Iranian Journal of Numerical Analysis and Optimization
Vol. 13, No. 3, 2023, pp 532–552
https://doi.org/10.22067/ijnao.2023.79051.1187
https://ijnao.um.ac.ir/

How to cite this article
Research Article

Nearest fuzzy number of type L-R to an arbitrary fuzzy number with applications to fuzzy linear system

S.M. Alavi

Abstract

The fuzzy operations on fuzzy numbers of type L-R are much easier than general fuzzy numbers. It would be interesting to approximate a fuzzy number by a fuzzy number of type L-R. In this paper, we state and prove two significant application inequalities in the monotonic functions set. These inequalities show that under a condition, the nearest fuzzy number of type L-R to an arbitrary fuzzy number exists and is unique. After that, the nearest fuzzy number of type L-R can be obtained by solving a linear system. Note that the trapezoidal fuzzy numbers are a particular case of the fuzzy numbers of type L-R. The proposed method can represent the nearest trapezoidal fuzzy number to a given fuzzy number. Finally, to approximate fuzzy solutions of a fuzzy linear system, we apply our idea to construct a framework to find solutions of crisp linear systems instead of the fuzzy linear system. The crisp linear systems give the nearest fuzzy numbers of type L-R to fuzzy solutions of a fuzzy linear system. The proposed method is illustrated with some examples.

AMS subject classifications (2020): Primary 47S40; Secondary 03E72, 47S40.

Keywords: Fuzzy number of type L-R; Nearest trapezoidal fuzzy number; Shape function; Fuzzy approximation.

1 Introduction

One of the most important topics related to fuzzy mathematics is to study fuzzy numbers. Fuzzy numbers were first introduced by Zadeh [22, 23, 24]

Received 5 October 2022; revised 24 January 2023; accepted 14 May 2023

Seyed Majid Alavi
Department of Mathematics, Faculty of Basic Sciences, Arak Branch, Islamic Azad University, Arak, Iran. e-mail: sm.alavi@iau.ac.ir

and then were developed by other researchers [16, 11]. Fuzzy numbers are important essential tools to represent uncertainty models, formalize fuzzy variable functions, defuzzification [3], and linguistic variables [4]. The researchers in [2] proposed an efficient approach to assigning a distance between fuzzy numbers and describing a pseudo-metric on the set of fuzzy numbers and a metric on the set of trapezoidal fuzzy numbers. Trapezoidal fuzzy numbers have a good application in dealing with uncertain information. In fact, to describe the specification of some uncertain events, we have to use fuzzy numbers. Voxman [20] represented two canonical representations of discrete fuzzy numbers. Nevertheless, some fuzzy numbers are too complicated. Hence, the approximation of general fuzzy numbers with regular fuzzy numbers like fuzzy numbers of type L-R, help decision-makers to make better decisions. Trapezoidal fuzzy numbers and triangular fuzzy numbers are particular cases of fuzzy numbers of type L-R. Abbasbandy et al. showed that the nearest trapezoidal fuzzy number to a given fuzzy number exists and is unique. Hajjari [12] used the concept of 0.5-Level and mean Core to approximate fuzzy numbers. In [13], authors introduced a trapezoidal approximation of an arbitrary fuzzy number by Core and support of the fuzzy numbers. Some researchers presented efficient methods to find the nearest trapezoidal fuzzy number or triangular fuzzy number to a given fuzzy number [8, 21, 5, 10]. Lucian Coroianu [9] proved that quadratic programs give the nearest trapezoidal approximation of general fuzzy numbers with respect to weighted metrics with or without additional constraints. Amirfakhrian and Bagherian [6, 7] represented a parametric distance and used it to find the nearest approximation of a given fuzzy Number. Zhou, Yang, and Wang [25] represented fuzzy arithmetic on L-R fuzzy numbers and showed that the proposed model could be transferred to an equivalent crisp programming model by the operational law and then solved with the aid of some well-developed optimization software packages. Ghanbari et al. [14] used an effective approximate multiplication operation on L-R fuzzy numbers and their application.

In this paper, we present two inequalities in monotonic function. These inequalities show that under a condition, the nearest fuzzy number of type L-R to a given fuzzy number exists and is unique. For this purpose, we represent a constrained optimization problem and prove that it has a unique solution. The unique solution is obtained by solving a linear system. Since the trapezoidal fuzzy numbers are a kind of fuzzy number of type L-R, our method can find the nearest trapezoidal fuzzy number to an arbitrary fuzzy number. Here some examples are given to illustrate the main results. Due to the presented method, it is easy to obtain the nearest fuzzy numbers of type L-R to the solutions of a fuzzy linear system, fuzzy linear differential equations, or fuzzy linear integral equations, etc. For instance, finding the nearest fuzzy numbers of type L-R to fuzzy solutions of a linear system is explained. An example is given to approximate a fuzzy solution to a 2×2 fuzzy linear system. In Section 2, we recall some notations and basic definitions of fuzzy sets and fuzzy numbers. In Section 3, two basic inequalities and theorems are stated and proved, and the proposed method is described. Section 5 represents our method with an example to approximate fuzzy solutions of a fuzzy linear system with L-R fuzzy numbers.

2 Notations and basic definitions

The concept of real numbers is generalized to the concept of fuzzy numbers. Fuzzy numbers have been defined based on their membership functions as below.

Definition 1. [17] A fuzzy number u is a fuzzy set of the real line with the following conditions:
(i) u is normal.
(ii) The support of u is bounded.
(iii) The membership function of u is continuous and convex.

The set of all such fuzzy numbers is represented by E^1. Considering four real numbers $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$, the membership function of fuzzy numbers can be introduced as the following form:

$$u(x) = \begin{cases} 0 & if \quad x < \alpha_1, \\ u_1(x) & if \ \alpha_1 \leq x < \alpha_2, \\ 1 & if \quad \alpha_2 \leq x \leq \alpha_3, \\ u_2(x) & if \ \alpha_3 \leq x < \alpha_4, \\ 0 & if \quad \alpha_4 < x, \end{cases}$$

in which $u_1 : [\alpha_1, \alpha_2] \to [0, 1]$ is a nondecreasing function and $u_2 : [\alpha_3, \alpha_4] \to [0, 1]$ is a nonincreasing function. The fuzzy number as the following form, completely characterized by four real numbers $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$, is called an *L-R fuzzy number*:

$$u(x) = \begin{cases} 0 & if \quad x < a - \alpha, \\ L(\frac{a-x}{\alpha}) & if \ a - \alpha \leq x < a, \\ 1 & if \quad a \leq x \leq b, \\ R(\frac{x-b}{\beta}) & if \ b \leq x \leq b + \beta, \\ 0 & if \quad x > b + \beta, \end{cases}$$

in which, $\alpha_1 = a - \alpha$, $\alpha_2 = a$, $\alpha_3 = b$, $\alpha_4 = b + \beta$, and

$$L : [0, 1] \to [0, 1], \qquad R : [0, 1] \to [0, 1]$$

are continuous and decreasing shape functions such that $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. It is mostly denoted in short as $u = (a, b, \alpha, \beta)_{LR}$.

If $L(x) = R(x)$, then u is denoted by $u = (a, b, \alpha, \beta)_L$. Suppose that $u = (a_1, b_1, \alpha_1, \beta_1)_L$ and $v = (a_2, b_2, \alpha_2, \beta_2)_L$ are two fuzzy numbers of type L-L and that $k \in R$. Then the following statements hold:

1) $u + v = (a_1 + a_2, b_1 + b_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_L$.

2) $ku(x) = \begin{cases} (ka_1, kb_1, k\alpha_1, k\beta_1)_L & if\ k \geq 0, \\ (kb_1, ka_1, k\beta_1, k\alpha_1)_L & if\ k < 0, \end{cases}$

3) $u - v = (a_1 - b_2, a_2 - b_1, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_L$.

Definition 2. Following [15], we show an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$; $0 \leq r \leq 1$, that satisfy the following conditions:

1) $\underline{u}(r)$ is a bounded left-continuous nondecreasing over $[0, 1]$.

2) $\overline{u}(r)$ is a bounded left-continuous nonincreasing over $[0, 1]$.

3) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

If $u = (\underline{u}(r), \overline{u}(r)$ and $v = (\underline{v}(r), \overline{v}(r)$ are two fuzzy numbers, then the following conditions holds

1.
$$\underline{u+v} = \underline{u} + \underline{v}, \qquad \overline{u+v} = \overline{u} + \overline{v}, \tag{1}$$

2.
$$\underline{ku} = k\underline{u}, \qquad \overline{ku} = k\overline{u} \quad if \quad k \geq 0, \tag{2}$$

3.
$$\underline{ku} = k\overline{u}, \qquad \overline{ku} = k\underline{u} \quad if \quad k < 0. \tag{3}$$

The parametric form of fuzzy number of type $u = (a, b, \alpha, \beta)_{LR}$ is represented as
$$u = (a - \alpha L^{-1}(r), b + \beta R^{-1}(r)).$$

Definition 3. [18, 3] Let A and B be two arbitrary fuzzy numbers. A distance between A and B is denoted by $D(A, B)$ and defined as below:

$$D(A, B) = \{\int_0^1 (\underline{A}(r) - \underline{B}(r))^2 dr + \int_0^1 (\overline{A}(r) - \overline{B}(r))^2 dr\}^{\frac{1}{2}}. \tag{4}$$

Note that $D(A, B)$ is metric in E^1 and (E^1, D) is a complete space. It is obviously that
$$A = B \Leftrightarrow \underline{A}(r) = \overline{B}(r), \qquad r \in [0, 1]. \tag{5}$$

3 Approximation of general fuzzy number by a given L-R fuzzy number

Let A be an arbitrary fuzzy number. We try to approximate it by a known fuzzy number of type L-R, $N(A) = (a, b, \alpha, \beta)_{LR}$ such that $N(A)$ is the nearest to A with respect to the certain distance in (4). Hence an optimal problem is considered, and then an optimal solution is obtained with an easy method. For this purpose, we denote $N(A)$ as

$$N(A) = (a, a + \xi_1, \xi_2, \xi_3)_{LR},$$

in which $\xi_i \geq 0$ for $i = 1, 2, 3$, and $b = a + \xi_1, \alpha = \xi_2, \beta = \xi_3$. For finding the nearest fuzzy number of type L-R to the arbitrary fuzzy number A, we have to solve the optimization problem as below:

$$\begin{cases} \min D(A, N(A))(a, \xi_1, \xi_2, \xi_3) \\ s.t. \\ \xi_i \geq 0, \quad i = 1, 2, 3, \end{cases} \tag{6}$$

in which

$$\begin{aligned} D(A, N(A))(a, \xi_1, \xi_2, \xi_3) = & \int_0^1 (\underline{A}(r) - a + \xi_1 L^{-1}(r))^2 dr \\ & + \int_0^1 (\overline{A}(r) - (a + \xi_2 + \xi_3 R^{-1}(r)))^2 dr. \end{aligned}$$

To discuss the existence and uniqueness of the solutions to the optimization problem (6), we need to represent some important inequalities as below:

Lemma 1. Let f and g be two integrable functions from an interval $[a, b]$ to R. Then the following inequalities hold:

1. If g is a nondecreasing (nonincreasing) function and f is a nonincreasing (nondecreasing) function, then

$$\int_a^b f(x)g(x)dx \leq \frac{\int_a^b f(x)dx \int_a^b g(x)dx}{b - a}. \tag{7}$$

2. If g and f are nonincreasing (nondecreasing) functions over $[a, b]$, then

$$\int_a^b f(x)g(x)dx \geq \frac{\int_a^b f(x)dx \int_a^b g(x)dx}{b - a}. \tag{8}$$

3. Regarding to part 1, if f and g are not constant functions over $[a, b]$, then inequality (7) becomes

$$\int_a^b f(x)g(x)dx < \frac{\int_a^b f(x)dx \int_a^b g(x)dx}{b-a}. \tag{9}$$

4. Regarding to part 2, if f and g are not constant functions over $[a,b]$, then inequality (8) becomes

$$\int_a^b f(x)g(x)dx > \frac{\int_a^b f(x)dx \int_a^b g(x)dx}{b-a}. \tag{10}$$

Proof. Since f and g are monotonic functions over $[a,b]$, they are integrable functions. Without loss of generality, assume that g is a nonincreasing and that f is a nondecreasing function over the interval $[a,b]$. For each $x, y \in [a,b]$, if $x \le y$, then $f(x) \le f(y)$ and $g(y) \le g(x)$. Similarly, if $y \le x$, then $f(x) \ge f(y)$ and i $g(y) \ge g(x)$. Thus for each $x, y \in [a,b]$, we have $(f(x) - f(y))(g(y) - g(x)) \ge 0$. we conclude that

$$\frac{1}{2}\{\int_a^b \int_a^b (f(x) - f(y))(g(y) - g(x))dydx\} \ge 0. \tag{11}$$

Hence

$$\frac{1}{2}\int_a^b \int_a^b (f(x)g(y) - f(x)g(x) - f(y)g(y) + f(y)g(x))dydx \ge 0. \tag{12}$$

Since

$$\begin{aligned}
\int_a^b \int_a^b f(x)g(y)dydx &= \int_a^b \int_a^b f(y)g(x)dydx = \int_a^b f(x)dx \int_a^b g(y)dy \\
&= \int_a^b f(y)dy \int_a^b g(x)dx, \\
\int_a^b \int_a^b f(x)g(x)dydx &= \int_a^b \int_a^b f(y)g(y)dydx = (b-a)\int_a^b f(y)g(y)dy \\
&= (b-a)\int_a^b f(x)g(x)dx,
\end{aligned}$$

then (12) yields the following inequality:

$$\begin{aligned}
\frac{1}{2}\{ &\int_a^b f(x)dx \int_a^b g(y)dy - (b-a)\int_a^b f(x)g(x)dx \\
&- (b-a)\int_a^b f(y)g(y)dy + \int_a^b f(y)dy \int_a^b g(x)dx\} \ge 0.
\end{aligned}$$

Therefore

$$\int_a^b f(x)dx\int_a^b g(y)dy-(b-a)\int_a^b f(x)g(x)dx\geq 0.$$

We conclude that

$$\int_a^b f(x)dx\int_a^b g(x)dx\geq (b-a)\int_a^b f(x)g(x)dx.$$

It results that

$$\int_a^b f(x)g(x)dx\leq \frac{\int_a^b f(x)dx\int_a^b g(x)dx}{b-a}.$$

Similarity, inequality (8) can be proved. If g is a nonincreasing function and f is a nondecreasing function over the interval $[a,b]$ such that they are not constant functions over $[a,b]$, then inequality (11) becomes

$$\frac{1}{2}\{\int_a^b\int_a^b (f(x)-f(y))(g(y)-g(x))dydx\}>0.$$

Similar to what was stated in the proof of the part 1, we obtain (9). Inequality (10) is provided in the same manner. □

Corollary 1. Suppose that $g:[0,1]\to[0,1]$ is integrable and decreasing shape functions such that $g(0)=1$ and $g(1)=0$. Then the following properties hold:

1 If f is a nondecreasing continuous function over $[0,1]$, then $\int_0^1 f(x)g(x)dx\leq \int_0^1 f(x)dx\int_0^1 g(x)dx$.

2 If f is a nonincreasing continuous function over $[0,1]$, then $\int_0^1 f(x)g(x)dx\geq \int_0^1 f(x)dx\int_0^1 g(x)dx$.

3 Since g is not constant, $\int_0^1 (g(x))^2dx>(\int_0^1 (g(x))dx)^2$.

Theorem 1. Let $A=(\overline{A}(r),\underline{A}(r))$ be the parametric form of the given fuzzy number A. The following inequalities hold:

1 $\int_0^1 \underline{A}(r)L^{-1}(r)dr\leq \int_0^1 \underline{A}(r)dr\int_0^1 L^{-1}(r)dr$,

2 $\int_0^1 \overline{A}(r)R^{-1}(r)dr\geq \int_0^1 \overline{A}(r)dr\int_0^1 R^{-1}(r)dr$,

3 $\int_0^1 (R^{-1}(r))^2dr>(\int_0^1 R^{-1}(r)dr)^2$,

4 $\int_0^1 (L^{-1}(r))^2dr>(\int_0^1 L^{-1}(r)dr)^2$.

Proof. We know that the following functions satisfy the hypotheses of Lemma 1 and Corollary 1:

a: $L:[0,1]\to[0,1]$ is a decreasing function,

b: $R : [0,1] \to [0,1]$ is a decreasing function,

c: $\underline{A}(r)$ is a nondecreasing function over $[0,1]$,

d: $\overline{A}(r)$ is a nonincreasing function over $[0,1]$.

Then the proof of theorem is deduced. $\square$

For simplicity in computations, we take

$$\begin{aligned} p &= \int_0^1 L^{-1}(r)dr, \qquad p' = \int_0^1 (L^{-1}(r))^2 dr, \\ q &= \int_0^1 R^{-1}(r)dr, \qquad q' = \int_0^1 (R^{-1}(r))^2 dr. \end{aligned} \tag{13}$$

Also, we take

$$\begin{aligned} \gamma_1 &= \int_0^1 \underline{A}(r)dr, \qquad \gamma_2 = \int_0^1 L^{-1}(r)\underline{A}(r)dr, \\ \gamma_3 &= \int_0^1 \overline{A}(r)dr, \qquad \gamma_4 = \int_0^1 R^{-1}(r)\overline{A}(r)dr. \end{aligned} \tag{14}$$

The last theorem immediately gives the following results:

$$p' - p^2 > 0, \quad q' - q^2 > 0, \quad \gamma_2 - p\gamma_1 \le 0, \quad \gamma_4 - q\gamma_3 \ge 0. \tag{15}$$

Now we prove that the optimization problem (6) has a unique solution, and then the optimal solution is obtained by solving a linear system

Theorem 2. Let A be an arbitrary fuzzy number. Then the nearest fuzzy number of type L-R to A exists and is unique.

Proof. To find the optimal solution for (6), we take

$$\begin{aligned} &\frac{\partial}{\partial a} D(A, N(A))(a, \xi_1, \xi_2, \xi_3) = 0, \\ &\frac{\partial}{\partial \xi_1} D(A, N(A))(a, \xi_1, \xi_2, \xi_3) = 0, \\ &\frac{\partial}{\partial \xi_2} D(A, N(A))(a, \xi_1, \xi_2, \xi_3) = 0, \\ &\frac{\partial}{\partial \xi_3} D(A, N(A))(a, \xi_1, \xi_2, \xi_3) = 0. \end{aligned}$$

Hence we obtain the following system:

$$\begin{cases} \int_0^1 (\underline{A}(r) - a + \xi_1 L^{-1}(r))dr + \int_0^1 (\overline{A}(r) - (a + \xi_2 + \xi_3 R^{-1}(r)))dr = 0, \\ \int_0^1 (\underline{A}(r) - a + \xi_1 L^{-1}(r))L^{-1}(r)dr = 0, \\ \int_0^1 (\overline{A}(r) - (a + \xi_2 + \xi_3 R^{-1}(r)))dr = 0, \\ \int_0^1 (\overline{A}(r) - (a + \xi_2 + \xi_3 R^{-1}(r)))R^{-1}(r)dr = 0. \end{cases}$$

The following linear system is obtained:

$$\begin{cases} 2a - \xi_1 \int_0^1 L^{-1}(r)dr + \xi_2 + \xi_3 \int_0^1 R^{-1}(r)dr = \int_0^1 (\overline{A}(r) + \underline{A}(r))dr, \\ a \int_0^1 L^{-1}(r)dr - \xi_1 \int_0^1 (L^{-1}(r))^2 dr = \int_0^1 \underline{A}(r)L^{-1}(r)dr, \\ a + \xi_2 + \xi_3 \int_0^1 R^{-1}(r)dr = \int_0^1 \overline{A}(r)dr, \\ a \int_0^1 R^{-1}(r)dr + \xi_2 \int_0^1 R^{-1}(r)dr + \xi_3 \int_0^1 (R^{-1}(r))^2 dr = \int_0^1 \overline{A}(r)R^{-1}(r)dr. \end{cases} \tag{16}$$

referring to (13) and (14), the linear equations system (16) becomes

$$\begin{pmatrix} 2 & -p & 1 & q \\ p & -p' & 0 & 0 \\ 1 & 0 & 1 & q \\ q & 0 & q & q' \end{pmatrix} \begin{pmatrix} a \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \gamma_1 + \gamma_3 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}.$$

We perform elementary row operations on the coefficient matrix and the right-hand side vector, the following system is obtained:

$$\begin{pmatrix} 1 & -p & 0 & 0 \\ p & -p' & 0 & 0 \\ 1 & 0 & 1 & q \\ q & 0 & q & q' \end{pmatrix} \begin{pmatrix} a \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}. \tag{17}$$

The coefficient matrix is a block matrix. Based on the definition of the determinate of matrices, it is obvious that

$$det \begin{pmatrix} 1 & -p & 0 & 0 \\ p & -p' & 0 & 0 \\ 1 & 0 & 1 & q \\ q & 0 & q & q' \end{pmatrix} = det \begin{pmatrix} 1 & -p \\ p & -p' \end{pmatrix} det \begin{pmatrix} 1 & q \\ q & q' \end{pmatrix} = (p^2 - p')(q' - q^2) \neq 0.$$

Due to the results in (15), the linear equations system (17) has a unique solution. To ensure that this linear system gives the optimal solution of (6), we need to prove that $\xi_1, i = 1, 2, 3$, are nonnegative. Consider the first and second equations of the linear equations system (17). We have

$$\begin{cases} a - p\xi_1 = \int_0^1 \underline{A}(r))dr, \\ \\ pa - p'\xi_1 = \int_0^1 \underline{A}(r)L^{-1}(r)dr. \end{cases} \tag{18}$$

To delete the variable a, we multiply the first row by $-p$ and added the results to the second row. The results give ξ_1 as below:

$$\begin{aligned} \xi_1 &= \frac{p\int_0^1 \underline{A}(r)dr - \int_0^1 \underline{A}(r)L^{-1}(r)dr}{p' - p^2} \\ &= \frac{\int_0^1 L^{-1}(r)dr \int_0^1 \underline{A}(r)dr - \int_0^1 \underline{A}(r)L^{-1}(r)dr}{p' - p^2}, \end{aligned} \tag{19}$$

By subsisting to (19) and using (15), we have

$$\xi_1 = \frac{p\gamma_1 - \gamma_2}{p' - p^2} \geq 0.$$

From the third and forth equations of (17), we have

$$\begin{cases} a + \xi_2 + q\xi_3 = \int_0^1 \overline{A}(r)dr, \\ \\ qa + q\xi_2 + q'\xi_3 = \int_0^1 \underline{A}(r)R^{-1}(r)dr. \end{cases}$$

Performing elementary row operations on the linear equations system, a and ξ_2 are deleted, and a new linear equation is obtained as below:

$$\begin{aligned} \xi_3 &= \frac{\int_0^1 \overline{A}(r)R^{-1}(r)dr - q\int_0^1 \overline{A}(r)dr}{q' - q^2} \\ &= \frac{\int_0^1 \overline{A}(r)R^{-1}(r)dr - \int_0^1 R^{-1}(r)dr \int_0^1 \overline{A}(r)dr}{\int_0^1 (R^{-1}(r))^2 dr - (\int_0^1 R^{-1}(r)dr)^2}. \end{aligned}$$

It means that

$$\xi_3 = \frac{\gamma_4 - q\gamma_3}{q' - q^2}.$$

Due to (15), we conclude that $\xi_3 \geq 0$. Consider the first and third linear equations of (17), then perform the elementary operation for deleting a. We have

$$\xi_2 = \int_0^1 \overline{A}(r)dr - \int_0^1 \underline{A}(r)dr - q\xi_3 - p\xi_1.$$

By substituting ξ_1 and ξ_3, we have

$$\xi_2 = \gamma_3 - \gamma_1 - q\frac{\gamma_4 - q\gamma_3}{q' - q^2} - p\frac{p\gamma_1 - \gamma_2}{p' - p^2}.$$

Due to the assumption and results in (15), we conclude that $\xi_2 \geq 0$. Then $N(A) = (a, a+\xi_2, \xi_1, \xi_3)_{LR}$, is the nearest fuzzy number to A. If $\gamma_3 - \gamma_1 < q\frac{\gamma_4 - q\gamma_3}{q' - q^2} - p\frac{p\gamma_1 - \gamma_2}{p' - p^2}$, then $\xi_2 < 0$ and (a, ξ_1, ξ_2, ξ_3) is a global optimal solution to $\min D(A, N(A))(a, \xi_1, \xi_2, \xi_3)$, but $N(A) = (a, a+\xi_2, \xi_1, \xi_3)_{LR}$, is not a fuzzy number. In this case, we use the quadratic penalty method on (6), as

$$\min D(A, N(A))(a, \xi_1, \xi_2, \xi_3) + c_j \sum_{j=1}^{3} \max\{0, -\xi_j\}, \tag{20}$$

where $c_j \geq 0$. Due to [19], the optimal solution to (20) exists and is obtained as (a, ξ_1, ξ_2, ξ_3), in which $\xi_1, \xi_2, \xi_3 \geq 0$. Then the nearest fuzzy number of type L-R to A is $N(A) = (a, a+\xi_1, \xi_2, \xi_3)_{LR}$. □

Corollary 2. Let $n > 0$. Then $A = (-(1-r)^n, (1-r)^n)$ is a fuzzy number. Consider $L^{-1} = R^{-1} = 1 - r$.

1: For $0 < n \leq 1$, we have $\xi_2 \geq 0$. Therefore, (17) gives $N(A) = (a, a+\xi_1, \xi_2, \xi_3)_{LR}$, which is the nearest trapezoidal fuzzy number to A with respect to distance (4). For instance, if $A = (-(1-r)^{0.5}, (1-r)^{0.5})$, then $N(A) = (-0.26, 0.53, 0.53, 0.8)_T$.

2: For $1 < n$, we have $\xi_2 < 0$. Therefore, (17) does not give the nearest trapezoidal fuzzy number to A. For instance, if $A = (-(1-r)^2, (1-r)^2)$ we cannot use linear system (17). Using quadratic penalty method, we have $N(A) = (0, 0, 0.73, 0.73)_T$ that is a triangular fuzzy number.

4 Results and examples

In this section, we represent three examples to show the efficiency of our method. The first example uses the fuzzy number of type L-L, the second example uses the trapezoidal fuzzy number, and the third example uses the fuzzy number of type L-R. Finally, we represent a kind of fuzzy number that may not have the nearest fuzzy number of type L-R.

Example 1. Let us consider the fuzzy number

$$A = \begin{cases} 1 - \frac{(x-5)^2}{4}, & 3 \leq x \leq 7, \\ \\ 0 & otherwise. \end{cases}$$

Take $L(x) = R(x) = \sqrt{1-x}$. Then $L^{-1}(r) = R^{-1}(r) = 1 - r^2$. Obviously the parametric form of A is

$$A = (5 - 2\sqrt{1-r}, 5 + 2\sqrt{1-r}).$$

Considering (17), to obtain the entries of the coefficient matrix, we take

$$p = q = \int_0^1 L^{-1}(r)dr = \int_0^1 (1-r^2)dr = 0.66,$$
$$p' = q' = \int_0^1 (L^{-1}(r))^2 dr = \int_0^1 (1-r^2)^2 dr = 0.53,$$

and entries of the right-hand side of the linear system (17) are obtained as below:

$$\gamma_1 = \int_0^1 \underline{A}(r)dr = \int_0^1 (5 - 2\sqrt{1-r})dr = 3.67,$$
$$\gamma_2 = \int_0^1 \underline{A}(r)L^{-1}(r)dr = \int_0^1 (5 - 2\sqrt{1-r})(1-r^2)dr = 2.30,$$
$$\gamma_3 = \int_0^1 \overline{A}(r)dr = \int_0^1 (5 + 2\sqrt{1-r})dr = 6.33,$$
$$\gamma_4 = \int_0^1 \overline{A}(r)R^{-1}(r)dr = \int_0^1 (5 + 2\sqrt{1-r})(1-r^2)dr = 4.36.$$

Hence the linear system is obtained as follows:

$$\begin{pmatrix} 1 & -0.66 & 0 & 0 \\ 0.66 & -0.53 & 0 & 0 \\ 1 & 0 & 1 & 0.66 \\ 0.66 & 0 & 0.66 & 0.53 \end{pmatrix} \begin{pmatrix} a \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 3.67 \\ 2.30 \\ 6.33 \\ 4.36 \end{pmatrix}.$$

Solving this linear system, we have $a = 4.54$, $\xi_1 = 1.32$, $\xi_2 = 0.72$, $\xi_3 = 1.60$. Therefore the nearest fuzzy number of type L-R to A is

$$N(A) = (3.18, 4.50, 0.72, 1.60)_L.$$

Figure 1 compares the graph of A and $N(A)$.

In particular, if $L(r) = R(r) = 1 - r$, then $p = q = \int_0^1 (1-r)dr = \frac{1}{2}$ and $p' = q' = \int_0^1 (1-r)^2 dr = \frac{1}{3}$. The linear equations system (17) becomes the following system and gives the nearest trapezoidal fuzzy number to the given fuzzy number $A = (\underline{A}(r), \overline{A}(r))$:

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 0 & 0 \\ 1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \int_0^1 \underline{A}(r)dr \\ \int_0^1 (1-r)\underline{A}(r)dr \\ \int_0^1 \overline{A}(r)dr \\ \int_0^1 (1-r)\overline{A}(r)dr \end{pmatrix}. \tag{21}$$

Example 2. Let us consider the fuzzy number

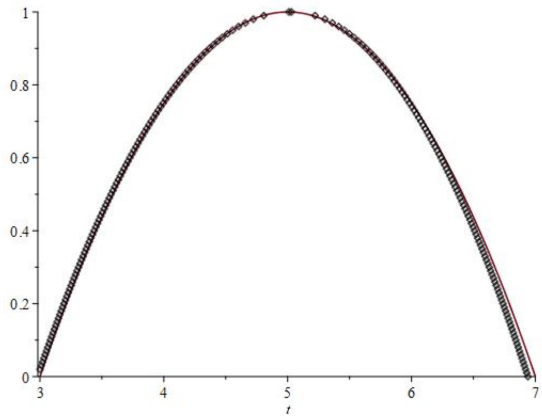

Figure 1: Membership functions of A (line) and $N(A)$ (point), where $N(A)$ is the nearest fuzzy number of type L-R with $L^{-1}(r) = R^{-1}(r) = 1 - r^2$ to A

$$A = \begin{cases} 1 - \frac{(x-5)^2}{4}, & 3 \leq x \leq 7, \\ \\ 0 & otherwise. \end{cases}$$

The parametric form of A is $A = (5 - 2\sqrt{1-r}, 5 + 2\sqrt{1-r})$. For finding the nearest trapezoidal fuzzy number to fuzzy number A, it is enough to take $L(x) = R(x) = 1 - x$ (($L^{-1}(r) = R^{-1}(r) = 1 - r$) and determine all entries of the right-hand side of linear system (21). Then

$$\begin{aligned} \gamma_1 &= \int_0^1 \underline{A}(r)dr = \int_0^1 (5 - 2\sqrt{1-r})dr = 3.67, \\ \gamma_2 &= \int_0^1 \underline{A}(r)L^{-1}(r)dr = \int_0^1 (5 - 2\sqrt{1-r})(1-r)dr = 1.70, \\ \gamma_3 &= \int_0^1 \overline{A}(r)dr = \int_0^1 (5 + 2\sqrt{1-r})dr = 6.33, \\ \gamma_4 &= \int_0^1 \overline{A}(r)R^{-1}(r)dr = \int_0^1 (5 + 2\sqrt{1-r})(1-r)dr = 3.30. \end{aligned}$$

Thus the linear system is obtained as

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 0 & 0 \\ 1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 3.67 \\ 1.70 \\ 6.33 \\ 3.30 \end{pmatrix}.$$

By solving this linear system, we have $a = 4.47$, $\xi_1 = 1.60$, $\xi_2 = 1.05$, $\xi_3 = 1.62$. Therefore the nearest trapezoidal fuzzy number to A is $N(A) = (4.47, 4.47 + \xi_1, \xi_2, \xi_3)_T$; that is,

$$N(A) = (4.47, 5.52, 1.60, 1.62)_T.$$

Figure 2 compares the graph of A and $N(A)$.

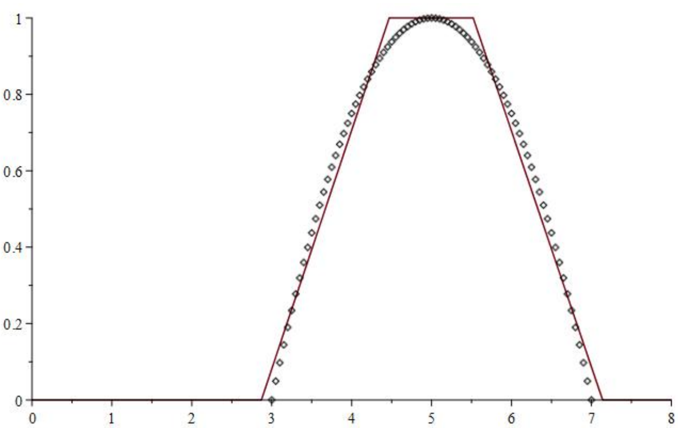

Figure 2: Membership functions of A (point) and $N(A)$ (line), where $N(A)$ is the nearest trapezoidal fuzzy number to A

Example 3. Consider the fuzzy number

$$A = \begin{cases} \sin(\frac{\pi x}{2}), & 0 \leq x \leq 1, \\ \frac{3-x}{2}, & 1 \leq x \leq 3, \\ 0, & otherwise. \end{cases}$$

Let $L(x) = \sqrt{1-x}$ and $R(x) = 1 - x$. We want to obtain the nearest fuzzy number of type L-R to A. The parametric form of A is $A = (\frac{2}{\pi}\arcsin(r), 3 - 2r)$. One can easily show that $L^{-1}(r) = 1 - r^2$ and $R^{-1}(r) = 1 - r$. referring to (14), we have $\gamma_1 = \int_0^1 \underline{A}(r)dr = \int_0^1 (\frac{2}{\pi} arcsin(r))dr = 0.363$, $\gamma_2 = \int_0^1 \underline{A}(r)L^{-1}(r)dr = 0.172$, $\gamma_3 = \int_0^1 \overline{A}(r)dr = 2$ and $\gamma_4 = \int_0^1 \overline{A}(r)R^{-1}(r)dr = 1.167$. Hence the linear system is obtained as

$$\begin{pmatrix} 1 & -0.667 & 0 & 0 \\ 0.667 & -0.533 & 0 & 0 \\ 1 & 0 & 1 & 0.500 \\ 0.500 & 0 & 0.500 & 0.333 \end{pmatrix} \begin{pmatrix} a \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0.363 \\ 0.172 \\ 2.000 \\ 1.167 \end{pmatrix}.$$

Solving this linear system, we have $a = 0.918$, $\xi_1 = 0.832$, $\xi_2 = 0.039$, $\xi_3 = 2.084$. Therefore the nearest fuzzy number of type L-R to A is $N(A) = (0.918, 0.918 + \xi_2, \xi_1, \xi_2)_{LR}$; that is,

$$N(A) = (0.918, 1.147, 0.832, 2.083)_{LR}.$$

Figure 3 compares the graph of A and $N(A)$.

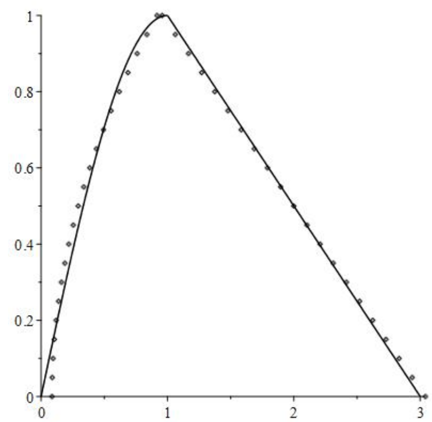

Figure 3: Membership functions of A (line) and $N(A)$ (point), where $N(A)$ is the nearest fuzzy number of type L-R to A with $L^{-1}(r) = 1 - r^2$ and $R^{-1}(r) = 1 - r$

5 Fuzzy linear system

In this section, we focus on the fuzzy linear system as $AX = b$, in which the entries of the right-hand side vector, b, are fuzzy numbers, and entries of the coefficient matrix, A, are real numbers. Using the proposed method, we approximate the fuzzy solutions by L-R fuzzy numbers.

Definition 4. The $n \times n$ linear system

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2, \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n, \end{cases} \tag{22}$$

is called a fuzzy linear system if $b_i = (\underline{b_i}(r), \overline{b_i}(r))$, for $i = 1, 2, \dots, n$, are fuzzy numbers such that $\xi_{b_j,2} \geq 0$ and $a_{i,j} \in \Re$.

Let $x_j = (\underline{x_j}, \overline{x_j})$ be the fuzzy solutions to (22). Due to (5), (1), (2), and (3), for each i, we have

$$\underline{a_{i,1}x_1} + \underline{a_{i,2}x_2} + \cdots + \underline{a_{i,n}x_n} = \underline{b_i},$$

$$\overline{a_{i,1}x_1} + \overline{a_{i,2}x_2} + \cdots + \overline{a_{i,n}x_n} = \overline{b_i},$$

and then

$$\begin{cases} \sum_{a_{i,j}\geq 0} a_{i,j}\underline{x_j}(r) + \sum_{a_{i,j}\leq 0} a_{i,j}\overline{x_j}(r) = \underline{b_i}(r), \quad i = 1, 2, \dots, n, \\ \sum_{a_{i,j}\geq 0} a_{i,j}\overline{x_j}(r) + \sum_{a_{i,j}\leq 0} a_{i,j}\underline{x_j}(r) = \overline{b_i}(r), \quad i = 1, 2, \dots, n. \end{cases} \tag{23}$$

Integrating with respect to r on the interval $[0, 1]$, we have

$$\begin{cases} \sum_{a_{i,j}\geq 0} a_{i,j} \int_0^1 \underline{x_j}(r)dr + \sum_{a_{i,j}\leq 0} a_{i,j} \int_0^1 \overline{x_j}(r)dr = \int_0^1 \underline{b_i}(r)dr, \\ \qquad\qquad\qquad\qquad i = 1, 2, \dots, n, \\ \sum_{a_{i,j}\geq 0} a_{i,j} \int_0^1 \overline{x_j}(r)dr + \sum_{a_{i,j}\leq 0} a_{i,j} \int_0^1 \underline{x_j}(r)dr = \int_0^1 \overline{b_i}(r)dr, \\ \qquad\qquad\qquad\qquad i = 1, 2, \dots, n. \end{cases} \tag{24}$$

Multiplying both sides of the equations (23) by $L^{-1}(r)$ and integrating, we obtain the following equations:

$$\begin{cases} \sum_{a_{i,j}\geq 0} a_{i,j} \int_0^1 L^{-1}(r)\underline{x_j}(r)dr + \sum_{a_{i,j}\leq 0} a_{i,j} \int_0^1 L^{-1}(r)\overline{x_j}(r)dr \\ \quad = \int_0^1 L^{-1}(r)\underline{b_i}(r)dr, \qquad i = 1, 2, \dots, n, \\ \sum_{a_{i,j}\geq 0} a_{i,j} \int_0^1 L^{-1}(r)\overline{x_j}(r)dr + \sum_{a_{i,j}\leq 0} a_{i,j} \int_0^1 L^{-1}(r)\underline{x_j}(r)dr \\ \quad = \int_0^1 L^{-1}(r)\overline{b_i}(r)dr, \qquad i = 1, 2, \dots, n. \end{cases} \tag{25}$$

Multiplying both sides of the equations (23) by $R^{-1}(r)$ and integrating, we obtain the following equations:

$$\begin{cases} \sum_{a_{i,j}\geq 0} a_{i,j} \int_0^1 R^{-1}(r)\underline{x_j}(r)dr + \sum_{a_{i,j}\leq 0} a_{i,j} \int_0^1 R^{-1}(r)\overline{x_j}(r)dr \\ \quad = \int_0^1 R^{-1}(r)\underline{b_i}(r)dr, \qquad i = 1, 2, \dots, n, \\ \sum_{a_{i,j}\geq 0} a_{i,j} \int_0^1 R^{-1}(r)\overline{x_j}(r)dr + \sum_{a_{i,j}\leq 0} a_{i,j} \int_0^1 R^{-1}(r)\underline{x_j}(r)dr \\ \quad = \int_0^1 R^{-1}(r)\overline{b_i}(r)dr, \qquad i = 1, 2, \dots, n. \end{cases} \tag{26}$$

We denote $\int_0^1 \underline{x_j}(r)dr$, $\int_0^1 L^{-1}(r)\underline{x_j}(r)dr$, $\int_0^1 \overline{x_j}(r)dr$,$\int_0^1 R^{-1}(r)\overline{x_j}(r)dr$, $\int_0^1 R^{-1}(r)\underline{x_j}(r)dr$, and $\int_0^1 L^{-1}(r)\overline{x_j}(r)dr$, respectively, by $\gamma_{j,1}$, $\gamma_{j,2}$, $\gamma_{j,3}$, $\gamma_{j,4}$, $\gamma_{j,5}$, and $\gamma_{j,6}$. Thus we can rewrite (24), (25), and (26), respectively, as below:

$$\begin{cases} \sum_{a_{i,j}\geq 0} a_{i,j}\gamma_{j,1} + \sum_{a_{i,j}\leq 0} a_{i,j}\gamma_{j,3} = \int_0^1 \underline{b_i}(r)dr, \quad i=1,2,\dots,n, \\ \sum_{a_{i,j}\geq 0} a_{i,j}\gamma_{j,3} + \sum_{a_{i,j}\leq 0} a_{i,j}\gamma_{j,1} = \int_0^1 \overline{b_i}(r)dr, \quad i=1,2,\dots,n, \end{cases} \tag{27}$$

$$\begin{cases} \sum_{a_{i,j}\geq 0} a_{i,j}\gamma_{j,2} + \sum_{a_{i,j}\leq 0} a_{i,j}\gamma_{j,6} = \int_0^1 L^{-1}(r)\underline{b_i}(r)dr, \quad i=1,2,\dots,n, \\ \\ \sum_{a_{i,j}\geq 0} a_{i,j}\gamma_{j,6} + \sum_{a_{i,j}\leq 0} a_{i,j}\gamma_{j,2} = \int_0^1 L^{-1}(r)\overline{b_i}(r)dr, \quad i=1,2,\dots,n, \end{cases} \tag{28}$$

and

$$\begin{cases} \sum_{a_{i,j}\geq 0} a_{i,j}\gamma_{j,5} + \sum_{a_{i,j}\leq 0} a_{i,j}\gamma_{j,4} = \int_0^1 R^{-1}(r)\underline{b_i}(r)dr, \quad i=1,2,\dots,n, \\ \\ \sum_{a_{i,j}\geq 0} a_{i,j}\gamma_{j,4} + \sum_{a_{i,j}\leq 0} a_{i,j}\gamma_{j,5} = \int_0^1 R^{-1}(r)\overline{b_i}(r)dr, \quad i=1,2,\dots,n. \end{cases} \tag{29}$$

The coefficient matrices of the linear equations (27), (28), and (29) are the same. Then it is enough to describe only one of them. Here we describe the linear system (27). Subtracting and adding the first n equations by the second n equations, we obtain two $n\times n$ linear systems as below [1]:

$$I)\begin{cases} a_{1,1}(\gamma_{1,3}+\gamma_{1,1}) + a_{1,2}(\gamma_{2,3}+\gamma_{2,1}) + \cdots + a_{1,n}(\gamma_{n,3}+\gamma_{n,1}) \\ \quad = \int_0^1 \overline{b_1}(r)dr + \int_0^1 \underline{b_1}(r)dr, \\ a_{2,1}(\gamma_{1,3}+\gamma_{1,1}) + a_{2,2}(\gamma_{2,3}+\gamma_{2,1}) + \cdots + a_{2,n}(\gamma_{n,3}+\gamma_{n,1}) \\ \quad = \int_0^1 \overline{b_2}(r)dr + \int_0^1 \underline{b_2}(r)dr, \\ \vdots \\ a_{n,1}(\gamma_{1,3}+\gamma_{1,1}) + a_{n,2}(\gamma_{2,3}+\gamma_{2,1}) + \cdots + a_{n,n}(\gamma_{n,3}+\gamma_{n,1}) \\ \quad = \int_0^1 \overline{b_n}(r)dr + \int_0^1 \underline{b_n}(r)dr, \end{cases} \tag{30}$$

$$II)\begin{cases} a^+_{1,1}(\gamma_{1,3}-\gamma_{1,1}) + a^+_{1,2}(\gamma_{2,3}-\gamma_{2,1}) + \cdots + a^+_{1,n}(\gamma_{n,3}-\gamma_{n,1}) \\ \quad = \int_0^1 \overline{b_1}(r)dr - \int_0^1 \underline{b_1}(r)dr, \\ a^+_{2,1}(\gamma_{1,3}-\gamma_{1,1}) + a^+_{2,2}(\gamma_{2,3}-\gamma_{2,1} + \cdots + a^+_{2,n}(\gamma_{n,3}-\gamma_{n,1}) \\ \quad = \int_0^1 \overline{b_2}(r)dr - \int_0^1 \underline{b_2}(r)dr, \\ \vdots \\ a^+_{n,1}(\gamma_{1,3}-\gamma_{1,1}) + a^+_{n,2}(\gamma_{2,3}-\gamma_{2,1}) + \cdots + a^+_{n,n}(\gamma_{n,3}-\gamma_{n,1}) \\ \quad = \int_0^1 \overline{b_n}(r)dr - \int_0^1 \underline{b_n}(r)dr, \end{cases} \tag{31}$$

in which $a^+_{i,j} = |a_{i,j}|$. Taking $\gamma^c_j = \gamma_{j,2} + \gamma_{j,1}$ and $\gamma^d_j = \gamma_{j,2} - \gamma_{j,1}$ and solving the linear equations (30) and (31), γ^c_j and γ^d_j are obtained for $j = 1, 2, \ldots, n$, provided that, coefficient matrices are nonsingular. After that, we conclude that $\gamma_{j,1} = 0.5(\gamma^c_j - \gamma^d_j)$ and $\gamma_{j,2} = 0.5(\gamma^c_j + \gamma^d_j)$.

Solving the linear equations (27), (28), and (29), the nearest fuzzy number of type L-R to each x_j are obtained as below:

$$N(x_j) = (a_j, a_j + \xi_{j,2}, \xi_{j,1}, \xi_{j,3})_{LR}, \quad j = 1, 2, \ldots, n,$$

in which

$$\xi_{j,1} = \frac{p\gamma_{j,1} - \gamma_{j,3}}{p' - p^2} \geq 0.$$

$$\xi_{j,2} = \gamma_{j,3} - \gamma_{j.1} - q\frac{\gamma_{j,4} - q\gamma_{j,3}}{q' - q^2} - p\frac{p\gamma_{j,1} - \gamma_{j,2}}{p' - p^2} \geq 0.$$

$$\xi_{j,3} = \frac{p\gamma_{j,1} - \gamma_{j,3}}{p' - p^2} \geq 0.$$

Due to (18), a_j is obtained as

$$a_j = p\xi_{j,1} + \gamma_{j,1}.$$

Example 4. Let us consider the 2×2 fuzzy linear system

$$\begin{cases} 2x_1 + x_2 = (r^2, 2 - r), \\ x_1 - x_2 = (5 - 2\sqrt{1 - r}, 5 + 2\sqrt{1 - r}). \end{cases} \tag{32}$$

For simplicity, suppose that $L^{-1}(r) = R^{-1}(r) = 1 - r$ is linear functions. Then the nearest fuzzy number of type L-R is trapezoidal fuzzy numbers. The exact solutions of system (32) are

$$x_1 = (\frac{5r^2 + r + 18}{12} + \sqrt{1 - r}, \frac{-r^2 - 5r + 30}{12} - \sqrt{1 - r}),$$

$$x_2 = (\frac{-r^2 - 5r - 30}{12} - 3\sqrt{1 - r}, \frac{5r^2 + r - 42}{12} + 3\sqrt{1 - r}).$$

One can see that $\underline{x_2}(r) \leq \overline{x_2}(r)$, but $\underline{x_1}(r) \geq \overline{x_1}(r)$. Then x_2 is a fuzzy solution and x_1 is not a fuzzy number. Then it is not a fuzzy solution to (32). Now represent x_1 as $(\frac{-r^2-5r+30}{12} - \sqrt{1 - r}, \frac{5r^2+r+18}{12} + \sqrt{1 - r})$ that is called weak fuzzy solution. We try to find the nearest trapezoidal fuzzy number to fuzzy solutions x_1 and x_2. Also, since $L^{-1}(r) = R^{-1}(r)$, we need to solve only two linear systems (27) and (28). Solving (27) and (28), the nearest trapezoidal fuzzy numbers to x_1 and x_2 are obtained, respectively, as below:

$$N(x_1) = (1.75, 2.180, 0.306, 0.318)_T,$$

$$N(x_2) = (-3.74, -2.313, 1.975, 1.987)_T.$$

Figures 4 and 5 represent the graphs of the exact fuzzy solutions x_1 and x_2 with their nearest trapezoidal fuzzy numbers.

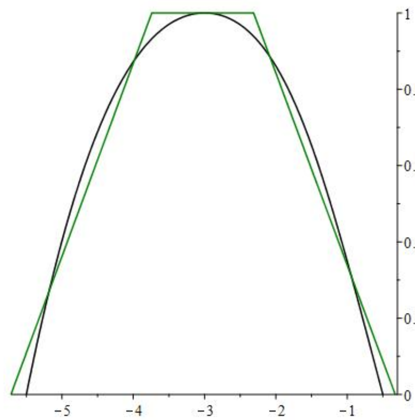

Figure 4: Membership functions of x_1 (curve) and $N(x_1)$ (trapezoidal), where $N(x_1)$ is the nearest fuzzy number of type L-R to x_1 with $L^{-1}(r) = R^{-1}(r) = 1 - r$

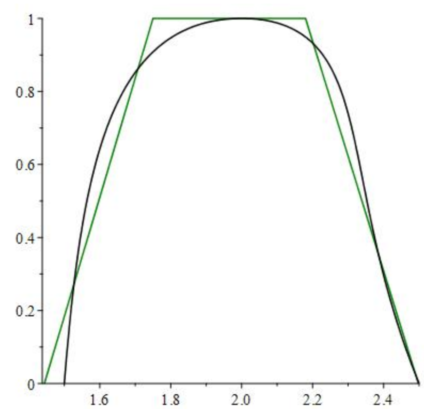

Figure 5: Membership functions of x_2 (curve) and $N(x_2)$ (trapezoidal), where $N(x_2)$ is the nearest trapezoidal fuzzy number to x_2

6 Conclusion

In this study, we focused on the approximate given general fuzzy numbers by fuzzy numbers of type L-R. Fuzzy numbers of type L-R, in particular, trapezoidal fuzzy numbers, play an essential role in the fuzzy environment.

We use a property of linear equations system to obtain the nearest trapezoidal fuzzy number to a given general fuzzy number with respect to the distance formula $D(\cdot,\cdot)$. The presented method is attractive, simple, and can be applied in any way.

References

[1] Abbasabandy, S. and Alavi, M. *A method for solving fuzzy linear systems*, Iran. J. Fuzzy Syst. 2(2) (2005) 37–43.

[2] Abbasbandy, S. and Amirfakhrian, M. *The nearest trapezoidal form of a generalized left right fuzzy number,* Internat. J. Approx. Reason. 43(2) (2006) 166–178.

[3] Abbasbandy, S.and Asady, B. *A note on ''A new approach for defuzzification''[Fuzzy Sets and Systems 111 (2000), no. 3, 351–356; 1 748 552] by M. Ma, A. Kandel and M. Friedman,* Fuzzy Sets Syst. 128 (2002) 131–132.

[4] Abbasbandy, S. and Asady, B. *The nearest trapezoidal fuzzy number to a fuzzy quantity*, Appl. Math. Comput. 156 (2004) 381–386.

[5] Abbasbandy, S. and Hajjari, T. *Weighted trapezoidal approximation-preserving cores of a fuzzy number,* Comput. Math. Appl. 59(9) (2010) 3066–3077.

[6] Amirfakhrian, M. A New Parametric Distance and Using it to Find The Nearest Approximation of A Fuzzy Number, Conference: 11th Iranian Conference of Fuzzy Systems, 2011.

[7] M. Amirfakhrian and S. Bagherian, *The nearest approximation of a fuzzy number by a new parametric distance,* Conference: 40th Annual Iranian Mathematics Conference, 2009.

[8] Ban, A.I and Coroianu, L. *Nearest interval, triangular and trapezoidal approximation of a fuzzy number preserving ambiguity,* Internat. J. Approx. Reason. 53 (5) (2012) 805–836.

[9] Coroianu, L. *Trapezoidal approximations of fuzzy numbers using quadratic programs,*Fuzzy Sets Syst. 417 (2021) 71–92.

[10] Coroianu, L., Gagolewski, M. and Grzegorzewski, P. *Piecewise linear approximation of fuzzy numbers: algorithms, arithmetic operations and stability of characteristics,* Soft Comput. 23 (2019) 9491–9505.

[11] Dubois, D. and Prade, H. *Fuzzy number: an overview,* Fuzzy number: an overview,

[12] T. Hajjari, Approximation of fuzzy numbers via 0.5-Level and Mean core, Int. J. Appl. Math. Stat. (22) (2011) 69–78.

[13] T. Hajjari and H. Mahzoon, *Approximation of a fuzzy number by two main characteristics,* Mathematical Inverse Problems, 1(1) (2014) 101–111.

[14] Ghanbari, M., Allahviranloo, T., Nuraei, R. and Pedrycz, W. *A new effective approximate multiplication operation on LR fuzzy numbers and its application,* Soft Comput. 26(9)(2022) 4103–4113.

[15] Giachetti, R.E. and Young, R.E. *A parametric representation of fuzzy numbers and their arithmetic operators,* Fuzzy Sets Syst. 91 (2) (1997) 185–202.

[16] Guerra, M.L. and Stefanini, L.*Approximate fuzzy arithmetic operations using monotonic interpolations,* Fuzzy Sets Syst. 150 (2005) 5–33.

[17] Klir, G.J., St. Clair, U. and Yuan, B., *Fuzzy set theory: foundations and applications,* Prentice-Hall Inc. 1997.

[18] Ma, M., Kandel, A. and Friedman, M. *Correction to a new approach for the difizification,* "[Fuzzy Sets and Systems 111 (2000) 351-356]." Fuzzy Sets Syst. 128 (1) (2002) 133–134.

[19] Zheng, Q. and Zhang, L. *Global minimization of constrained problems with discontinuous penalty functions, Global optimization, control, and games, III,* Comput. Math. Appl. 37 (1999), no. 4-5, 41–58.

[20] Voxman, W. *Canonical representation of discrete fuzzy numbers,* Fuzzy Set Syst. 118 (3) (2001) 557–466.

[21] Yeh, C.-T. *Existence of interval, triangular, and trapezoidal approximations of fuzzy numbers under a general condition,* Fuzzy Sets Syst. 310 (2017) 1–13.

[22] Zadeh, L.A. *The concept of a linguistic variable and its application to approximate reasoning-I*, Inf. Sci. 8 (1975) 199–249.

[23] Zadeh L.A. *The concept of a linguistic variable and its application to approximate reasoning-II,* Inf. Sci. 8 (1975) , 301–357.

[24] Zadeh L.A. *The concept of a linguistic variable and its application to approximate reasoning-III*, Inf. Sci. 9 (1975) 43–80.

[25] Zhou, J., Yang, F. and Wang, K. *Fuzzy arithmetic on LR fuzzy numbers with applications to fuzzy programming,* J. Intell. Fuzzy Syst. 30 (2016) 71–87.

How to cite this article

Alavi, S.M., Nearest fuzzy number of type L-R to an arbitrary fuzzy number with applications to fuzzy linear system. *Iran. j. numer. anal. optim.*, 2023; 13(3): 532-552. https://doi.org/10.22067/ijnao.2023.79051.1187