



Differential transform method for conformable fractional partial differential equations

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Abstract

We expand a new generalization of the two-dimensional differential transform method. The new generalization is based on the two-dimensional differential transform method, fractional power series expansions, and conformable fractional derivative. We use the new method for solving a nonlinear conformable fractional partial differential equation and a system of conformable fractional partial differential equation. Finally, numerical examples are presented to illustrate the preciseness and effectiveness of the new technique.

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Keywords: Conformable fractional derivative; Differential transform method; two-dimensional differential transform method.

1 Introduction

Fractional partial differential equations and systems of partial differential equations arise in many areas of mathematics, engineering, and the physical science, which make it very important to find efficient methods for solving the partial fractional differential equations.

The differential transform method is an analytical method for solving differential equations. The concept of differential transform method was first introduced by Zhou in solving linear and nonlinear initial value problems in

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electrical circuit analysis [16]. The method provides an analytical solution in the form of a polynomial.

Fractional calculus [9, 13] is a field of Mathematics in which derivatives and integrals of arbitrary orders are discussed. Studies show many physical systems compatible with fraction derivatives theory. Therefore, there have been significant interest in this subject trying to introduce a definition for fractional derivative in most of which integral forms have been widely used. The two most commonly used definitions are Riemann–Liouville, and Caputo.

Recently, a new kind of fractional derivative called a conformable fractional derivative was proposed by Khalil et al. [8]. This new definition satisfies formulas of derivative of product and quotient of two functions. In addition to the conformable fractional derivative definition, the conformable fractional integral definition, Rolle theorem, and mean value theorem for the conformable fractional differentiable functions were given. Since then, many researchers have made a huge number of scientific articles on the topic. As an instance, Abdeljawad [1] provided fractional versions of the chain rule, exponential functions, Gronwall’s inequality, integration by parts, Taylor power series expansions, and Laplace transforms. Cenesiz and Kurt [4, 10] found the solutions of time and space-fractional heat differential equations by the conformable fractional derivative and the approximate analytical solution of the time conformable fractional Burger’s equation via homotopy analysis method. Acan et al. [2, 3] introduced a new type reduced differential transform method called the conformable fractional reduced differential transform method and conformable variational iteration method based on the new defined fractional derivative. Masalmeh [11] applied the series solution for a case of conformable fractional Riccati differential equation with variable coefficients and Ilie et al. [11] introduced a general solution to conformable fractional differential equations. Therefore, it is clearly deduced that further studies and explanations can be made on this newly introduced fractional derivative.

2 Basic definition

Definition 1. [8] Given a function $f : [a, \infty] \rightarrow R$. Then the conformable fractional derivative of f order α is defined by

$$({}_a T_t^\alpha f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon(t - a)^{1-\alpha}) - f(t)}{\varepsilon}, \quad (1)$$

for all $t > 0$ and $\alpha \in (0, 1]$.

Definition 2. [1] Given a function $f : [a, \infty] \rightarrow R$. Let $n < \alpha \leq n + 1$ and $\beta = \alpha - n$. Then the conformable (left) fractional of f order α , where exists, is defined by

$$(T_\alpha^a f)(t) = (T_\beta^a f^{(n)})(t).$$

Definition 3. [15] Assume that $f(x)$ is an infinite-differentiable function for some $\alpha \in (0, 1]$. The conformable fractional differential transform of $f(x)$ is defined as

$$F_\alpha(k) = \frac{1}{\alpha^k k!} \left[(T_\alpha^{x_0} f)^{(k)}(x) \right]_{x=x_0}. \tag{2}$$

Definition 4. The inverse conformable fractional differential transform of $F_\alpha(k)$ is defined as [15]

$$f(x) = \sum_{k=0}^{\infty} F_\alpha(k)(x - x_0)^{k\alpha}. \tag{3}$$

3 Two-dimensional conformable fractional differential transform method

Consider a function of two variables $u(x, t)$, and suppose that it can be represented as a product of two single variable functions, that is, $u(x, t) = f(x).g(t)$; see [12]. On the basis of the properties of one-dimensional conformable fractional differential transform [15], the function $u(x, t)$ can be represented as

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{\infty} F_\alpha(k)(x - x_0)^{k\alpha} \times \sum_{h=0}^{\infty} G_\beta(h)(t - t_0)^{h\beta} \\ &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(k, h)(x - x_0)^{k\alpha}(t - t_0)^{h\beta}, \end{aligned} \tag{4}$$

where $0 < \alpha, \beta \leq 1$, $U_{\alpha,\beta}(k, h) = F_\alpha(k)G_\beta(h)$ is called the spectrum of $u(x, t)$.

The conformable fractional differential transform of a function $u(x, t)$ is as follows:

$$U_{\alpha,\beta}(k, h) = \frac{1}{\alpha^k k! \beta^h h!} \left[({}_{x_0}T_x^\alpha)^{(k)} ({}_{t_0}T_t^\beta)^{(h)} u(x, t) \right]_{(x_0, t_0)}. \tag{5}$$

Based on (4) and (5), we have the following results.

Theorem 1. Suppose that $U_{\alpha,\beta}(k, h)$, $V_{\alpha,\beta}(k, h)$, and $W_{\alpha,\beta}(k, h)$ are the conformable fractional differential transformations of the functions $u(x, t)$, $v(x, t)$, and $w(x, t)$, respectively. Then

1. if $u(x, t) = v(x, t) \pm w(x, t)$, then $U_{\alpha, \beta}(k, h) = V_{\alpha, \beta}(k, h) \pm W_{\alpha, \beta}(k, h)$;
2. if $u(x, t) = C.v(x, t)$, $C \in R$, then $U_{\alpha, \beta}(k, h) = C.V_{\alpha, \beta}(k, h)$;
3. if $u(x, t) = v(x, t).w(x, t)$, then

$$U_{\alpha, \beta}(k, h) = \sum_{r=0}^k \sum_{s=0}^h V_{\alpha, \beta}(r, h-s).W_{\alpha, \beta}(k-r, s);$$

4. if $u(x, t) = v(x, t).w(x, t).z(x, t)$, then

$$U_{\alpha, \beta}(k, h) = \sum_{r=0}^k \sum_{t=0}^{k-r} \sum_{s=0}^h \sum_{p=0}^{h-s} V_{\alpha, \beta}(r, h-s-p).W_{\alpha, \beta}(t, s)Z_{\alpha, \beta}(k-r-t, p);$$

5. if $u(x, t) = (x - x_0)^m(t - t_0)^n$, then $U_{\alpha, \beta}(k, h) = \delta(k - \frac{m}{\alpha})\delta(h - \frac{n}{\beta})$;

6. if $u(x, t) = e^{\lambda(\frac{(x-x_0)^\alpha}{\alpha} + \frac{(t-t_0)^\beta}{\beta})}$, where λ is constant, then $U_{\alpha, \beta}(k, h) = \frac{\lambda^{k+h}}{\alpha^k k! \beta^h h!}$;

Proof. These items can be proved by using (4) and (5). □

Theorem 2. If $u(x, t) = {}_{x_0}T_x^\alpha v(x, t)$, $0 < \alpha \leq 1$, then

$$U_{\alpha, \beta}(k, h) = \alpha(k+1).V_{\alpha, \beta}(k+1, h). \quad (6)$$

Proof. From (5), we have

$$\begin{aligned} U_{\alpha, \beta}(k, h) &= \frac{1}{\alpha^k k! \beta^h h!} \left[({}_{x_0}T_x^\alpha)^k ({}_{t_0}T_t^\beta)^h {}_{x_0}T_x^\alpha v(x, t) \right]_{(x_0, t_0)} \\ &= \frac{1}{\alpha^k k! \beta^h h!} \left[({}_{x_0}T_x^\alpha)^{k+1} ({}_{t_0}T_t^\beta)^h v(x, t) \right]_{(x_0, t_0)} \\ &= \alpha(h+1)V_{\alpha, \beta}(k+1, h). \end{aligned} \quad (7)$$

□

Theorem 3. If $u(x, t) = {}_{t_0}T_t^\gamma v(x, t)$, $m < \gamma \leq m+1$, then

$$U_{\alpha, \beta}(k, h) = V_{\alpha, \beta}(k, h + \frac{\gamma}{\beta}). \frac{\Gamma(h\beta + \gamma + 1)}{\Gamma(h\beta + \gamma - m)}. \quad (8)$$

Proof. Consider the initial condition of the problem,

$$\begin{aligned} T_t^\gamma v(x, t) &= {}_{t_0}T_t^\gamma \left[v(x, t) - f(x) \times \sum_{h=0}^m \frac{g^{(h)}(0)}{h!} (t - t_0)^h \right] \\ &= {}_{t_0}T_t^\gamma \left[f(x) \times \sum_{h=0}^{\infty} G_\beta(h)(t - t_0)^{h\beta} - f(x) \times \sum_{h=0}^m \frac{g^{(h)}(0)}{h!} (t - t_0)^h \right]. \end{aligned}$$

Substituting $h\beta$ in place of k in the second series and considering the initial conditions, we obtain

$$\begin{aligned} T_t^\gamma v(x, t) &= {}_{t_0}T_t^\gamma \left[f(x) \times \sum_{h=0}^{\infty} G_\beta(h)(t - t_0)^{h\beta} - f(x) \times \sum_{h=0}^{\frac{\gamma}{\beta}-1} \frac{g^{(h\beta)}(0)}{(h\beta)!} (t - t_0)^{h\beta} \right] \\ &= {}_{t_0}T_t^\gamma \left[f(x) \times \sum_{h=0}^{\infty} G_\beta(h)(t - t_0)^{h\beta} - f(x) \times \sum_{h=0}^{\frac{\gamma}{\beta}-1} G_\beta(h)(t - t_0)^{h\beta} \right] \\ &= {}_{t_0}T_t^\gamma \left[f(x) \times \sum_{h=\frac{\gamma}{\beta}}^{\infty} G_\beta(h)(t - t_0)^{h\beta} \right] \\ &= f(x) \times \sum_{h=\frac{\gamma}{\beta}}^{\infty} G_\beta(h) \frac{\Gamma(h\beta + 1)}{\Gamma(h\beta - m)} (t - t_0)^{h\beta - \gamma} \\ &= f(x) \times \sum_{h=0}^{\infty} G_\beta\left(h + \frac{\gamma}{\beta}\right) \frac{\Gamma(h\beta + \gamma + 1)}{\Gamma(h\beta + \gamma - m)} (t - t_0)^{h\beta} \\ &= \sum_{k=0}^{\infty} F_\alpha(k)(x - x_0)^{k\alpha} \times \sum_{h=0}^{\infty} G_\beta\left(h + \frac{\gamma}{\beta}\right) \frac{\Gamma(h\beta + \gamma + 1)}{\Gamma(h\beta + \gamma - m)} (t - t_0)^{h\beta} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(h\beta + \gamma + 1)}{\Gamma(h\beta + \gamma - m)} U_{\alpha, \beta}\left(k, h + \frac{\gamma}{\beta}\right) (x - x_0)^{k\alpha} (t - t_0)^{h\beta}. \end{aligned}$$

According to this result, we have

$$U_{\alpha, \beta}(k, h) = V_{\alpha, \beta}\left(k, h + \frac{\gamma}{\beta}\right) \frac{\Gamma(h\beta + \gamma + 1)}{\Gamma(h\beta + \gamma - m)}.$$

□

4 Applications

In this section, we will apply the new method to nonlinear conformable fractional partial differential equations and systems of nonlinear fractional partial differential equations. Also, we compare these solutions with the exact solu-

tions.

Example 1. Consider the fractional model of nonlinear Schrödinger equation [7]

$$i \frac{\partial^\beta u(x, t)}{\partial t^\beta} = -\frac{1}{2} \frac{\partial^2 u(x, t)}{\partial x^2} + u \cos^2(x) + |u|^2 u, \quad 0 < \beta \leq 1, \quad (9)$$

subject to the initial condition $u(x, 0) = \sin(x)$. Taking the two-dimensional conformable fractional differential transform of (9), then we have

$$\begin{aligned} i\beta(h+1)U_{1,\beta}(k, h+1) & \quad (10) \\ &= \frac{\Gamma(k+2+1)}{\Gamma(k+2-1)}U_{1,\beta}(k+2, h) - \frac{U_{1,\beta}(k, h)}{2} \\ &\quad - \frac{1}{2} \sum_{r=0}^k \sum_{s=0}^h \frac{2^r}{r!} \cos\left(\frac{r\pi}{2}\right) U_{1,\beta}(k-r, s) \\ &\quad - \sum_{r=0}^k \sum_{t=0}^{k-r} \sum_{s=0}^h \sum_{p=0}^{h-s} U_{1,\beta}(r, h-s-p) \overline{U_{1,\beta}(t, s)} U_{1,\beta}(k-r-t, p) \end{aligned}$$

and from the initial condition, we get

$$U_{1,\beta}(k, 0) = \frac{1}{k!} \sin\left(\frac{k\pi}{2}\right), \quad k = 0, 1, \dots \quad (11)$$

Substituting (10) to (11), the following series solution is obtained:

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{1,\beta}(k, h) x^k t^{h\beta} \\ &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \times \left(1 - \frac{3it^\beta}{2\beta} - \frac{9}{4} \frac{t^{2\beta}}{\alpha^2 2!} + \frac{27}{8} \frac{it^{3\beta}}{\beta^3 3!} - \frac{81}{16} \frac{t^{4\beta}}{\beta^4 4!} - \dots \right) \\ &= \sin(x) e^{-\frac{3it^\beta}{2\beta}}, \end{aligned}$$

which is the exact solution.

Example 2. We consider the Klein–Gordon equation

$$\frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^2 u}{\partial x^2} + u^2 = -x \sin\left(\frac{t^\beta}{\beta}\right) + x^2 \cos^2\left(\frac{t^\beta}{\beta}\right), \quad 0 < \beta \leq 1, \quad (12)$$

subject to the initial conditions

$$u(x, 0) = x. \quad (13)$$

Taking the two-dimensional conformable fractional differential transform of (12), then we have

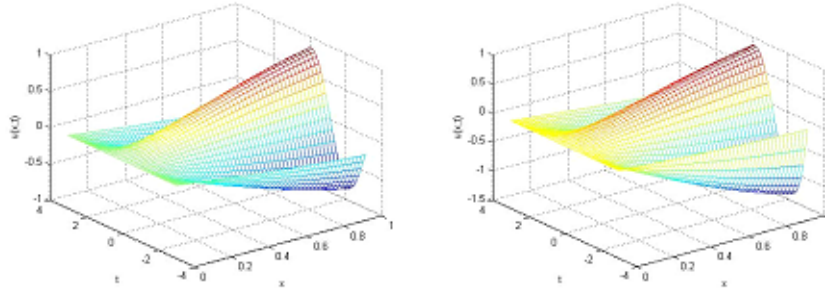


Figure 1: Graphs of the $u(x, t)$ for $\beta = 1$ and $\beta = 0.95$, when $N = 10$ (the number of terms), form left to right

$$\begin{aligned}
 \beta(h+1)U_{1,\beta}(k, h+1) &= \frac{\Gamma(k+2+1)}{\Gamma(k+2-1)}U_{1,\beta}(k+2, h) \\
 &\quad - \sum_{r=0}^k \sum_{s=0}^h U_{1,\beta}(r, h-s)U_{1,\beta}(k-r, s) \\
 &\quad - \delta(k-1)\frac{1}{h!}\sin\left(\frac{h\pi}{2}\right) + \frac{1}{2}\delta(k-2) \\
 &\quad + \frac{1}{2}\delta(k-2)\cdot\frac{2^h}{h!}\cos\left(\frac{h\pi}{2}\right). \tag{14}
 \end{aligned}$$

From the initial conditions (13), we can write

$$U_{1,\beta}(0, 0) = 0, \quad U_{1,\beta}(1, 0) = 1 \quad U_{1,\beta}(k, 0) = 0, \quad k = 2, 3, \dots \tag{15}$$

Substituting (15) into (14), we obtain the series solution as

$$\begin{aligned}
 u(x, t) &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{1,\beta}(k, h)x^k t^{h\beta} \\
 &= x \left(1 - \frac{t^{2\beta}}{\beta 2!} + \frac{t^{4\beta}}{\beta^2 4!} - \dots \right) = x \cdot \cos\left(\frac{t^\beta}{\beta}\right),
 \end{aligned}$$

which is the exact solution.

Example 3. Consider the nonlinear time fractional Fokker–Planck equation [14]

$$\frac{\partial^\beta u}{\partial t^\beta} = \left[-\frac{\partial}{\partial x} \left(3u - \frac{x}{2} \right) + \frac{\partial^2}{\partial x^2} (xu) \right] u, \tag{16}$$

where $x > 0$, $t > 0$, $0 < \beta \leq 1$, and the initial condition is

$$u(x, 0) = x. \tag{17}$$

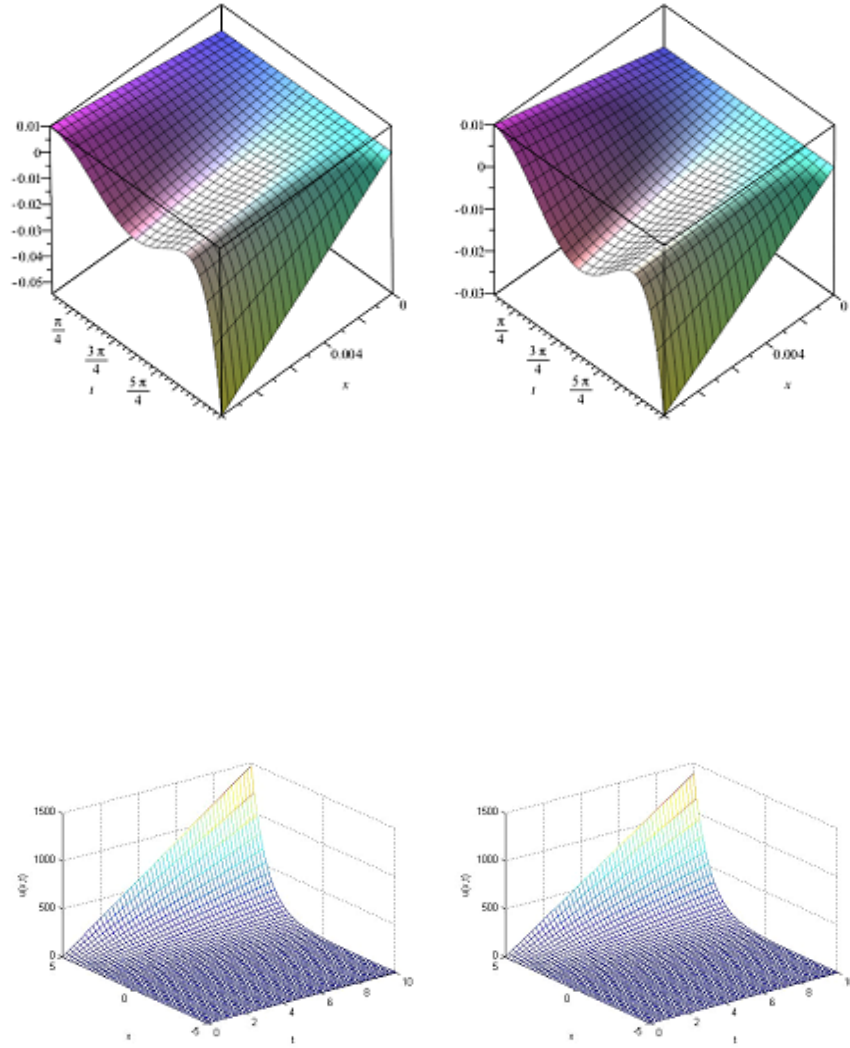


Figure 2: Graphs of the $u(x, t)$ for $\beta = 1$ and $\beta = 0.95$, when $N = 10$ (the number of terms), from left to right

Selecting $\alpha = 1$ and taking the two-dimensional conformable fractional differential transform of (16), we have

$$\begin{aligned} &\beta(h+1)U_{1,\beta}(k, h+1) \\ &= -(k+1) \left\{ 3 \sum_{r=0}^{k+1} \sum_{s=0}^h U_{1,\beta}(r, h-s)U_{1,\beta}(k+1-r, s) - \frac{1}{2}U_{1,\beta}(k, h) \right\} \\ &\quad + (k+2)(k+1) \sum_{r=0}^{k+1} \sum_{s=0}^h U_{1,\beta}(r, h-s)U_{1,\beta}(k+1-r, s). \end{aligned} \tag{18}$$

By using the initial condition (17), we write

$$U_{1,\beta}(k, 0) = \delta(k-1). \tag{19}$$

Substituting (19) in (18), we obtain the closed form series solution as

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{1,\beta}(k, h)x^k t^{h\beta} \\ &= x \left(1 + \frac{t^\beta}{\beta} + \frac{t^{2\beta}}{2\beta^2} + \frac{t^{3\beta}}{6\beta^3} + \dots \right) = x \sum_{h=0}^{\infty} \frac{t^{h\beta}}{\beta^h h!} = x e^{\frac{t^\beta}{\beta}}, \end{aligned}$$

which is the exact solution.

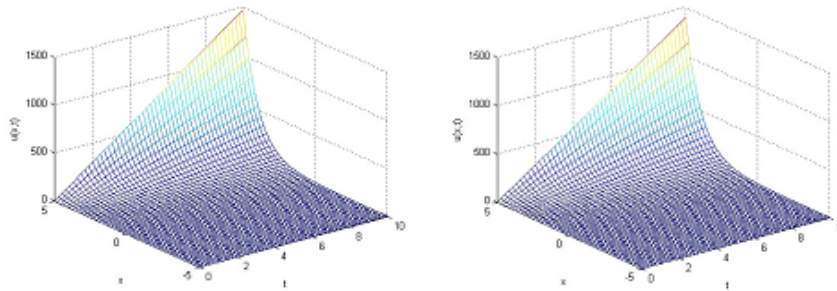


Figure 3: Graphs of the $u(x, t)$ for $\beta = 1$ and $\beta = 0.98$, when $N = 10$ (the number of terms), from left to right.

Example 4. Consider the time fractional inhomogeneous nonlinear system of PDEs [5]

$$\begin{cases} \frac{\partial^\beta u}{\partial t^\beta} + v \frac{\partial u}{\partial x} + u = 1, \\ \frac{\partial^\beta v}{\partial t^\beta} - u \frac{\partial v}{\partial x} - v = 1, \end{cases} \quad 0 < \beta \leq 1, \tag{20}$$

subject to the initial conditions

$$u(x, 0) = e^x, v(x, 0) = e^{-x}. \quad (21)$$

Taking the two-dimensional conformable fractional differential transform of (20), we have

$$\left\{ \begin{array}{l} \beta(h+1)U(k, h+1) = -\sum_{r=0}^k \sum_{s=0}^h (k-r+1)V(r, h-s)U(k-r+1, s) \\ \quad - U(k, h) + \delta(k)\delta(h) \\ \beta(h+1)V(k, h+1) = \sum_{r=0}^k \sum_{s=0}^h (k-r+1)U(r, h-s)V(k-r+1, s) \\ \quad + V(k, h) + \delta(k)\delta(h) \end{array} \right. \quad (22)$$

By using the initial conditions (21), we write

$$\left\{ \begin{array}{l} U(k, 0) = \frac{1}{k!}, \\ V(k, 0) = \frac{(-1)^k}{k!}. \end{array} \right. \quad (23)$$

Substituting (23) in (22), we obtain the closed form series solution as

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h)x^k t^{h\beta} \\ &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(1 - \frac{t^\alpha}{\alpha} + \frac{t^{2\alpha}}{\alpha^2 2!} - \frac{t^{3\alpha}}{\alpha^3 3!} + \dots\right) = e^{x - \frac{t^\beta}{\beta}} \\ v(x, t) &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V(k, h)x^k t^{h\beta} \\ &= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) \left(1 + \frac{t^\alpha}{\alpha} + \frac{t^{2\alpha}}{\alpha^2 2!} + \frac{t^{3\alpha}}{\alpha^3 3!} + \dots\right) = e^{-x + \frac{t^\beta}{\beta}} \end{aligned}$$

which are the exact solutions.

5 Conclusion

In this paper, we presented a two-dimensional conformable fractional differential transform method. Then we apply the new method to some conformable fractional partial differential equations and system of conformable fractional partial differential equations. Comparison of the results obtained by using the new method with exact solution reveals that the present method is very effective and convenient for solving nonlinear partial differential equations and system of partial differential equations of fractional order.

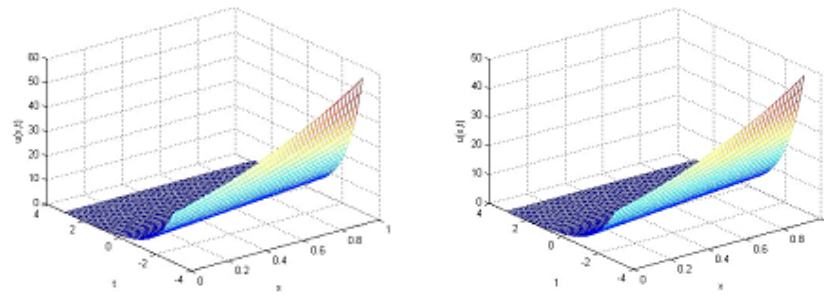


Figure 4: Graphs of the $u(x, t)$ for $\beta = 1$ and $\beta = 0.95$, when $N = 10$ (the number of terms), from left to right.

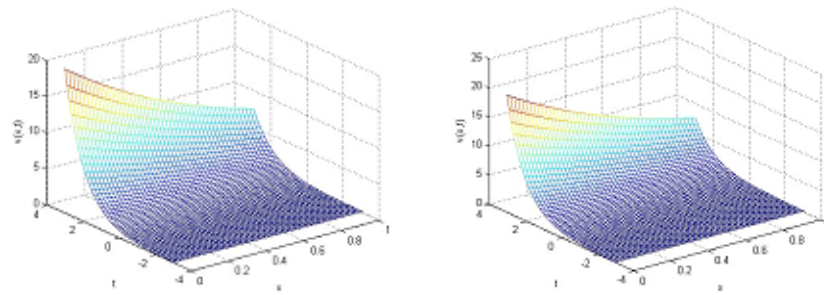


Figure 5: Graphs of the $v(x, t)$ for $\beta = 1, \beta = 0.95$ when $N=10$ (the number of terms), from left to right.

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