Iranian Journal of Numerical Analysis and Optimization
Vol. 12, No. 3 (Special Issue), 2022, pp 513–532
DOI:10.22067/ijnao.2022.70466.1035
https://ijnao.um.ac.ir/

How to cite this article
Research Article

Using shifted Legendre orthonormal polynomials for solving fractional optimal control problems

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Abstract

Shifted Legendre orthonormal polynomials (SLOPs) are used to approximate the numerical solutions of fractional optimal control problems. To do so, first, the operational matrix of the Caputo fractional derivative, the SLOPs, and Lagrange multipliers are used to convert such problems into algebraic equations. Also, the method is proposed for solving multidimensional problems, and its convergence is proved. This method is tested on some nonlinear examples. The results indicate that the technique can efficiently solve multidimensional problems.

AMS subject classifications (2020): 49M25; 49J30; 34A08.

Keywords: Shifted Legendre orthonormal polynomials (SLOPs); Fractional optimal control problem (FOCP); Caputo fractional derivative

1 Introduction

For the first time, fractional calculus was introduced in the 17th century. Liouville, Grünwald, Letnikov, Riemann, and Caputo substantially contributed

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Received 18 May 2021; revised 5 March 2022; accepted 9 March 2022

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to the development of its theoretical foundations [6]. They worked on mass and heat transfer problems using the terms semi-derivative and semi-integral. The first book on fractional calculus was written by Oldham and Spanier [27]. Further details on fractional calculus and some of its applications can be found in [11, 12, 21, 22].

In recent years, the applications of fractional calculus in engineering and sciences, including mathematics, fluid dynamics, and physics, have attracted considerable attentions. Fractional calculus is used to extend the usual notions of derivative and integral to ones with real orders and is based on the concepts of fractional derivative in the sense of Caputo and fractional integral in the sense of Riemann–Liouville [22, 27].

When we use a term involving fractional-order derivative(s) in differential equations of optimal control problems, we obtain *fractional optimal control problems* (FOCPs). Many scientific studies confirm the applications of FOCPs in mathematics, mechanics, medicine, and engineering [13, 23]. For example, such problems have been used to obtain numerical solutions of the fractional models of some diseases, such as the fractional-order tumor-immune model, HIV epidemic, and the glucose-insulin system [2, 15, 24].

Orthonormal polynomials have been applied in various linear and nonlinear problems, because they can be used to convert these problems into easy-to-solve algebraic equations. They have many useful properties that facilitate the solution of mathematical problems and provide a way for solving, expanding, and interpreting solutions in some types of differential equations [1, 5, 10, 12].

In this article, we use the SLOPs as the basis functions of the method proposed to solve fractional differential equations. The common approach adopted in the past studies was to solve the one-dimensional problem. Moreover, most of the studies like [5, 4, 10], just obtained the error bound of the operational matrix in fractional derivatives. Hence, none of them proved the convergence of the method under consideration.

Therefore, we aim to develop the method for multidimensional problems in this paper. Moreover, we prove the convergence of the method. The outputs reveal that the method is efficient for multidimensional problems.

We organized the paper as follows. In Section 2, we present the important properties of shifted Legendre polynomials, some preliminary definitions from fractional calculus, and the operational matrix of fractional derivatives. In Section 3, we explain the method and the necessary conditions for the FOCPs. Section 4 discusses the convergence of the proposed technique. In Section 5, we compare our results with those of the previous researches for nonlinear and multidimensional examples. Finally, in Section 6, we present the conclusion.

2 Shifted Legendre orthonormal polynomials

Definition 1. [5] For a function $\xi(t)$, the *Riemann–Liouville fractional integral* of order $\alpha \geq 0$ is defined by

$$I^{\alpha}\xi(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1}\xi(z)\,dz, & \alpha > 0, \quad t > 0, \\ \xi(t), & \alpha = 0, \end{cases} \tag{1}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} z^{\alpha-1} e^{-z} dz,$$

denotes the gamma function.

Definition 2. [5] For a function $\xi(t)$, the *Caputo fractional derivative* of order α is defined by

$$D^{\alpha}\xi(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-z)^{n-\alpha-1} \frac{d^n}{dz^n}\,\xi(z)\,dz, \qquad n-1 < \alpha \leqslant n, \quad t > 0, \tag{2}$$

where n is an integer.

Some properties of these operators can be written as

$$D^{\alpha}\, c = 0, \qquad c \text{ is a constant}, \tag{3}$$

$$I^{\alpha}\left(D^{\alpha}\,\xi(t)\right) = \xi(t) - \sum_{k=0}^{n-1} \xi^{(k)}(0)\,\frac{t^k}{k!}, \tag{4}$$

$$D^{\alpha}\, t^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\delta+1-\alpha)}\, t^{\delta-\alpha}, \tag{5}$$

and

$$D^{\alpha}\left(\beta\,\xi(t) + \gamma\,\tau(t)\right) = \beta\, D^{\alpha}\,\xi(t) + \gamma\, D^{\alpha}\,\tau(t), \tag{6}$$

where δ, β, and γ are scalar coefficients.

Definition 3. [3] The *Legendre polynomial* of degree i, $p_i(z)$, is defined on the interval $[-1,1]$ by the recurrence relation

$$p_{i+1}(z) = \frac{2i+1}{i+1}\, z\, p_i(z) - \frac{i}{i+1}\, p_{i-1}(z), \qquad i \geqslant 1, \tag{7}$$

where

$$p_0(z) = 1, \qquad p_1(z) = z. \tag{8}$$

We obtain the *shifted Legendre polynomials* $p_i^*(t)$ on $[0,1]$ if we use the change of variable $z = 2t - 1$:

$$p_{i+1}^*(t) = \frac{2i+1}{i+1}\,(2t-1)\,p_i^*(t) - \frac{i}{i+1}\,p_{i-1}^*(t), \qquad i \geqslant 1, \tag{9}$$

$$p_0^*(t) = 1, \qquad p_1^*(t) = 2t - 1. \tag{10}$$

These polynomials are orthogonal, in the sense that

$$\langle p_j^*(t), p_i^*(t)\rangle = \int_0^1 p_j^*(t)\,p_i^*(t)\,dt = \begin{cases} \frac{1}{2i+1}, & j = i, \\ 0, & j \neq i. \end{cases} \tag{11}$$

As shown in [3], if we introduce the SLOPs $\widehat{p}_i(t) \equiv \sqrt{2i+1}\,p_i^*(t)$, then

$$\int_0^1 \widehat{p}_i(t)\,\widehat{p}_j(t)dt = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases} \tag{12}$$

and

$$\widehat{p}_i(t) = \sqrt{2i+1}\sum_{k=0}^{i}(-1)^{i+k}\,\frac{(i+k)!}{(i-k)!\,(k!)^2}\,t^k. \tag{13}$$

Assume that ζ is any element of $L^2[0,1]$ and

$$\rho_M = \operatorname{span}\{\widehat{p}_0(t), \widehat{p}_1(t), \dots, \widehat{p}_M(t)\}. \tag{14}$$

Now, for any $h \in \rho_M$, we can write $h \simeq \sum_{i=0}^{M} d_i\,\widehat{p}_i(t)$, where the coefficients d_i are determined as follows:

$$d_i = \int_0^1 h(t)\,\widehat{p}_i(t)\,dt, \quad i = 0, 1, \dots, M. \tag{15}$$

We call $\zeta_\rho \in \rho_M$ *the best approximation* of ζ out of ρ_M whenever

$$\text{for all } h \in \rho_M : \ \|\zeta - \zeta_\rho\|_2 \leqslant \|\zeta - h\|_2. \tag{16}$$

Since $\zeta_\rho \in \rho_M$, there exist coefficients $c_i, i = 0, 1, \dots, M$, such that

$$\zeta_\rho(t) \simeq \sum_{i=0}^{M} c_i\,\widehat{p}_i(t). \tag{17}$$

So, the matrix form of $\zeta_\rho(t)$ is

$$\zeta_\rho(t) \simeq F^T \triangle_M(t), \tag{18}$$

where

$$F = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_M \end{pmatrix}, \qquad \triangle_M(t) = \begin{pmatrix} \widehat{p}_0(t) \\ \widehat{p}_1(t) \\ \vdots \\ \widehat{p}_M(t) \end{pmatrix}. \tag{19}$$

Theorem 1. For the SLOPs vector $\triangle_M(t)$, the fractional derivative of order α, in the sense of Caputo, is defined as follows:

$$D^\alpha \triangle_M(t) = D_{(\alpha)} \triangle_M(t). \tag{20}$$

Herein, $D_{(\alpha)}$ denotes the $(M+1)\times(M+1)$ operational matrix of the fractional derivative, given by

$$D_{(\alpha)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ W_\alpha(n,0) & W_\alpha(n,1) & W_\alpha(n,2) & \cdots & W_\alpha(n,M) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_\alpha(M,0) & W_\alpha(M,1) & W_\alpha(M,2) & \cdots & W_\alpha(M,M) \end{bmatrix},$$

where

$$W_\alpha(k,j) = \sqrt{(2j+1)\,(2k+1)} \sum_{i=n}^{k} \sum_{l=0}^{j} \frac{(-1)^{k+j+i+l}\,(k+i)!\,(l+j)!}{(k-i)!\,i!\,\Gamma(i-\alpha+1)\,(j-l)!\,(l!)^2\,(i+l-\alpha+1)}, \tag{21}$$

and rows 0 to n-1 are zero.

Proof. See [3]. □

3 The numerical method

To solve the following problem, we use the operational matrix of fractional derivatives, the SLOPs and Lagrange multipliers.

$$\min\ J = \int_{t_0}^{t_1} f\big(t, x(t), u(t)\big)\, dt, \tag{22}$$

$$D^\alpha x(t) = \phi\big(t, x(t), u(t)\big), \qquad n-1 < \alpha \leqslant n,\ t \in [t_0, t_1], \tag{23}$$

$$D^{(k)}\, x(t_0) = x_k, \qquad k = 0, 1, \ldots, n-1. \tag{24}$$

Here, $\phi\big(t, x(t), u(t)\big) = g\big(t, x(t)\big) + b(t)\, u(t)$, and S is the feasible solution set. Also, $u(t)$ and $x(t)$ denote the control and state variables, respectively, $u(t)$ is continuous, $x(t)$ is continuously differentiable, $g\big(t, x(t)\big)$, $f\big(t, x(t), u(t)\big)$, and $b(t)$ are smooth functions, $b(t)$ is invertible, $f\big(t, x(t), u(t)\big)$ and $\phi\big(t, x(t), u(t)\big)$ are convex functions, S is a convex set, and $f\big(t, x(t), u(t)\big)$ is integrable on $I = [t_0, t_1]$. Moreover, $f\big(t, x(t), u(t)\big)$ and $g\big(t, x(t)\big)$ satisfy the Lipschitz property. In fact,

$$\|f\big(t, x_1(t), u_1(t)\big) - f\big(t, x_2(t), u_2(t)\big)\| \leq L(\|x_1(t) - x_2(t)\| + \|u_1(t) - u_2(t)\|), \tag{25}$$

and

$$\|g\big(t, x_1(t)\big) - g\big(t, x_2(t)\big))\| \leq K(\|x_1(t) - x_2(t)\|), \tag{26}$$

where L and K are positive constants. Approximate $x(t)$ by the SLOPs $\widehat{p}_i(t)$ as

$$\overline{x}_M(t) = C^T \triangle_M(t), \tag{27}$$

where C^T is an unknown scalar coefficient vector given by

$$C^T = \big(c_0 \; c_1 \; \cdots \; c_M\big). \tag{28}$$

We defined $\widehat{p}_i(t)$ and $\triangle_M(t)$ in (10) and (19), respectively. By (27), we can rewrite the dynamic constraint (23) as

$$C^T D_{(\alpha)} \triangle_M(t) = g\big(t, C^T \triangle_M(t)\big) + b(t)\, u(t). \tag{29}$$

So, we obtain

$$u(t) = \frac{1}{b(t)} \big(C^T D_{(\alpha)} \triangle_M(t) - g\big(t, C^T \triangle_M(t)\big)\big). \tag{30}$$

Then, we can rewrite the initial conditions (24) in the form

$$C^T D_{(k)} \triangle_M(t_0) - x_k = 0, \qquad k = 0, 1, \ldots, n-1. \tag{31}$$

Due to (27), (30) and (31), the performance index J can be approximated by

$$J_M\,[C^T] = \int_{t_0}^{t_1} \widehat{f}\big(t, \overline{x}_M(t), D^\alpha \overline{x}_M(t)) dt + \sum_{k=0}^{n-1} \big(C^T D_{(k)} \triangle_M(t_0) - x_k\big)\lambda_k, \tag{32}$$

where

$$\widehat{f}\big(t, \overline{x}_M(t), D^{\alpha}\overline{x}_M(t)\big) = f\big(t, C^T \triangle_M(t), \frac{1}{b(t)} \left(C^T D_{(\alpha)} \triangle_M(t) - g(t, C^T \triangle_M(t))\right)\big), \quad (33)$$

and λ_k denotes the Lagrange multiplier, which should be determined [11].

The necessary conditions for the optimality of (22) are subject to the dynamic constraints (23) and (24) in the form

$$\frac{\partial J_M}{\partial c_i} = 0,\ i = 0, 1, \ldots, M, \qquad \frac{\partial J_M}{\partial \lambda_k} = 0, \quad k = 0, 1, \ldots, n-1. \quad (34)$$

We can use any standard iterative method to solve the aforementioned system for c_i, $i = 0, 1, \ldots, M$, and λ_k, $k = 0, 1, \ldots, n-1$. As a result, we obtain $x(t)$ and $u(t)$ as given in (27) and (30), respectively [3].

4 Convergence analysis

The use of SLOPs operates as a proof of convergence in three steps. In the first step, we show that the usage is indeed justifiable. In the second step, we show that the functional derivative of a shifted Legendre polynomial is a proper approximation for the same derivative. In the third step, we indicate the difference between the target function for any optimized solution and the value of the target function of the shifted Legendre approximation, tends to zero as the number of the shifted Legendre orthonormal basis increases. We complete these steps by the hypotheses, Lemmas 1 and 2. To find an upper bound for the operational matrix errors in fractional derivatives and to prove the convergence, we use the following theorems.

Theorem 2. Let $\mathcal{H}$ be a Hilbert space, and let Y be a finite-dimensional subspace of $\mathcal{H}$. Also, assume that $\{y_1, y_2, \ldots, y_M\}$ is any basis for Y. Given any x in $\mathcal{H}$, let y_0 denotes the unique best approximation of x out of Y. Then,

$$\|x - y_0\|_2^2 = \frac{G(x, y_1, y_2, \ldots, y_M)}{G(y_1, y_2, \ldots, y_M)}, \quad (35)$$

where

$$G(x, y_1, y_2, \cdots, y_M) = \begin{vmatrix} \langle x, x\rangle & \langle x, y_1\rangle & \cdots & \langle x, y_M\rangle \\ \langle y_1, x\rangle & \langle y_1, y_1\rangle & \cdots & \langle y_1, y_M\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle y_M, x\rangle & \langle y_M, y_1\rangle & \cdots & \langle y_M, y_M\rangle \end{vmatrix}, \quad (36)$$

and

$$G(y_1, y_2, \cdots, y_M) = \begin{vmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_M \rangle \\ \vdots & \vdots & \vdots \\ \langle y_M, y_1 \rangle & \cdots & \langle y_M, y_M \rangle \end{vmatrix}. \tag{37}$$

Proof. See [5]. □

We show that the upper bound of operational matrix errors in fractional derivatives $D^{(\alpha)}$ can be obtained as

$$\varepsilon_D^\alpha := D^{(\alpha)} \triangle_M(t) - \widehat{D}^\alpha \triangle_M(t), \tag{38}$$

where $\widehat{D}^\alpha$ is an approximation of the operator $D^{(\alpha)}$ and

$$\varepsilon_D^\alpha = \begin{pmatrix} \varepsilon_{D,0}^\alpha \\ \varepsilon_{D,1}^\alpha \\ \vdots \\ \varepsilon_{D,M}^\alpha \end{pmatrix}. \tag{39}$$

As mentioned in [18], for each element of ε_D^α, an upper bound for the error related to $D^{(\alpha)}$ can be written as follows:

$$\|\varepsilon_{D,k}^\alpha\|_2 \leqslant \sqrt{2k+1} \sum_{i=1}^{k} \left| \frac{(k+i)!}{(k-i)!\, i!\, \Gamma(i-\alpha+1)} \right| \times \left(\frac{G(t^{i-1}, \widehat{p}_0(t), \ldots, \widehat{p}_M(t))}{G(\widehat{p}_0(t), \ldots, \widehat{p}_M(t))} \right)^{\frac{1}{2}}, \quad 0 \leqslant k \leqslant M. \tag{40}$$

By Theorem 2 and (40), we conclude that ε_D^α tends to zero as the number of the shifted Legendre orthonormal basis increases [5].

Lemma 1. Let $x(t)$ be a continuously differentiable function, and let $\overline{x}_M(t)$ denote the approximation of $x(t)$ by the $SLOPs$. Then,

$$\|x(t) - \overline{x}_M(t)\| \to 0 \quad as \quad M \to \infty. \tag{41}$$

Proof. See [15]. □

Lemma 2. For $x(t)$ and $\overline{x}_M(t)$ as in Lemma 1, when $M \to \infty$,

$$\|D^\alpha x(t) - D^\alpha \overline{x}_M(t)\| \to 0, \tag{42}$$

$$|D^k \overline{x}_M(t_0) - x_k| = 0, \quad k = 0, 1, \ldots, n-1, \tag{43}$$

$$\|\dot{x}(t) - \dot{x}_m(t)\| \to 0. \tag{44}$$

Proof. See [5]. □

We define $J1\,[C^T]$ as follows:

$$J1\,[C^T] = \int_{t_0}^{t_1} f\big(t, x(t), \frac{1}{b(t)}\left(D_{(\alpha)}\, x(t) - g\big(t, x(t)\big)\right) dt + \sum_{k=0}^{n-1} \left(D_{(k)}\, x(t_0) - x_k\right) \lambda_k. \tag{45}$$

Theorem 3. Consider problems (22)–(24), and let $x^*(t)$ be an optimal solution of min $J1\,[C^T]$. Then,

$$\left|J_M\,[C^T] - J1\,[C^T]\right| \to 0 \quad as \quad M \to \infty. \tag{46}$$

Proof. Using (27) and (30) we obtain

$$\begin{aligned}\left|J_M\,[C^T] - J1\,[C^T]\right| =& \Big| \int_{t_0}^{t_1} f\big(t, C^T \triangle_M(t), \frac{1}{b(t)}\left(C^T D_{(\alpha)} \triangle_M(t) - g(t, C^T \triangle_M(t))\right)\big) dt \\ &+ \sum_{k=0}^{n-1} \left(C^T D_{(k)} \triangle_M(t_0) - x_k\right)\lambda_k \\ &- \int_{t_0}^{t_1} f(t, x^*(t), \frac{1}{b(t)}\left(D_{(\alpha)}\, x^*(t) - g(t, x^*(t))\right)\big) dt \\ &- \sum_{k=0}^{n-1} \left(D_{(k)}\, x^*(t_0) - x_k\right)\lambda_k \Big|.\end{aligned}$$

According to (24), (31), and Lemmas 1 and 2, we know that

$$\sum_{k=0}^{n-1} \left(C^T D_{(k)} \triangle_M(t_0) - x_k\right)\lambda_k = 0$$

and that $\sum_{k=0}^{n-1} \left(D_{(k)}\, x^*(t_0) - x_k\right)\lambda_k = 0$. So,

$$\begin{aligned}&\left|J_M\,[C^T] - J1\,[C^T]\right| \\ &= \Big| \int_{t_0}^{t_1} \Big(f\big(t, C^T \triangle_M(t), \frac{1}{b(t)}\left(C^T D_{(\alpha)} \triangle_M(t) - g(t, C^T \triangle_M(t))\right)\big) \\ &\quad - f\big(t, x(t), \frac{1}{b(t)}\left(D_{(\alpha)}\, x(t) - g(t, x(t))\right)\big)\Big) dt\Big|\end{aligned}$$

We know that f satisfies the Lipschitz condition. Therefore,

$$\begin{aligned}&\left|J_M\,[C^T] - J1\,[C^T]\right| \\ &\le \int_{t_0}^{t_1} \Big(L\left(\|C^T \triangle_M(t) - x(t)\|\right) \\ &\quad + \|\frac{1}{b(t)}\left(C^T D_{(\alpha)} \triangle_M(t) - g\big(t, C^T \triangle_M(t)\big) - D_{(\alpha)}\, x(t) + g(t, x(t))\right)\|\Big) dt.\end{aligned}$$

By the Schwartz inequality and separating integrals, we obtain

$$\begin{aligned}
&\left|J_M\left[C^T\right]-J1\left[C^T\right]\right| \\
&\quad\leq L\int_{t_0}^{t_1}\left(\|C^T\triangle_M(t)-x(t))\|\right)dt \\
&\qquad+\frac{1}{|b(t)|}\int_{t_0}^{t_1}\left((\|C^T D_{(\alpha)}\triangle_M(t)-D_{(\alpha)}x(t)\|\right)dt \\
&\qquad+\frac{1}{|b(t)|}\int_{t_0}^{t_1}\left(\|g(t,x(t))-g(t,C^T\triangle_M(t))\|)\right)dt.
\end{aligned}$$

We write the upper bounds of integrals and note that g satisfies the Lipschitz condition. Then,

$$\begin{aligned}
\left|J_M\left[C^T\right]-J1\left[C^T\right]\right|\leq L(t_1-t_0)\big(\|C^T\triangle_M(t)-x(t)\| \\
+\frac{(t_1-t_0)}{|b(t)|}\big(\|C^T D_{(\alpha)}\triangle_M(t)-D_{(\alpha)}x(t)\big)\| \\
+\frac{K\,(t_1-t_0)}{|b(t)|}\,\|x(t)-C^T\triangle_M(t)\|.
\end{aligned}$$

If $M\to\infty$, then Lemma 1 shows that the first and third terms tend to zero. Also, the second term tends to zero by Lemma 2. Consequently, $J_M\left[C^T\right]\to J1[C^T]$. □

Through Theorem 3, we observed that the difference between the value of the target function for any optimized solution of min $J1\left[C^T\right]$ and that of the target function for the approximate value of Legendre tends to zero as $M\to\infty$. Having (27)–(32) in mind, min $J1\left[C^T\right]$ is equivalent to (22). Hence, the difference between the value of target function (22) and that of the Legendre approximate target function tends to zero.

5 Numerical experiments

In this section, we prove the accuracy of the proposed technique by providing some examples and then comparing our achievements with the numerical results obtained in other papers by the computer with Intel Core i7 CPU up to 3.5 GHz, RAM 12GB, and the codes written with Wolfram Mathematica 11.

Example 1. Consider the problem

$$\min\ J=\int_0^1\left((x(t)-t^2)^2+(u(t)+t^4-\frac{20\,t^{\frac{9}{10}}}{9\,\Gamma(\frac{9}{10})})^2\right)dt, \tag{47}$$

subject to dynamic constraints

$$D^{1.1}\, x(t) = t^2\, x(t) + u(t), \tag{48}$$

$$x(0) = \dot{x}(0) = 0. \tag{49}$$

Due to (48), we obtain $u(t)$ and rewrite (47) as

$$u(t) = D^{1.1}\, x(t) - t^2 x(t),$$

$$\begin{aligned} \min\ J = \int_0^1 & \big((C^T \triangle_M(t) - t^2)^2 \\ & + (D^{1.1}\, C^T \triangle_M(t) - t^2\, C^T \triangle_M(t) + t^4 - \frac{20\, t^{\frac{9}{10}}}{9\, \Gamma(\frac{9}{10})})^2\big)\, dt \\ & + \big(C^T D_{(0)} \triangle_M(t_0) - x(0)\big)\lambda_0 + \big(C^T D_{(1)} \triangle_M(t_0) - \dot{x}(0)\big)\lambda_1. \end{aligned}$$

The functional J is minimized by $x^*(t) = t^2$ and $u^*(t) = \frac{20\, t^{\frac{9}{10}}}{9\, \Gamma(\frac{9}{10})} - t^4$, with minimum equal to zero. Table 2 presents the approximate values of J, which are obtained by the proposed method and the methods utilized in [21, 3], with different values of M. As the results indicate, our approach is better than the ones used in [21, 3].

Table 1: Approximations of J with different values of M

M	The method	The method used in [21]	The method used in [3]
4	1.66202×10^{-6}	6.07530×10^{-6}	4.76932×10^{-6}
6	2.44576×10^{-7}	5.91532×10^{-7}	5.37825×10^{-7}
8	5.90947×10^{-8}	1.21966×10^{-7}	1.06099×10^{-7}
9	3.26447×10^{-8}	7.03371×10^{-8}	5.44304×10^{-8}

Table 3 presents the absolute values of errors for the control and state variables for various values of t. Also, in Figure 6, the approximate and exact values of the control and state variables are plotted for $M = 6$.

Table 2: Absolute errors of $x(t)$ and $u(t)$ at $M = 6$

t	$\lvert x^*(t) - x(t)\rvert$	$\lvert u^*(t) - u(t)\rvert$
0.1	1.60241×10^{-7}	1.72334×10^{-5}
0.2	2.35607×10^{-7}	4.57424×10^{-4}
0.3	9.96796×10^{-8}	2.85637×10^{-4}
0.4	6.68032×10^{-8}	2.89849×10^{-4}
0.5	7.86075×10^{-8}	1.79588×10^{-4}
0.6	9.06389×10^{-8}	2.80773×10^{-4}
0.7	2.84397×10^{-7}	1.15197×10^{-4}
0.8	2.78471×10^{-7}	2.69036×10^{-4}
0.9	3.55721×10^{-8}	2.73064×10^{-4}

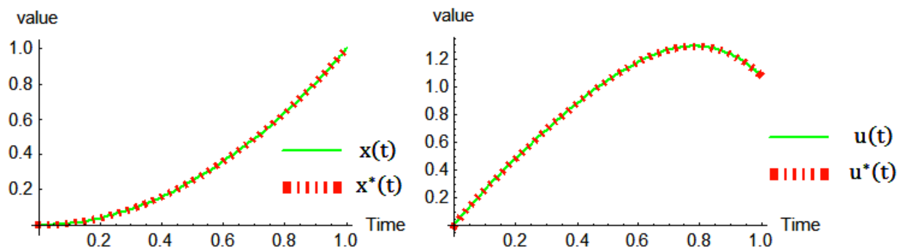

Figure 1: Approximate and exact values of the control and state variables for $M = 6$

Example 2. Consider the two-dimensional problem

$$\min J = \int_0^1 \big((x_1(t) - t^2)^2 + (x_2(t) - t^3)^2 + (u_1(t) - t^4 + \frac{\Gamma(4)}{6\,\Gamma(2.9)}\, t^{1.9} - \frac{\Gamma(3)}{3\,\Gamma(1.9)}\, t^{0.9})^2 + (u_2(t) - t^5 + \frac{\Gamma(4)}{2\,\Gamma(2.9)}\, t^{1.9})^2\big)\, dt, \tag{50}$$

subject to dynamic constraints

$$D^{1.1}\, x_1(t) = 3\, u_1(t) - 3\, t^2 x_1(t) + t^2\, x_2(t) - u_2(t), \tag{51}$$

$$D^{1.1}\, x_2(t) = -2\, u_2(t) + (2\, t^2 - 1)\, x_2(t) + t\, x_1(t), \tag{52}$$

$$x_1(0) = \dot{x_1}(0) = 0, \tag{53}$$

and

$$x_2(0) = \dot{x_2}(0) = 0. \tag{54}$$

By (51) and (52), we obtain $u_1(t)$ and $u_2(t)$ as follows:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ 0 & -\frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} D^{1.1}\, x_1(t) \\ D^{1.1}\, x_2(t) \end{bmatrix} - \begin{bmatrix} -3\,t^2 x_1(t) + t^2\, x_2(t) \\ (2\,t^2-1)\, x_2(t) + t\, x_1(t) \end{bmatrix} \right).$$

We define

$$x_1(t) = C_1^T \triangle_M(t), \quad C_1^T = \left(c_{10}\; c_{11}\; \cdots\; c_{1M}\right),$$
$$x_2(t) = C_2^T \triangle_M(t), \quad C_2^T = \left(c_{20}\; c_{21}\; \cdots\; c_{2M}\right),$$

and rewrite (50) as

$$\begin{aligned} \min J = \int_0^1 & \big((C_1^T \triangle_M(t) - t^2)^2 + (C_2^T \triangle_M(t) - t^3)^2 \\ & + \big(\frac{1}{3}(D^{1.1}\, C_1^T \triangle_M(t) + 3\,t^2\,(C_1^T \triangle_M(t)) - t^2\,(C_2^T \triangle_M(t))) \\ & - \frac{1}{6}(D^{1.1}\, C_2^T \triangle_M(t) - (2\,t^2-1)\,(C_2^T \triangle_M(t)) - t\,(C_1^T \triangle_M(t))) - t^4 \\ & + \frac{\Gamma(4)}{6\,\Gamma(2.9)}\, t^{1.9} - \frac{\Gamma(3)}{3\,\Gamma(1.9)}\, t^{0.9}\big)^2 + \big(-\frac{1}{2}(D^{1.1}\, C_2^T \triangle_M(t) \\ & - (2\,t^2-1)\,(C_2^T \triangle_M(t)) - t\,(C_1^T \triangle_M(t))) - t^5 + \frac{\Gamma(4)}{6\,\Gamma(2.9)}\, t^{1.9}\big)^2\big) dt \\ & + \big(C_1^T\, D_{(0)} \triangle_M(t_0) - x_1(0)\big)\lambda_0 + \big(C_1^T\, D_{(1)} \triangle_M(t_0) - \dot{x_1}(0)\big)\lambda_1 \\ & + \big(C_2^T\, D_{(0)} \triangle_M(t_0) - x_2(0)\big)\lambda_0 + \big(C_2^T\, D_{(1)} \triangle_M(t_0) - \dot{x_2}(0)\big)\lambda_1. \end{aligned}$$

The functions $x_1^*(t) = t^2, x_2^*(t) = t^3$ and $u_1^*(t) = t^4 - \frac{\Gamma(4)}{6\,\Gamma(2.9)}\, t^{1.9} + \frac{\Gamma(3)}{3\,\Gamma(1.9)}\, t^{0.9}, u_2^*(t) = t^5 - \frac{\Gamma(4)}{6\,\Gamma(2.9)}\, t^{1.9}$ minimize the functional J, and the minimum value is zero. In Table 4, we present the approximate values of J with different values of M.

Table 3: Approximate values of J with different values of M

M	J
4	2.39801×10^{-7}
6	3.03043×10^{-8}
8	6.97336×10^{-9}
9	6.97321×10^{-9}

Table 4 presents the absolute values of errors for the state and control variables for various values of t.

Also, in Figures 2 and 3, the approximate and exact values of the state and

Table 4: Absolute errors of $x_1(t)$, $x_2(t)$, $u_1(t)$, and $u_2(t)$ at $M=6$

t	$\lvert x_1^*(t)-x_1(t)\rvert$	$\lvert x_2^*(t)-x_2(t)\rvert$	$\lvert u_1^*(t)-u_1(t)\rvert$	$\lvert u_2^*(t)-u_2(t)\rvert$
0.1	7.19262×10^{-7}	1.74666×10^{-7}	6.4603×10^{-6}	9.51622×10^{-6}
0.2	1.0357×10^{-6}	2.48769×10^{-7}	1.60228×10^{-4}	2.89678×10^{-5}
0.3	3.70976×10^{-7}	7.82014×10^{-8}	1.03983×10^{-4}	1.19302×10^{-5}
0.4	4.54804×10^{-7}	1.37132×10^{-7}	1.05124×10^{-4}	1.82481×10^{-5}
0.5	5.92208×10^{-7}	1.84041×10^{-7}	6.48507×10^{-5}	8.03613×10^{-6}
0.6	9.51419×10^{-8}	1.81842×10^{-8}	1.04023×10^{-4}	1.65065×10^{-5}
0.7	9.14941×10^{-7}	2.02377×10^{-7}	3.7991×10^{-5}	6.07151×10^{-6}
0.8	8.57316×10^{-7}	2.32216×10^{-7}	9.89654×10^{-5}	1.59221×10^{-5}
0.9	2.67307×10^{-7}	2.16467×10^{-9}	9.10531×10^{-5}	1.77034×10^{-5}

control variables are plotted at $M=6$.

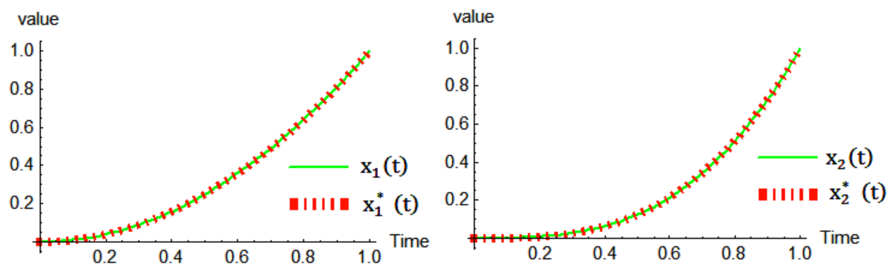

Figure 2: Approximate and exact values of the state variable at $M=6$

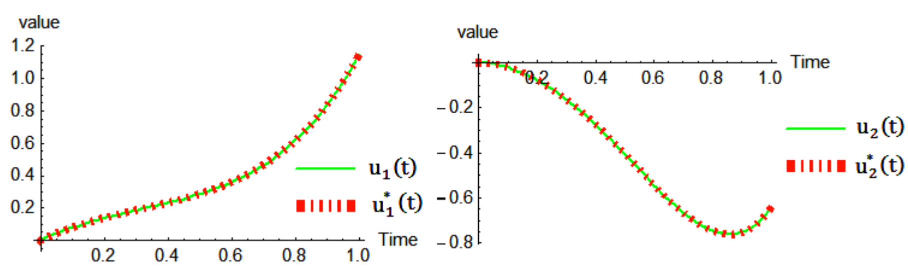

Figure 3: Approximate and exact values of the control variable at $M=6$

We can apply this method to another category of problems. In fact, if in problems (22)–(24), we replace (23) by

$$\varphi\, D^{\alpha}\, x(t) + \psi\, \dot{x}(t) = g\big(t, x(t)\big) + b(t)\, u(t), \qquad n-1<\alpha\leqslant n,\ b(t)\neq 0,\ t\in[t_0,t_1], \tag{55}$$

then the method still converges according to (44), where φ and ψ are scalar coefficients. Let us present one example of this form.

Example 3. Recall from [28] the problem

$$\min J = \int_0^1 \big(u(t) - x(t)\big)^2 \, dt, \tag{56}$$

subject to dynamic constraints

$$\dot{x}(t) + D^{\alpha} x(t) = u(t) - x(t) + \frac{6\, t^{\alpha+2}}{\Gamma(\alpha+3)} + t^3, \tag{57}$$

and

$$x(0) = 0. \tag{58}$$

By (57), we can find $u(t)$:

$$u(t) = \dot{x}(t) + D^{\alpha} x(t) + x(t) - \frac{6\, t^{\alpha+2}}{\Gamma(\alpha+3)} - t^3,$$

$$\min\ J = \int_0^1 \Big(C^T\, \dot{\triangle}_M(t) + D^{\alpha}(C^T \triangle_M(t)) - \frac{6\, t^{\alpha+2}}{\Gamma(\alpha+3)} - t^3\Big)^2 \, dt$$
$$+ \big(C^T\, D_{(0)} \triangle_M(t_0) - x(0)\big)\lambda_0.$$

The functions $x^*(t) = \frac{6\, t^{\alpha+3}}{\Gamma(\alpha+4)}$ and $u^*(t) = \frac{6\, t^{\alpha+3}}{\Gamma(\alpha+4)}$ minimize the functional J, and the minimum value is zero. In Table 5, we present the approximate values of J with different values of M.

Table 5: Approximate values of J at $\alpha = 0.9$ with different values of M

M	J
4	2.32302×10^{-7}
6	2.32786×10^{-10}
8	2.98816×10^{-12}

Table 6 presents the absolute values of errors for the control and state variables for various values of t.
Also, in Figure 3, the approximate and exact values of the control and state variables are plotted for $M = 6$. Tables 3 and 8 present the maximum errors of $u(t)$ and $x(t)$ with different values of M.
Also, in Figure 5, the control and state variables are plotted for $M = 5$ and different values of α.

Table 6: Absolute errors of $x(t)$ and $u(t)$ at $M = 6$

t	$\lvert x^*(t) - x(t)\rvert$	$\lvert u^*(t) - u(t)\rvert$
0.1	3.22688×10^{-7}	2.3951×10^{-5}
0.2	4.89573×10^{-7}	1.18457×10^{-5}
0.3	5.31838×10^{-7}	1.52362×10^{-5}
0.4	6.51328×10^{-7}	5.73914×10^{-6}
0.5	1.48297×10^{-7}	1.58438×10^{-5}
0.6	6.3336×10^{-7}	2.83551×10^{-5}
0.7	1.34478×10^{-7}	1.45402×10^{-5}
0.8	5.49314×10^{-7}	7.44278×10^{-6}
0.9	1.0371×10^{-7}	1.81787×10^{-5}

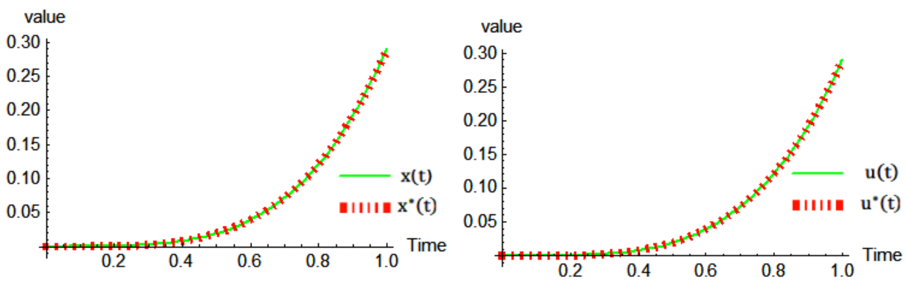

Figure 4: Approximate and exact values of the state and control variables at $M = 6$

Table 7: Maximum errors of $x(t)$ and $u(t)$ at $M = 3$.

$M = 3$	Maximum errors of $x(t)$	Maximum errors of $u(t)$
The method	2.36519×10^{-3}	2.30757×10^{-2}
Algorithm 1 in [28]	8.8025×10^{-3}	8.8025×10^{-3}
Algorithm 2 in [28]	5.1966×10^{-3}	4.3260×10^{-2}

Table 8: Maximum errors of $x(t)$ and $u(t)$ at $M = 5$.

$M = 5$	Maximum errors of $x(t)$	Maximum errors of $u(t)$
Our method	2.21121×10^{-5}	4.7773×10^{-4}
Algorithm 1 in [28]	1.0903×10^{-4}	1.0903×10^{-4}
Algorithm 2 in [28]	4.5321×10^{-5}	6.3134×10^{-4}

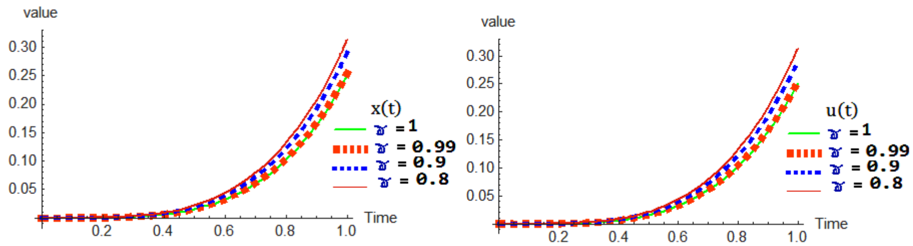

Figure 5: Control and state variables for $M = 5$ and different values of α

6 Conclusion

In this paper, we applied a numerical method to solve a class of fractional optimal control problems. We used the SLOPs and the operational matrix of fractional derivatives. Then, we used the Newton iterative technique to solve these problems. We obtained the error bound of the operational matrix in fractional derivatives and proved the convergence of the method. We focused on multidimensional problems, which have never been solved by this technique. To show the efficiency of the method for multidimensional problems, we provided some nonlinear examples. Comparison of our results with those obtained by other techniques in previous studies revealed the accuracy of the proposed technique for nonlinear and multidimensional problems.

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How to cite this article

R. Naseri, A. Heydari and A.S. Bagherzadeh Using shifted Legendre orthonormal polynomials for solving fractional optimal control problems. *Iranian Journal of Numerical Analysis and Optimization*, 2022; 12(3 (Special Issue), 2022): 513-532. doi: 10.22067/ijnao.2022.70466.1035.