors and the corresponding order of convergence before extrapolation for Example 1

	Number of Intervals $N/\Delta\theta$					
ε	$128/\frac{1}{10}$	$256/\frac{1}{20}$	$512/\frac{1}{40}$	$1024/\frac{1}{80}$	$2048/\frac{1}{160}$	$4096/\frac{1}{320}$
10^{-2}	$1.3707e-02$	$7.5039e-03$	$3.9359e-03$	$2.0188e-03$	$1.0231e-03$	$5.1505e-04$
	0.8692	0.9310	0.9632	0.9805	0.9902	
10^{-4}	$3.2822e-02$	$1.7760e-02$	$9.1788e-03$	$4.7257e-03$	$2.3904e-03$	$1.2045e-03$
	0.8860	0.9523	0.9578	0.9832	0.9889	
10^{-6}	$4.8001e-02$	$2.6420e-02$	$1.3866e-02$	$7.1029e-03$	$3.5934e-03$	$1.8063e-03$
	0.8614	0.9302	0.9650	0.9831	0.9923	
$\mathbb{E}^{N,\triangle\theta}$	$4.8001e-02$	$2.6420e-02$	$1.3866e-02$	$7.1029e-03$	$3.5934e-03$	$1.8063e-03$
$\mathbb{P}^{N,\triangle\theta}$	0.8614	0.9302	0.9650	0.9831	0.9923	

Results in [18]

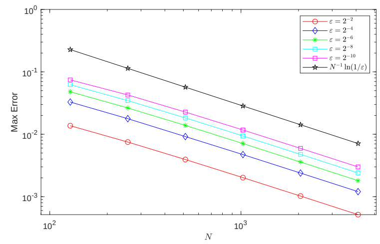
	Number of Intervals N					
ε	32	64	128	256	512	1024
10^{-2}	$4.3463e-02$	$2.9874e-02$	$1.8761e-02$	$1.1330e-02$	$6.5980e-03$	$3.7392e-03$
	0.5408	0.6711	0.7276	0.7799	0.8192	
10^{-4}	$4.2785e-02$	$2.9447e-02$	$1.8513e-02$	$1.1186e-02$	$6.5158e-03$	$3.6933e-03$
	0.5389	0.6695	0.7269	0.7796	0.8190	
10^{-6}	$4.2779e-02$	$2.9443e-02$	$1.8511e-02$	$1.1184e-02$	$6.5150e-03$	$3.6928e-03$
	0.5389	0.6695	0.7269	0.7796	0.8190	
$\mathbb{E}^{N,\triangle\theta}$	$4.2779e-02$	$2.9443e-02$	$1.8511e-02$	$1.1184e-02$	$6.5150e-03$	$3.6928e-03$
$\mathbb{P}^{N,\triangle\theta}$	0.5389	0.6695	0.7269	0.7796	0.8190	

Table 6: Comparison of maximum point-wise errors and the corresponding order of convergence after extrapolation for Example 1

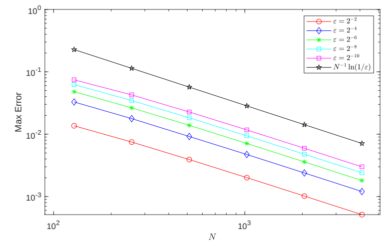
ε	Number of Intervals $N/\Delta\theta$ $128/\frac{1}{10}$	$256/\frac{1}{20}$	$512/\frac{1}{40}$	$1024/\frac{1}{80}$	$2048/\frac{1}{160}$	$4096/\frac{1}{320}$
10^{-2}	$1.8300e-03$	$6.7491e-04$	$2.1138e-04$	$6.1563e-05$	$1.6744e-05$	$4.3943e-06$
	1.4390	1.6748	1.7797	1.8784	1.9299	
10^{-4}	$1.3132e-03$	$4.8091e-04$	$2.4003e-04$	$1.6554e-04$	$1.2649e-04$	$8.9042e-05$
	1.4492	1.0025	1.5360	1.3881	1.5065	
10^{-6}	$2.5422e-03$	$7.2236e-04$	$1.9374e-04$	$5.1776e-05$	$1.4992e-05$	$5.6506e-06$
	1.8153	1.8986	1.9038	1.7881	1.4077	
$\mathbb{E}^{N,\triangle\theta}$	$2.5422e-03$	$7.2236e-04$	$1.9374e-04$	$5.1776e-05$	$1.4992e-05$	$5.6506e-06$
$\mathbb{P}^{N,\triangle\theta}$	1.8153	1.8986	1.9038	1.7881	1.4077	

Results in [18]

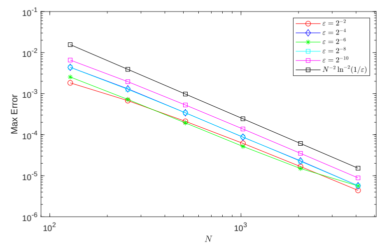
ε	Number of Intervals N 32	64	128	256	512	1024
10^{-2}	$6.7626e-03$	$2.9552e-03$	$1.2260e-03$	$4.4704e-03$	$1.5112e-04$	$4.8606e-05$
	1.1943	1.2693	1.4555	1.5646	1.6365	
10^{-4}	$6.7299e-03$	$2.9308e-03$	$1.2146e-03$	$4.4264e-04$	$1.4950e-04$	$4.8044e-05$
	1.1993	1.2708	1.4563	1.5660	1.6377	
10^{-6}	$6.7297e-03$	$2.9306e-03$	$1.2145e-03$	$4.4260e-04$	$1.4949e-04$	$4.8039e-05$
	1.1993	1.2708	1.4563	1.5660	1.6378	
$\mathbb{E}^{N,\triangle\theta}$	$6.7297e-03$	$2.9306e-03$	$1.2145e-03$	$4.4260e-04$	$1.4949e-04$	$4.8039e-05$
$\mathbb{P}^{N,\triangle\theta}$	1.1993	1.2708	1.4563	1.5660	1.6378	



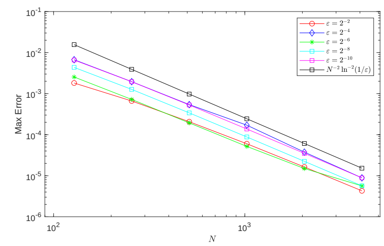
(a)



(b)



(c)



(d)

Figure 6: Before extrapolation and after extrapolation Loglog plot of the maximum point-wise errors for Example 1 and Example 2. (a) Before extrapolation for Example 1 (b) Before extrapolation for Example 2 (c) After extrapolation for Example 1 (d) After extrapolation for Example 2.

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