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Research Article

Finite element analysis for microscale heat equation with Neumann boundary conditions

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Abstract

In this paper, we explore the numerical analysis of the microscale heat equation. We present the characteristics of numerical solutions obtained through both semi- and fully-discrete linear finite element methods. We establish a priori estimates and error bounds for both semi-discrete and fully-discrete finite element approximations. Additionally, the existence and uniqueness of the semi-discrete and fully-discrete finite element approximations have been confirmed. The study explores error bounds in various spaces, comparing the semi-discrete to the exact solutions, the semi-discrete against the fully-discrete solutions, and the fully-discrete solutions

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with the exact ones. A practical algorithm is introduced to address the system emerging from the fully-discrete finite element approximation at every time step. Additionally, the paper presents numerical error calculations to further demonstrate and validate the results.

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1 Introduction

The microscale heat transport equation holds significance as a crucial model in microtechnology. Diverging from the classical heat diffusion model, microscale heat transport incorporates temperature derivatives of second and third order concerning time and space, introducing a more intricate representation of heat transfer dynamics at the microscale. The equation governing the microscale heat transport, capturing the thermal characteristics of thin films and other microstructures, is expressed as follows: [32]:

$$\frac{1}{\alpha}(\gamma_t + \gamma_q \gamma_{tt}) = \gamma_q \Delta\gamma_t + \Delta\gamma + s, \qquad \text{in} \quad \Im \times [0, \Re], \tag{1}$$

$$\frac{\partial \gamma}{\partial \nu} = 0, \qquad \text{on} \quad \partial\Im \times [0, \Re], \tag{2}$$

$$\gamma(\cdot, 0) = \gamma_1^0, \qquad \gamma_t(\cdot, 0) = \gamma_2^0 \qquad \text{in} \quad \Im, \tag{3}$$

where $\Im$ is an open bounded domain in $\mathbb{R}^n$ $(n = 1, 2, 3)$, γ is the temperature, α and γ_q are positive constants. Here α is the thermal diffusivity. Also, γ_q's represent the time lag of the heat flux and the temperature gradient [31].

The microscale heat transport equation serves as a mathematical representation to elucidate heat transfer phenomena occurring at extremely small scales, predominantly in micro- and nanoscale systems. This equation holds significance as a foundational tool in the realm of microscale heat transfer, finding applications in diverse areas such as the phonon-electron interaction model [25], the single energy equation [27, 28], the phonon scattering model [20], the phonon radiative transfer model [21], and the lagging behavior model

[27, 26]. These models can be effectively described and analyzed using the microscale heat transport equation.

A limited number of researchers have addressed the numerical solution of the one-dimensional microscale heat transport equation. Qui and Tien [24] employed the Crank–Nicolson technique to solve the phonon-electron interaction model. Joshi and Majumdar [21] utilized an explicit upstream difference method to address the phonon radiative transfer model in a one-dimensional medium. Zhang and Zhao [31] tackled the one-dimensional microscale heat transport equation using a fourth-order compact scheme, demonstrating its unconditional stability. In their works [33, 32], they extended their approach to solving the two- and three-dimensional microscale heat transport equations with second-order accuracy in both time and space. Additionally, a compact finite difference scheme with fourth-order spatial accuracy and second-order temporal accuracy for the three-dimensional microscale heat transport equation was developed by Harfash [11].

Recently, a compact difference scheme for the microscale heat transport equation was formulated in [6]. This scheme demonstrated superior precision compared to a previously suggested method, offering higher-order accuracy. The study established the unconditional stability and convergence of the developed compact difference scheme. In the work presented in [22], the application of the localized radial basis function partition of the unity method has been investigated for solving the microscale heat transport equation. The proposed algorithm involves a two-phase discretization of the unknown solution. In the study documented in [1], the focus was on developing a novel meshless numerical approach for solving the heat transport equation. To achieve this objective, the Crank–Nicolson finite difference method was employed to discretize the time derivative. The time-discrete scheme's unconditional stability and convergence were subsequently verified through an energy method.

It is worth noting that all previous numerical studies to solve the microscale heat transport equation were not concerned with performing theoretical analysis of the numerical solutions in terms of studying the solution spaces and the convergence of the numerical solution to the analytical solution. Also, the study of error in previous studies was incomplete due to the failure to specify the numerical solution spaces. In this study, the system will be approximated by the finite element method for space and the finite difference method for time. A comprehensive study of the numerical solution will be conducted. The study includes finding the spaces of the numerical solution and studying the error. In addition, we perform the convergence analysis of the approximate weak form to the continuous weak form. Many recent studies have involved the use of the finite element method to solve various problems [5, 29, 23, 4, 10, 30].

To address our approaches, we see that under the transformation $\psi = \gamma + \gamma_q \gamma_t$, the system (1)–(3) becomes as follows:
(Λ) Find $\{\gamma, \psi\}$ such that

$$\frac{1}{\alpha}\partial_t \psi = \Delta\psi + s, \qquad \text{in } \Im \times [0, \Re], \tag{4}$$

$$\gamma_q \partial_t \gamma = \psi - \gamma, \qquad \text{in } \Im \times [0, \Re], \tag{5}$$

$$\frac{\partial \gamma}{\partial \nu} = 0, \qquad \text{on } \partial\Im \times [0, \Re], \tag{6}$$

$$\gamma(\cdot, 0) = \gamma_1^0, \qquad \psi(\cdot, 0) = \gamma_1^0 + \gamma_q \gamma_2^0 := \psi^0, \qquad \text{in } \Im. \tag{7}$$

Next, we introduce a weak formulation of the system (4) and (5) in the following form:
(Λ) Find $\psi(\mathbf{x}, t), \gamma(\mathbf{x}, t) \in H^1(\Im)$ such that $\psi(\mathbf{x}, 0) = \psi^0(\mathbf{x}), \gamma(\mathbf{x}, 0) = \gamma_1^0(\mathbf{x})$ and for a.e. $t \in (0, \Re)$ and for all $\Upsilon \in H^1(\Im)$,

$$\frac{1}{\alpha}(\partial_t \psi, \Upsilon) + (\nabla\psi, \nabla\Upsilon) = s(1, \Upsilon), \qquad \text{in } \Im \times [0, \Re], \tag{8}$$

$$\gamma_q(\partial_t \gamma, \Upsilon) + (\gamma, \Upsilon) = (\psi, \Upsilon), \qquad \text{in } \Im \times [0, \Re]. \tag{9}$$

This passage outlines the structure of the paper. Section 2 is dedicated to defining the notation used throughout this study. In section 3, the semi-discrete approximation of Problem (Λ) is discussed, with subsection 3.1 focusing on global existence and subsection 3.2 on uniqueness. Subsection 3.3 presents the error bounds associated with the semi-discrete approximation. The fully-discrete finite element approximation of the Problem (Λ) is covered in section 4, where stability bounds are also derived. The following subsections, 4.1, 4.2, and 4.3, analyze the existence, uniqueness, and convergence of the solution, respectively, and subsection 4.4 examines the error estimates for the fully-discrete approximation. Lastly, section 5 describes a numerical algorithm for implementing the fully-discrete approximation of the Problem (Λ), including calculations of numerical errors.

2 Finite element spaces and associated results

Consider the finite element space $S^{\hbar}$, which is a subset of $H^1(\Im)$, defined by

$$S^{\hbar} := \{\varepsilon \in C(\overline{\Im}) : \varepsilon|_{\tau} \text{ is linear } \text{ for all } \tau \in \mathcal{T}^{\hbar}\}.$$

The set $\{\kappa_j\}_{j=1}^{J}$ represents the standard basis functions for $S^{\hbar}$, which adhere to the property that $\kappa_j(\varsigma_i) = \delta_{ij}$ for all $i, j = 1, \ldots, J$. Here, $\mathcal{N}^{\hbar} := \{\varsigma_j\}_{j=0}^{J}$ denotes the collection of nodes corresponding to the partition $\mathcal{T}^{\hbar}$. Additionally, we introduce

$$S^{\hbar}_{\geq 0} := \{\varepsilon \in S^{\hbar} : \varepsilon(\varsigma_j) \geq 0, j = 1, \ldots, J\}$$
$$\subset H^1_{\geq 0} := \{\varepsilon \in H^1(\Im) : \varepsilon \geq 0 \ a.e. \in \Im\}.$$

The operator $\Pi^{\hbar} : C(\overline{\Im}) \to S^{\hbar}$ represents the Lagrange interpolation operator, which can also be referred to as the piecewise linear interpolant. This operator ensures that

$$\Pi^{\hbar}\varepsilon(\varsigma_j) := \varepsilon(\varsigma_j), \quad \text{for } j = 1, \ldots, J.$$

Furthermore, we introduce a discrete L^2 inner (or semi-inner) product on $S^{\hbar}(C(\overline{\Im}))$ as

$$(u, v)^{\hbar} := \int_{\Im} \Pi^{\hbar}(u(\mathbf{x})v(\mathbf{x}))d\mathbf{x} = \sum_{j=1}^{J} \widehat{M}_{jj}\ u(\varsigma_j)\, v(\varsigma_j), \tag{10}$$

where $\widehat{M}_{jj} = (\kappa_j, \kappa_j)^{\hbar} = (1, \kappa_j) > 0$. By observing (10), it is straightforward to confirm that

$$(\varepsilon_1, \varepsilon_2)^{\hbar} = (\Pi^{\hbar}\varepsilon_1, \varepsilon_2)^{\hbar} = (\Pi^{\hbar}\varepsilon_1, \Pi^{\hbar}\varepsilon_2)^{\hbar} \qquad \text{for all } \varepsilon_1, \varepsilon_2 \in C(\overline{\Im}).$$

In the context of the finite element space $S^{\hbar}$, several established results are notable. The induced discrete semi-norm on $C(\overline{\Im})$ and the norm on $S^{\hbar}$ are both represented by $|\cdot|_h$, which is defined by the expression $[(\cdot,\cdot)^\hbar]^{1/2}$. It has been proven that the semi-norm $|\cdot|_h$ is equivalent to the norm $\|\cdot\|_0$, which is defined as $[(\cdot,\cdot)]^{1/2}$. This relationship can be described as

$$\|\vartheta\|_0^2 \le |\vartheta|_\hbar^2 \le (\ell+2)\|\vartheta\|_0^2 \qquad \text{for all } \vartheta \in S^\hbar. \tag{11}$$

The Poincaré inequality, given that $\hbar$ is sufficiently small, can be expressed in the following form:

$$((\zeta,\zeta)^\hbar)^{\frac{1}{2}} = |\zeta|_\hbar \le C_p(|\zeta|_1 + |(\zeta,1)^\hbar|). \tag{12}$$

We define, for any $\lambda(x) \in S^\hbar$, that

$$|\lambda^\hbar|_{\hbar,\zeta} := (\int_{\Im} \Pi^\hbar\{|\lambda(x)^\hbar|^\zeta dx)^{\frac{1}{\zeta}} \equiv (\sum_{i=0}^{k} \widehat{M}_{jj}\lambda(x_i)^\hbar|^\zeta)^{\frac{1}{\zeta}} \quad if \quad 0 \le \zeta < \infty,$$

and

$$|\lambda^\hbar|_{\hbar,\zeta} := \max_{0\le j\le k} |\lambda(x_j)^\hbar| \qquad\qquad if \quad \zeta = \infty.$$

We now revisit some well-established results concerning the space $S^\hbar$ under the assumption that $\mathcal{T}^\hbar$ forms a quasi-uniform partitioning: For any $\tau \in \mathcal{T}^\hbar, \xi \in S^\hbar, 1 \le p, q \le \infty$ and $m, l \in \{0,1\}$ with $l \le m$, we have

$$\|\xi\|_{m,p,\text{"}\tau\text{"}} \le C\hbar_{\text{"}\tau\text{"}}^{l-m+\ell\min(0,\frac{1}{p}-\frac{1}{q})}\|\xi\|_{l,q,\text{"}\tau\text{"}},$$

where the abbreviation "τ" means "with" or "without" τ. The inequality stated above is commonly referred to as "the inverse inequality, " as documented in . 75–771[9]. Additionally, it remains valid when replacing $\|\cdot\|$ with $|\cdot|$, as indicated in . 140–1421[7].

For future reference, we introduce the subsequent inverse inequalities, derived from the quasi-uniform condition as outlined in of [7, Theorem 3.2.6],

$$|\xi|_{1,p,\text{"}\tau\text{"}} \le C\hbar_{\text{"}\tau\text{"}}^{-1}|\xi|_{0,p,\text{"}\tau\text{"}}, \quad 1 \le p \le \infty,$$

$$|\xi|_{m,p,\text{"}\tau\text{"}} \le C\hbar_{\text{"}\tau\text{"}}^{-\ell(\frac{1}{q}-\frac{1}{p})}|\xi|_{m,q,\text{"}\tau\text{"}}, \quad 1 \le q \le p \le \infty, \quad m \in \{0,1\}.$$

We also need the following interpolation results for every $\xi \in W^{1,s}(\Im)$, where $s \in [2,\infty]$ if $\ell = 1$, and $s \in (\ell,\infty]$ if ℓ is either 2 or 3:

$$|(I-\Pi^{\hbar})\xi|_{m,s} \leq C\hbar^{1-m}|\xi|_{1,s}, \quad m \in \{0,1\}, \tag{13}$$

$$\lim_{\hbar\to 0} |(I-\Pi^{\hbar})\xi|_{1,s} = 0, \tag{14}$$

(see [9, Theorem 1.103 and Corollary 1.110], respectively). We also bring to mind the following useful result (e.g., [8]): For any $\xi_1, \xi_2 \in S^{\hbar}$, we have

$$|(\xi_1,\xi_2) - (\xi_1,\xi_2)^{\hbar}| \leq C\hbar^{1+m}|\xi_1|_{m,n_1}|\xi_2|_{1,n_2}, \tag{15}$$

for $m \in \{0,1\}$ and $1 \leq n_1, n_2 \leq \infty$ with $\frac{1}{n_1} + \frac{1}{n_2} = 1$.

We can conveniently introduce the "inverse Laplacian Green's operator" denoted as $\mathcal{G} : (H^1(\Im))' \to H^1(\Im)$ such that

$$(\nabla\mathcal{G}\breve{v}_1, \nabla\breve{v}_2) = \langle \breve{v}_1, \breve{v}_2 \rangle \qquad \text{for all} \quad \breve{v}_2 \in H^1(\Im),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(H^1(\Im))'$ and $H^1(\Im)$ such that

$$\langle \hbar, \breve{v} \rangle = (\hbar, \breve{v}) \quad \text{for all} \quad \hbar \in L^2(\Im) \text{ and } \breve{v} \in H^1(\Im). \tag{16}$$

3 A semi-discrete approximation

We define the following semi-discrete approximation of the system (4)–(7): $(\Lambda^{\hbar})$ Find $\{\gamma^{\hbar}, \psi^{\hbar}\} \in S^{\hbar} \times S^{\hbar}$ such that for a.e. $t \in (0, \Re)$

$$\frac{1}{\alpha}(\frac{\partial \psi^{\hbar}}{\partial t}, \Upsilon^{\hbar})^{\hbar} + (\nabla\psi^{\hbar}, \nabla\Upsilon^{\hbar}) = s(1, \Upsilon^{\hbar})^{\hbar} \qquad \text{for all } \Upsilon^{\hbar} \in S^{\hbar}, \tag{17}$$

$$\gamma_q(\frac{\partial \gamma^{\hbar}}{\partial t}, \Upsilon^{\hbar}) + (\gamma^{\hbar}, \Upsilon^{\hbar}) = (\psi^{\hbar}, \Upsilon^{\hbar}) \qquad \text{for all } \Upsilon^{\hbar} \in S^{\hbar}, \tag{18}$$

$$\gamma^{\hbar}(\mathbf{x}, 0) = \mathbb{P}^{\hbar}\gamma(\mathbf{x}). \tag{19}$$

3.1 Global existence

Theorem 1. Suppose that $\Im \subset \mathbb{R}^d$ (with $d \leq 3$) is an open, bounded, convex domain. Let $\gamma^0 \in H^1(\Im)$. Then, the system (17)–(19) admits a solution $\psi^{\hbar}, \gamma^{\hbar}$ that satisfies

$$\psi^\hbar \in L^\infty(0,\Re;H^1(\Im)) \cap L^\infty(0,\Re;L^2(\Im)) \cap L^2(0,\Re;H^1(\Im)) \cap L^2(\Im_\Re),$$

$$\gamma^\hbar \in L^2(0,\Re;H^1(\Im)) \cap L^\infty(0,\Re;L^2(\Im)) \cap L^\infty(\Im_\Re),$$
$$\frac{\partial \psi^\hbar}{\partial t} \cap \frac{\partial \gamma^\hbar}{\partial t} \in L^2(\Im_\Re).$$

Proof. Selecting $\Upsilon^\hbar = \psi^\hbar$ in (17) and $\Upsilon^\hbar = \gamma^\hbar$ in (18), we obtain

$$\frac{1}{\alpha}(\frac{\partial \psi^\hbar}{\partial t}, \psi^\hbar)^\hbar + (\nabla\psi^\hbar, \nabla\psi^\hbar) = s(1, \psi^\hbar)^\hbar, \tag{20}$$

$$\gamma_q(\frac{\partial \gamma^\hbar}{\partial t}, \gamma^\hbar) + (\gamma^\hbar, \gamma^\hbar) = (\psi^\hbar, \gamma^\hbar). \tag{21}$$

Now, by utilizing (20) and (21), we derive the following result:

$$\frac{1}{2\alpha}\frac{d}{dt}|\psi^\hbar|_\hbar^2 + \frac{\gamma_q}{2}\frac{d}{dt}\|\gamma^\hbar\|_0^2 + \|\gamma^\hbar\|_0^2 + |\nabla\psi^\hbar|_0^2 = \int_\Im \Pi^\hbar(\psi^\hbar, \gamma^\hbar)\, d\mathbf{x} + s\int_\Im \Pi^\hbar \psi^\hbar\, d\mathbf{x}.$$

Through the application of Young's inequality, Hölder's inequality, and (11), it can be established that

$$\begin{aligned}\frac{d}{2dt}\left[\frac{1}{\alpha}|\psi^\hbar|_\hbar^2 + \gamma_q\|\gamma^\hbar\|_0^2\right] + \|\gamma^\hbar\|_0^2 + \|\nabla\psi^\hbar\|_0^2 &\le C\left[\|\psi^\hbar\|_0^2 + \|\gamma^\hbar\|_0^2\right] + C(|\Im|, s)\\ &\le C\left[|\psi^\hbar|_\hbar^2 + \|\gamma^\hbar\|_0^2\right] + C(|\Im|, s).\end{aligned}$$

By applying the Grönwall lemma and integrating equation (21) over the time interval $(0,\Re)$, we obtain

$$\begin{aligned}&\frac{1}{2\alpha}|\psi^\hbar(\Re)|_\hbar^2 + \frac{\gamma_q}{2}\|\gamma^\hbar(\Re)\|_0^2 + \int_0^\Re |\gamma^\hbar|_\hbar^2\, dt + \int_0^\Re |\nabla\psi^\hbar|_\hbar^2\, dt\\ &\le \frac{1}{2\alpha}|\psi^\hbar(0)|_\hbar^2 + \frac{\gamma_q}{2}\|\gamma^\hbar(0)\|_0^2 + C(|\Im|, s).\end{aligned} \tag{22}$$

From (22), it follows that

$$\begin{aligned}\|\psi^\hbar\|_{L^\infty(0,\Re;L^2(\Im))} \le C, \qquad \|\gamma^\hbar\|_{L^\infty(0,\Re;L^2(\Im))} \le C,\\ \|\gamma^\hbar\|_{L^2(\Im_\Re)} \le C, \qquad \|\psi^\hbar\|_{L^2(0,\Re;H^1(\Im))} \le C.\end{aligned} \tag{23}$$

Now, by choosing $\Upsilon^\hbar = \frac{\partial \psi^\hbar}{\partial t}$ in (17), we have

$$\frac{1}{\alpha}\|\frac{\partial \psi^\hbar}{\partial t}\|_0^2 + \frac{d}{2dt}\|\nabla\psi^\hbar\|_0^2 = s(1, \frac{\partial \psi^\hbar}{\partial t})^\hbar.$$

Through the application of Hölder and Young's inequalities, we can state that

$$\frac{1}{2\alpha}\|\frac{\partial\psi^{\hbar}}{\partial t}\|_0^2+\frac{d}{2dt}\|\nabla\psi^{\hbar}\|_0^2\leq C(s^2,|\Im|,\alpha). \tag{24}$$

Integrating both sides of (24) from 0 to t results in

$$\frac{1}{2\alpha}\int_0^{\Re}\|\frac{\partial\psi^{\hbar}}{\partial t}\|_0^2\,dt+\frac{1}{2}\|\nabla\psi^{\hbar}(T)\|_0^2\leq\frac{1}{2}\|\nabla\psi^{\hbar}(0)\|_0^2+C(s^2,|\Im|,\alpha). \tag{25}$$

From (25) and considering the assumptions that $\psi^0\in H^1(\Im)$, it can be concluded that

$$\|\frac{\partial\psi^{\hbar}}{\partial t}\|_{L^2(\Im_{\Re})}\leq C,\qquad \|\psi^{\hbar}\|_{L^2(0,\Re;H^1(\Im))}\leq C. \tag{26}$$

Now, by choosing $\Upsilon^{\hbar}=\frac{\partial\gamma^{\hbar}}{\partial t}$ in (18), we obtain that

$$\gamma_q\|\frac{\partial\gamma^{\hbar}}{\partial t}\|_0^2+\frac{d}{2dt}\|\gamma^{\hbar}\|_0^2\leq(\psi^{\hbar},\frac{\partial\gamma^{\hbar}}{\partial t})^{\hbar}. \tag{27}$$

Through the application of Young's inequality, we can express that

$$\frac{\gamma_q}{2}\|\frac{\partial\gamma^{\hbar}}{\partial t}\|_0^2+\frac{d}{2dt}\|\gamma^{\hbar}\|_0^2\leq\frac{1}{2\gamma_q}|\psi^{\hbar}|_{\hbar}^2. \tag{28}$$

Integrating (28) over $(0,1)$ and utilizing (28), along with the observation that $L^{\infty}(0,\Re,L^2(\Im))\hookrightarrow L^2(\Im_{\Re})$, we deduce that

$$\frac{\gamma_q}{2}\int_0^{\Re}\|\frac{\partial\gamma^{\hbar}}{\partial t}\|_0^2\,dt+\frac{1}{2}\|\gamma^{\hbar}(\Re)\|_0^2\leq\frac{1}{2}\|\gamma^{\hbar}(0)\|_0^2+\frac{1}{2\gamma_q}\int_0^{\Re}|\psi^{\hbar}|_{\hbar}^2\,dt. \tag{29}$$

From (29) and considering the assumptions that $\gamma_1^0\in H^1(\Im)$, it can be concluded that

$$\|\frac{\partial\gamma^{\hbar}}{\partial t}\|_{L^2(\Im_{\Re})}\leq C,\quad \|\gamma^{\hbar}\|_{L^{\infty}(\Im_{\Re})}\leq C.$$

Now, by setting $\Upsilon^{\hbar}=\Delta\gamma^{\hbar}$ in (18), it follows that

$$\frac{\gamma_q}{2}\frac{d}{dt}\|\nabla\gamma^{\hbar}\|_0^2+\|\nabla\gamma^{\hbar}\|_0^2=(\nabla\psi^{\hbar},\nabla\gamma^{\hbar}).$$

By applying Young's inequality, we arrive at the conclusion that

$$\frac{\gamma_q}{2}\frac{d}{dt}\|\nabla\gamma^{\hbar}\|_0^2+\frac{1}{2}\|\nabla\gamma^{\hbar}\|_0^2\leq\frac{1}{2}\|\nabla\psi^{\hbar}\|_0^2. \tag{30}$$

When integrating both sides of (30) from 0 to t, the result is

$$\gamma_q \|\nabla \gamma^\hbar(\Re)\|_0^2 + \int_0^\Re \|\nabla \gamma^\hbar\|_0^2 dt \leq \int_0^\Re \|\nabla \psi^\hbar\|_0^2 dt + \gamma_q \|\nabla \gamma^\hbar(0)\|_0^2. \tag{31}$$

From (23) and (31), considering the assumptions that $\gamma_1^0 \in H^1(\Im)$, it can be concluded that

$$\|\psi^\hbar\|_{L^\infty(0,\Re;H^1(\Im))} \leq C, \quad \|\gamma^\hbar\|_{L^2(0,\Re;H^1(\Im))} \leq C.$$

□

3.2 Uniqueness

Let $\psi_1^\hbar, \psi_2^\hbar$ and $\gamma_1^\hbar, \gamma_2^\hbar$ represent two sets of solutions from the semi-discrete approximations given by (17) and (18), respectively. By defining $\psi^\hbar = \psi_1^\hbar - \psi_2^\hbar$ and $\gamma^\hbar = \gamma_1^\hbar - \gamma_2^\hbar$ and subsequently subtracting the semi-discrete approximations $(\Im^\hbar)$, we obtain the following result:

$$\frac{1}{\alpha}(\frac{\partial \psi^\hbar}{\partial t}, \Upsilon^\hbar)^\hbar + (\nabla \psi^\hbar, \nabla \Upsilon^\hbar) = 0, \tag{32}$$

and

$$\gamma_q(\frac{\partial \gamma^\hbar}{\partial t}, \Upsilon^\hbar) + (\gamma^\hbar, \Upsilon^\hbar) = (\psi^\hbar, \Upsilon^\hbar). \tag{33}$$

By selecting $\Upsilon^\hbar = \alpha\psi^\hbar$ for (32) and $\Upsilon^\hbar = \frac{1}{\gamma_q}\gamma^\hbar$ for (33), respectively, we deduce the that

$$\frac{d}{2dt}|\psi^\hbar|_\hbar^2 + \alpha|\psi^\hbar|_1^2 = 0, \tag{34}$$

and

$$\frac{d}{2dt}\|\gamma^\hbar\|_0^2 + \frac{1}{\gamma_q}\|\gamma^\hbar\|_0^2 = \frac{1}{\gamma_q}(\psi^\hbar, \gamma^\hbar). \tag{35}$$

From (34) and (35), it follows that

$$\frac{d}{2dt}|\psi^\hbar|_\hbar^2 + \frac{d}{2dt}\|\gamma^\hbar\|_0^2 + \alpha|\psi^\hbar|_1^2 + \frac{1}{\gamma_q}\|\gamma^\hbar\|_0^2 = \frac{1}{\gamma_q}(\psi^\hbar, \gamma^\hbar).$$

Applying Young's inequality along with (11), we deduce that

$$\frac{d}{2dt}[|\psi^\hbar|_\hbar^2 + \|\gamma^\hbar\|_0^2] + \alpha|\psi^\hbar|_1^2 + \frac{1}{\gamma_q}\|\gamma^\hbar\|_0^2 \leq \frac{1}{2\gamma_q}\|\psi^\hbar\|_0^2 + \frac{1}{2\gamma_q}\|\gamma^\hbar\|_0^2$$

$$\leq \frac{1}{2\gamma_q}|\psi^\hbar|_\hbar^2 + \frac{1}{2\gamma_q}\|\gamma^\hbar\|_0^2.$$

The application of Grönwall's lemma leads to the conclusion as

$$|\psi^\hbar|_\hbar^2 + \|\gamma^\hbar\|_0^2 + \alpha\int_0^{\Re} |\psi^\hbar|_1^2 dt + \frac{1}{\gamma_q}\int_0^{\Re} \|\gamma^\hbar\|_0^2 dt \leq e^{C\Re}[|\psi^\hbar(0)|_\hbar^2 + \|\gamma^\hbar(0)\|_0^2].$$

Therefore, if $\psi_1^\hbar(0) = \psi_2^\hbar(0)$ and $\gamma_1^\hbar(0) = \gamma_2^\hbar(0)$, then we can deduce the uniqueness of the solutions such that $\psi_1^\hbar(t) = \psi_2^\hbar(t)$ and $\gamma_1^\hbar(t) = \gamma_2^\hbar(t)$ for all t.

3.3 Error estimate

Theorem 2. Assuming that the conditions of Theorem 1 are met and given

$$\|\psi\|_{L^2(0,\Re;H^2(\Im))} \leq C, \tag{36}$$

then the solution $\{\psi, \gamma\}$ adheres to the specified error constraints

$$\|e_\psi\|_{L^\infty(0,\Re;L^2(\Im))} + \|e_\gamma\|_{L^\infty(0,\Re;L^2(\Im))} + \|e_\psi\|_{L^2(0,\Re;H^1(\Im))} + \|e_\gamma\|_{L^2(\Im_\Re)} \leq C\hbar^2, \tag{37}$$

where

$$e_\psi = \psi - \psi^\hbar, \quad e_\gamma = \gamma - \gamma^\hbar. \tag{38}$$

Proof. To begin with, we establish the following definitions:

$$e_\psi^A = \psi - \Pi^\hbar\psi, \quad e_\gamma^A = \gamma - \Pi^\hbar\gamma, \tag{39}$$

$$e_\psi^\hbar = \Pi^\hbar\psi - \psi^\hbar, \quad e_\gamma^\hbar = \Pi^\hbar\gamma - \gamma^\hbar. \tag{40}$$

From (38)–(40), we have that

$$e_\psi^\hbar = e_\psi - e_\psi^A, \quad e_\gamma^\hbar = e_\gamma - e_\gamma^A. \tag{41}$$

By applying (40) and (38)–(40), it is straightforward to determine that

$$\|e_\psi^A\|_0 \leq C\hbar^2\|\psi\|_2, \quad \|e_\gamma^A\|_0 \leq C\hbar^2\|\gamma\|_2,$$

$$|e_\psi^A|_1 \leq C\hbar\|\psi\|_2, \quad |e_\gamma^A|_1 \leq C\hbar\|\gamma\|_2, \tag{42}$$

$$\|e_\psi^A\|_1 \leq C\hbar\|\psi\|_2, \quad \|e_\gamma^A\|_1 \leq C\hbar\|\gamma\|_2,$$

$$\|e_\psi^\hbar\|_0 \leq \|e_\psi\|_0 + C\hbar^2\|\psi\|_2, \quad \|e_\gamma^\hbar\|_0 \leq \|e_\gamma\|_0 + C\hbar^2\|\gamma\|_2,$$

$$|e_\psi^\hbar|_1 \leq |e_\psi|_1 + C\hbar\|\psi\|_2, \quad |e_\gamma^\hbar|_1 \leq |e_\gamma|_1 + C\hbar\|\gamma\|_2,$$

$$\|e_\psi^\hbar\|_1 \leq \|e_\psi\|_0 + |e_\psi|_1 + C\hbar\|\psi\|_2, \quad \|e_\gamma^A\|_1 \leq \|e_\gamma\|_0 + |e_\gamma|_1 + C\hbar\|\gamma\|_2. \quad (43)$$

Subtracting (8) and (9) from (17) and (18), respectively, and choosing $\Upsilon = \Upsilon^\hbar$ yields that

$$(\partial_t\psi, \Upsilon) - (\partial_t\psi^\hbar, \Upsilon)^\hbar + \alpha(\nabla e_\psi, \nabla\Upsilon) = 0, \quad (44)$$

and

$$(\partial_t\gamma, \Upsilon) - (\partial_t\gamma^\hbar, \Upsilon)^\hbar + \frac{1}{\gamma_q}[(\gamma, \Upsilon) - (\gamma^\hbar, \Upsilon)^\hbar] = \frac{1}{\gamma_q}[(\psi, \Upsilon) - (\psi^\hbar, \Upsilon)^\hbar]. \quad (45)$$

Selecting $\Upsilon = e_\psi^\hbar$ in (44) and $\Upsilon = e_\gamma^\hbar$ in (45) results in

$$(\partial_t\psi, e_\psi^\hbar) - (\partial_t\psi^\hbar, e_\psi^\hbar)^\hbar + \alpha(\nabla e_\psi, \nabla e_\psi^\hbar) = 0, \quad (46)$$

and

$$(\partial_t\gamma, e_\gamma^\hbar) - (\partial_t\gamma^\hbar, e_\gamma^\hbar) + \frac{1}{\gamma_q}[(\gamma, e_\gamma^\hbar) - (\gamma^\hbar, e_\gamma^\hbar)] = \frac{1}{\gamma_q}[(\psi, e_\gamma^\hbar) - (\psi^\hbar, e_\gamma^\hbar)]. \quad (47)$$

By employing (41) and adding and subtracting the terms $(\partial_t\psi^\hbar, e_\psi^\hbar)$ and $(\partial_t\gamma^\hbar, e_\gamma^\hbar)$ to (46), as well as the terms $(\gamma^\hbar, e_\gamma^\hbar)$ and $(\psi^\hbar, e_\gamma^\hbar)$ to (47), it can be deduced that

$$(\partial_t e_\psi, e_\psi^\hbar) + \{(\partial_t\psi^\hbar, e_\psi^\hbar) - (\partial_t\psi^\hbar, e_\psi^\hbar)^\hbar\} + \alpha(\nabla e_\psi, \nabla e_\psi) = \alpha(\nabla e_\psi, \nabla e_\psi^A), \quad (48)$$

and

$$(\partial_t\gamma, e_\gamma^\hbar) - (\partial_t\gamma^\hbar, e_\gamma^\hbar) + \frac{1}{\gamma_q}[(\gamma, e_\gamma^\hbar) - (\gamma^\hbar, e_\gamma^\hbar)] = \frac{1}{\gamma_q}[(\psi, e_\gamma^\hbar) - (\psi^\hbar, e_\gamma^\hbar)]. \quad (49)$$

From (48) and (49), it can be observed that

$$\frac{d}{2dt}\|e_\psi\|_0^2 + \frac{d}{2dt}\|e_\gamma\|_0^2 + \alpha|e_\psi|_1^2 + \frac{1}{\gamma_q}\|e_\gamma\|_0^2$$
$$\leq \alpha(\nabla e_\psi, \nabla e_\psi^A) + [(\partial_t\psi^\hbar, e_\psi^\hbar)^\hbar - (\partial_t\psi^\hbar, e_\psi^\hbar)] + \frac{1}{\gamma_q}[(\psi, e_\gamma^\hbar) - (\psi^\hbar, e_\gamma^\hbar)]. \quad (50)$$

We bound each term on the right-hand side of (50) separately. Initially, by employing Cauchy–Schwarz and Young inequalities, along with (42), it can be inferred that

$$\alpha(\nabla e_\psi, \nabla e_\psi^A) \le \frac{\alpha}{4}|e_\psi|_1^2 + C\hbar^2\|\psi\|_2^2. \tag{51}$$

By applying (15), Young's inequality, and (43), it can be established that

$$[(\partial_t\psi^\hbar, e_\psi^\hbar)^\hbar - (\partial_t\psi^\hbar, e_\psi^\hbar)] \le C\hbar^2\|\partial_t\psi^\hbar\|_0^2 + \frac{1}{2}\|e_\psi\|_0^2 + \frac{\alpha}{4}|e_\psi|_1^2 + C\hbar^2\|\psi\|_2^2. \tag{52}$$

By employing (40), Young's inequality, and (43), we determine that

$$\frac{1}{\gamma_q}[(\psi, e_\gamma^\hbar) - (\psi^\hbar, e_\gamma^\hbar)^\hbar] = \frac{1}{\gamma_q}(e_\psi^\hbar, e_\gamma^\hbar) \le \frac{1}{2}\|e_\psi^\hbar\|_0^2 + C\|e_\gamma^\hbar\|_0^2. \tag{53}$$

By inserting equations (51)–(53) into (50), it can be deduced that

$$\begin{aligned}&\frac{d}{2dt}[\|e_\psi\|_0^2 + \|e_\gamma\|_0^2] + \frac{\alpha}{2}|e_\psi|_1^2 + \frac{1}{\gamma_q}\|e_\gamma\|_0^2\\ &\le C\|e_\psi\|_0^2 + C\|e_\gamma\|_0^2 + C\hbar^2\|\partial_t\psi^\hbar\|_0^2 + C\hbar^2\|\psi\|_2^2.\end{aligned} \tag{54}$$

Multiplying (54) by 2 and applying Grönwall's lemma, we obtain that

$$\begin{aligned}&\|e_\psi(\Re)\|_0^2 + \|e_\gamma(\Re)\|_0^2 + \alpha\int_0^\Re |e_\psi|_1^2 dt + \frac{2}{\gamma_q}\int_0^\Re \|e_\gamma\|_0^2 dt\\ &\le e^{2C\Re}[\|e_\psi(0)\|_0^2 + \|e_\gamma(0)\|_0^2] + e^{2C\Re}[C\hbar^2\|\partial_t\psi^\hbar\|_{L^2(\Im_\Re)}\\ &\quad + C\hbar^2\|\psi\|_{L^2(0,\Re;H^2(\Im))}].\end{aligned} \tag{55}$$

In order to bound the right-hand side of (55), initially, we observe from (54) and taking into account that $\psi^0, \gamma_1^0 \in H^1$, yielding that

$$\|e_\psi(0)\|_0^2 \equiv \|\psi^0 - p^\hbar\psi^0\|_0^2 \le C\hbar^2\psi^0 \le C\hbar^2,$$

and

$$\|e_\gamma(0)\|_0^2 \equiv \|\gamma_1^0 - p^\hbar\gamma_1^0\|_0^2 \le C\hbar^2\gamma_1^0 \le C\hbar^2.$$

Additionally, based on Theorem 1, the remaining terms on the right-hand side of (55) are bounded. Consequently, we ultimately obtain that

$$\|e_\psi\|_{L^\infty(0,\Re;L^2(\Im))} + \|e_\gamma\|_{L^\infty(0,\Re;L^2(\Im))} + \|e_\psi\|_{L^2(0,\Re;H^1(\Im))} + \|e_\gamma\|_{L^2(\Im_\Re)} \le C\hbar^2.$$

□

4 A fully-discrete approximation

Suppose that N is a positive integer, and let $\Delta t := \frac{\Re}{N}$ represent the time step. We investigate the following fully-discrete finite element approximation of the system (1)–(7):

$(\Lambda^{h,\Delta t})$ For $n \geq 1$ find $\{\Psi^n,\ \Gamma^n\} \in [S^\hbar]^2$ such that for all $\Upsilon \in S^\hbar$

$$\frac{1}{\alpha}\left(\frac{\Psi^n - \Psi^{n-1}}{\Delta t},\ \Upsilon\right)^\hbar + (\nabla\Psi^n,\ \nabla\Upsilon) = s(1,\Upsilon)^\hbar \qquad \text{for all } \Upsilon \in S^\hbar, \quad (56)$$

$$\gamma_q\left(\frac{\Gamma^n - \Gamma^{n-1}}{\Delta t},\ \Upsilon\right) + (\Gamma^n,\ \Upsilon) = (\Psi^n,\ \Upsilon) \qquad \text{for all } \Upsilon \in S^\hbar, \quad (57)$$

$$\Psi^0 = \mathbb{P}^\hbar\psi, \qquad \Gamma_1^0 = \mathbb{P}^\hbar\gamma.$$

Theorem 3. Suppose ψ^0, $\gamma_1^0 \in L^2(\Im)$ with $|\psi^0(\cdot)| \leq 1$ almost everywhere in $\Im$ and $\Gamma^0 \in H^1(\Im) \cap L^2(\Im)$. Then, for all $\Delta t \leq \frac{\gamma_q(1-\theta)}{1+\alpha s\gamma_q}$, where $\theta \in (0,1)$, the problem $(\Lambda^{h,\Delta t})$ has a solution Ψ^n, Γ^n, $n = 1,\dots,N$, satisfying that

$$\max_{m=1,\dots,N}[\|\Psi^n\|_\hbar^2 + \|\nabla\Psi^n\|_0^2 + \|\Gamma^n\|_\hbar^2] + \sum_{n=1}^{m}\Delta t\|\nabla\Psi^n\|_0^2 + \sum_{n=1}^{N}\Delta t\|\Gamma^n\|_\hbar^2$$
$$+ \sum_{n=1}^{m}[\|\Psi^n - \Psi^{n-1}\|_\hbar^2 + \|\nabla(\Psi^n - \Psi^{n-1})\|_\hbar^2 + \|\Gamma^n - \Gamma^{n-1}\|_\hbar^2] \leq C.$$

Proof. Selecting $\Upsilon = \alpha\Delta t\Psi^n$ and $\Upsilon = \frac{\Delta t}{\gamma_q}\Gamma^n$ in (56) and (57), respectively, we derive that

$$(\Psi^n - \Psi^{n-1},\ \Psi^n)^\hbar + \alpha\Delta t(\nabla\Psi^n,\ \nabla\Psi^n) = \alpha\Delta t s(1,\Psi^n)^\hbar, \quad (58)$$

$$(\Gamma^n - \Gamma^{n-1},\ \Gamma^n) + \frac{\Delta t}{\gamma_q}(\Gamma^n,\ \Gamma^n) = \frac{\Delta t}{\gamma_q}(\Psi^n,\ \Gamma^n). \quad (59)$$

By inserting (58) into (59), we find that

$$(\Psi^n - \Psi^{n-1},\ \Psi^n)^\hbar + (\Gamma^n - \Gamma^{n-1},\ \Gamma^n) + \frac{\Delta t}{\gamma_q}(\Gamma^n,\ \Gamma^n) + \alpha\Delta t(\nabla\Psi^n,\ \nabla\Psi^n)$$
$$= \alpha\Delta t s(1,\Psi^n)^\hbar + \frac{\Delta t}{\gamma_q}\Gamma^n(\Psi^n,\ \Gamma^n), \quad (60)$$

Utilizing the subsequent straightforward identity,

$$2x(x-y) = x^2 - y^2 + (x-y)^2 \qquad \text{for all } x\,, y \in R, \tag{61}$$

Hölder and Young's inequalities, and (59)–(60), we find that

$$[1 - \Delta t(\frac{1}{\gamma_q} + \alpha s)]\|\Psi^n\|_\hbar^2 + \|\Gamma^n\|_\hbar^2 + \|\Psi^n - \Psi^{n-1}\|_\hbar^2$$
$$+ \|\Gamma^n - \Gamma^{n-1}\|_\hbar^2 + 2\alpha\Delta t\|\nabla\Psi^n\|_0^2 + \frac{\Delta t}{\gamma_q}|\Gamma^n|_\hbar^2$$
$$\leq \|\Psi^{n-1}\|_\hbar^2 + \|\Gamma^{n-1}\|_\hbar^2 + C(s, \Delta t, \alpha, |\Im|).$$

Subsequently, it can be concluded that

$$[1 - \Delta t(\frac{1+\alpha\gamma_q s}{\gamma_q})][\|\Psi^n\|_\hbar^2 + \|\Gamma^n\|_\hbar^2] + \|\Psi^n - \Psi^{n-1}\|_\hbar^2$$
$$+ \|\Gamma^n - \Gamma^{n-1}\|_\hbar^2 + 2\alpha\Delta t\|\nabla\Psi^n\|_0^2 + \frac{\Delta t}{\gamma_q}|\Gamma^n|_\hbar^2$$
$$\leq \|\Psi^{n-1}\|_\hbar^2 + \|\Gamma^{n-1}\|_\hbar^2 + C(s, \Delta t, \alpha, |\Im|). \tag{62}$$

Since $\Psi^n, \Gamma^n \geq 0, \quad \Delta t \leq \frac{\gamma_q(1-\theta)}{1+\alpha s\gamma_q}$, thus we have $\theta \leq 1 - \Delta t(\frac{1+\alpha\gamma_q s}{\gamma_q})$. Then we can find that

$$\frac{1}{1 - \Delta t(\frac{1+\alpha\gamma_q s}{\gamma_q})} = 1 + \frac{\Delta t(\frac{1+\alpha\gamma_q s}{\gamma_q})}{1 - \Delta t(\frac{1+\alpha\gamma_q s}{\gamma_q})} \leq 1 + \frac{\Delta t(1+\alpha\gamma_q s)}{\gamma_q\theta}.$$

It is deduced from (62) that

$$\|\Psi^n\|_\hbar^2 + \|\Gamma^n\|_\hbar^2 + 2\alpha\Delta t\|\nabla\Psi^n\|_0^2 + \frac{\Delta t}{\gamma_q}\|\Gamma^n\|_\hbar^2 + \|\Psi^n - \Psi^{n-1}\|_\hbar^2 + \|\Gamma^n - \Gamma^{n-1}\|_\hbar^2$$
$$\leq \frac{1}{1 - \Delta t(\frac{1+\alpha\gamma_q s}{\gamma_q})}[\|\Psi^{n-1}\|_\hbar^2 + \|\Gamma^{n-1}\|_\hbar^2 + C]$$
$$\leq 1 + \frac{\Delta t(1+\alpha\gamma_q s)}{\gamma_q\theta}[\|\Psi^{n-1}\|_\hbar^2 + \|\Gamma^{n-1}\|_\hbar^2 + C].$$

Summing the given equation over $n = 1, \ldots, m$ for $m \leq N$ and incorporating assumptions about initial conditions, lead to

$$\max_{m=1,\ldots,N}[\|\Psi^m\|_\hbar^2 + \|\Gamma^m\|_\hbar^2] + 2\alpha\sum_{n=1}^{m}\Delta t\|\nabla\Psi^n\|_0^2 + \frac{1}{\gamma_q}\sum_{n=1}^{N}\Delta t|\Gamma^n|_\hbar^2$$
$$+ \sum_{n=1}^{m}[\|\Psi^n - \Psi^{n-1}\|_\hbar^2 + \|\Gamma^n - \Gamma^{n-1}\|_\hbar^2] \leq e^{\frac{\Re(1+\alpha\gamma_q s)}{\gamma_q\theta}}[|\Psi^0|_\hbar^2 + |\Gamma^0|_\hbar^2 + C] \leq C.$$

Opting for $\Upsilon = \alpha \frac{\Psi^n - \Psi^{n-1}}{\Delta t}$ in (56), we deduce that

$$|\frac{\Psi^n - \Psi^{n-1}}{\Delta t}|_\hbar^2 + \frac{1}{\Delta t}(\nabla\Psi^n,\ \nabla(\Psi^n - \Psi^{n-1})) = \frac{s}{\Delta t}(1, \Psi^n - \Psi^{n-1})^\hbar.$$

Utilizing Hölder and Young's inequalities, along with (61), we obtain that

$$|\frac{\Psi^n - \Psi^{n-1}}{\Delta t}|_\hbar^2 + \frac{\alpha}{\Delta t}|\nabla\Psi^n|_\hbar^2 + \frac{\alpha}{\Delta t}|\nabla(\Psi^n - \Psi^{n-1})|_\hbar^2 \le \frac{\alpha}{\Delta t}|\nabla\Psi^{n-1}|_\hbar^2 + C(\Delta t, |\Im|, s\alpha).$$

Summing the aforementioned equation over $n = 1, \ldots, m$, where $m \le N$ and employing assumptions regarding initial conditions, result in

$$\max_{m=1,\ldots,N}[\|\nabla\Psi^m\|_\hbar^2] + \sum_{n=1}^{m}[|\frac{\Psi^n - \Psi^{n-1}}{\Delta t}|_\hbar^2 + \frac{\alpha}{\Delta t}|\nabla(\Psi^n - \Psi^{n-1})|_\hbar^2]$$
$$\le \frac{\alpha}{\Delta t}|\nabla\Psi^0|_\hbar^2 + C(\Delta t, |\Im|, s, \alpha) \le C.$$

□

4.1 Existence of the approximation

Following the methodology similar to what is described in [12, 13, 14, 15, 18, 16, 19, 2, 17, 3], we introduce the functions as follows: $A_\psi : S^\hbar \times S^\hbar \to S^\hbar$ and $A_\gamma : S^\hbar \times S^\hbar \to S^\hbar$ such that for all $\Upsilon \in S^\hbar$, we have

$$(A_\psi(\Psi,\Gamma),\ \Upsilon)^\hbar = (\Psi - \Psi^{n-1},\ \Upsilon)^\hbar + \alpha\Delta t(\nabla\Psi,\ \nabla\Upsilon) - s\alpha\Delta t(1,\ \Upsilon), \quad (63)$$

$$(A_\gamma(\Psi,\Gamma),\ \Upsilon) = (\Gamma - \Gamma^{n-1},\ \Upsilon) + \frac{\Delta t}{\gamma_q}(\Gamma,\ \Upsilon) - \frac{\Delta t}{\gamma_q}(\Psi,\ \Upsilon), \quad (64)$$

respectively. We initially observe that the continuous piecewise linear functions A_ψ and A_γ are distinctly determinable based on their values at the nodal points $\mathcal{N}^\hbar$. This uniqueness becomes evident when we set $\Upsilon \equiv \kappa_j$, where $j = 0, \ldots, J$, in (63) and (64). Subsequently, we derive solvable square matrix systems as follows:

$$\widehat{M}A_\psi(\Psi,\Gamma) = S_1, \qquad \widehat{M}A_\gamma(\Psi,\Gamma) = S_2,$$

where $\widehat{M}$ is the lumped mass matrix introduced and S_1 and S_2 are given vectors in terms of the nodal values of Ψ, Γ, Ψ^{n-1}, and Γ^{n-1}. Thus, the functions A_ψ and A_γ are well defined.

By considering (63) and (64), the problem $(\Lambda^{h,\Delta t})$ can be reformulated as
For given $\{\Psi^0, \Gamma^0\} \in S^\hbar \times S^\hbar$, find $\{\Psi^n, \Gamma^n\} \in S^\hbar \times S^\hbar$, $n \geq 1$ such that

$$A_\psi(\Psi, \Gamma) = 0, \qquad A_\gamma(\Psi, \Gamma) = 0.$$

Lemma 1. For any given $R > 0$, the functions $A_\psi : [S^\hbar]^2 R \to S^\hbar$ and $A_\gamma : [S^\hbar]^2_R \to S^\hbar$ exhibit continuity in the following manner

$$[S^\hbar]^2_R = \left\{ \{\Upsilon_1, \Upsilon_2\} \in S^\hbar \times S^\hbar : |\Upsilon_1|^2_\hbar + |\Upsilon_2|^2_\hbar \leq R^2 \right\}.$$

Proof. Consider $\Psi_1, \Gamma_1, \Psi_2, \Gamma_2 \in [S^\hbar]^2$. From (63), for all $\Upsilon \in S^\hbar$, we have

$$(A_\psi(\Psi_1, \Gamma_1) - A_\psi(\Psi_2, \Gamma_2),\ \Upsilon)^\hbar = (\Psi_1 - \Psi_2,\ \Upsilon)^\hbar + \alpha \Delta t (\nabla \Psi_1 - \nabla \Psi_2,\ \nabla \Upsilon). \tag{65}$$

Choosing $\Upsilon = A_\Psi(\Psi_1, \Gamma_1) - A_\Psi(\Psi_2, \Gamma_2)$ in (65) yields on noting the Cauchy–Schwarz inequality, (**??**), and (11), that

$$\begin{aligned}
&|A_\psi(\Psi_1, \Gamma_1) - A_\psi(\Psi_2, \Gamma_2)|^2_\hbar = (\Psi_1 - \Psi_2,\ A_\gamma(\Psi_1, \Gamma_1) - A_\psi(\Psi_2, \Gamma_2))^\hbar \\
&+ \Delta t \alpha (\nabla \Psi_1 - \nabla \Psi_2,\ \nabla (A_\psi(\Psi_1, \Gamma_1) - A_\psi(\Psi_2, \Gamma_2))) \\
&\quad \leq |\Psi_1 - \Psi_2|_\hbar |A_\psi(\Psi_1, \Gamma_1) - A_\psi(\Psi_2, \Gamma_2|_\hbar \\
&\qquad + \Delta t \alpha |\Psi_1 - \Psi_2|_1 |A_\psi(\Psi_1, \Gamma_1) - A_\psi(\Psi_2, \Gamma_2|_1 \\
&\quad \leq |\Psi_1 - \Psi_2|_\hbar |A_\psi(\Psi_1, \Gamma_1) - A_\psi(\Psi_2, \Gamma_2|_\hbar \\
&\qquad + C\hbar^{-1} \Delta t \alpha [|\Psi_1 - \Psi_2|_\hbar |A_\psi(\Psi_1, \Gamma_1) - A_\psi(\Psi_2, \Gamma_2|_\hbar] \\
&\qquad + (1 + C\hbar^{-1} \Delta t \alpha) [|\Psi_1 - \Psi_2|_\hbar |A_\psi(\Psi_1, \Gamma_1) - A_\psi(\Psi_2, \Gamma_2|_\hbar].
\end{aligned} \tag{66}$$

Based on (66), it follows that

$$|A_\psi(\Psi_1, \Gamma_1) - A_\psi(\Psi_2, \Gamma_2)|_\hbar \leq C(\hbar^{-1}, \Delta t, \alpha) |\Psi_1 - \Psi_2|_\hbar.$$

From (64), we deduce for all $\Upsilon \in S^\hbar$ that

$$(A_u(\Psi_1,\Gamma_1)-A_\gamma(\Psi_2,\Gamma_2),\Upsilon)=(\Gamma_1-\Gamma_2,\Upsilon)+\frac{\Delta t}{\gamma_q}(\Gamma_1-\Gamma_2,\Upsilon)-\frac{\Delta t}{\gamma_q}(\Psi_1-\Psi_2,\Upsilon). \tag{67}$$

By defining $\Upsilon = A_\gamma(\Psi_1,\Gamma_1) - A_\gamma(\Psi_2,\Gamma_2)$ and utilizing the Cauchy–Schwarz inequality along with (11), we obtain

$$\begin{aligned}
&\|A_\gamma(\Psi_1,\Gamma_1) - A_\gamma(\Psi_2,\Gamma_2)\|_0^2 \\
&\le (1+\frac{\Delta t}{\gamma_q})\|\Gamma_1-\Gamma_2\|_0\|A_\gamma(\Psi_1,\Gamma_1) - A_\gamma(\Psi_2,\Gamma_2)\|_0 \\
&\quad + \frac{\Delta t}{\gamma_q}\|\Psi_1-\Psi_2\|_0\|A_\gamma(\Psi_1,\Gamma_1) - A_\gamma(\Psi_2,\Gamma_2)\|_0.
\end{aligned}$$

By employing (11), it follows that

$$\begin{aligned}
|A_\gamma(\Psi_1,\Gamma_1) - A_\gamma(\Psi_2,\Gamma_2)|_\hbar^2 \le & C|\Gamma_1-\Gamma_2|_\hbar|A_\gamma(\Psi_1,\Gamma_1) - A_\gamma(\Psi_2,\Gamma_2)|_\hbar \\
&+ C|\Psi_1-\Psi_2|_\hbar|A_\gamma(\Psi_1,\Gamma_1) - A_\gamma(\Psi_2,\Gamma_2)|_\hbar.
\end{aligned}$$

Hence, we can conclude that

$$|A_\gamma(\Psi_1,\Gamma_1) - A_\gamma(\Psi_2,\Gamma_2)|_\hbar \le C[|\Gamma_1-\Gamma_2|_\hbar + |\Psi_1-\Psi_2|_\hbar]. \tag{68}$$

The outcomes (67) and (68) demonstrate that A_ψ and A_γ, respectively, are Lipschitz continuous. □

Theorem 4. Let $\{\Psi^{n-1},\Gamma^{n-1}\} \in S^\hbar \times S^\hbar$ be given solution to the $(n-1)$th step of $(\Lambda^{h,\Delta t})$ for some $n = 1,2,\dots,N$. Then, for all $h > 0$, and for all $\Delta t \le \frac{1+2\alpha s\gamma_q}{4\gamma_q}$, there exists a solution $\{\Psi^n,\Gamma^n\} \in [S^\hbar]^2_R$ to the nth step of $(\Lambda^{h,\Delta t})$.

Proof. By way of contradiction, assume that for $R > 0$, there is no $\Psi^n,\Gamma^n \in S^\hbar \times S^\hbar$ such that $A_\psi(\Psi,\Gamma) = A_\gamma(\Psi,\Gamma) = 0$. Noting the continuity of $A_\psi(\Psi,\Gamma)$ and $A_\gamma(\Psi,\Gamma)$ on $[S^\hbar]^2R$, we define the continuous function $\mathtt{B} : [S^\hbar]^2R \to [S^\hbar]^2_R$ as

$$\mathtt{B}(\Psi,\Gamma) = (\mathtt{B}_\psi(\Psi,\Gamma),\mathtt{B}_\gamma(\Psi,\Gamma)),$$

where $\mathtt{B}_\psi(\Psi,\Gamma)$ and $\mathtt{B}_\gamma(\Psi,\Gamma)$ are given by

$$\begin{aligned} \mathtt{B}_\psi(\Psi,\Gamma) &:= \frac{-RA_\psi(\Psi,\Gamma)}{|(A_\psi(\Psi,\Gamma), A_\gamma(\Psi,\Gamma))|_{S^\hbar\times S^\hbar}}, \\ \mathtt{B}_\gamma(\Psi,\Gamma) &:= \frac{-RA_\psi(\Psi,\Gamma)}{|(A_\psi(\Psi,\Gamma), A_\gamma(\Psi,\Gamma))|_{S^\hbar\times S^\hbar}}. \end{aligned} \tag{69}$$

where $|(\cdot,\cdot)|_{[S^\hbar]^2_R}$ is the standard norm on $[S^\hbar]^2_R$ defined by

$$|(\mathtt{X}_1,\mathtt{X}_2)|_{S^\hbar\times S^\hbar} = (\sum_{i=1}^{2} |\mathtt{X}_i|_\hbar^2)^{\frac{1}{2}}.$$

Observing the continuity of A_ψ and A_γ, as indicated in Lemma 1, we can deduce that the function $\mathtt{B}$ is continuous. Therefore, considering that $[S^\hbar]^2R$ is a convex and compact subset of $S^\hbar \times S^\hbar$, Schauder's theorem implies the existence of $\Psi,\Gamma \in [S^\hbar]^2R$ as a fixed point of $\mathtt{B}$, i.e.,

$$\mathtt{B}(\Psi,\Gamma) = (\mathtt{B}_\psi(\Psi,\Gamma), \mathtt{B}_\gamma(\Psi,\Gamma)) = (\Psi,\Gamma).$$

Additionally, we observe from (69) that the fixed point Ψ,Γ satisfies

$$|\Psi|_\hbar^2 + \|\Gamma\|_0^2 = \|\mathtt{B}_\psi(\Psi,\Gamma)\|_0^2 + |\mathtt{B}_\gamma(\Psi,\Gamma)|_\hbar^2 = (\Psi,\Gamma) = R^2. \tag{70}$$

To establish a contradiction for R sufficiently large, we select $\Upsilon \equiv \Psi$ in (63) and $\Upsilon \equiv \Gamma$ in (64), leading to the determination that

$$(A_\psi(\Psi,\Gamma),\ \Psi)^\hbar = (\Psi - \Psi^{n-1},\ \Psi)^\hbar + \alpha\Delta t(\nabla\Psi,\ \nabla\Psi) - s\alpha\Delta t(1,\ \Psi)^\hbar, \tag{71}$$

$$(A_\gamma(\Psi,\Gamma),\ \Gamma) = (\Gamma - \Gamma^{n-1},\ \Gamma) + \frac{\Delta t}{\gamma_q}(\Gamma,\ \Gamma) - \frac{\Delta t}{\gamma_q}(\Psi,\ \Gamma). \tag{72}$$

By merging (71) and (72) and considering (61), (11), as well as Hölder and Young's inequalities, we obtain, for R sufficiently large, that

$$\begin{aligned} &(A_\psi(\Psi,\Gamma),\ \Psi)^\hbar + (A_\psi(\Psi,\Gamma),\ \Gamma)^\hbar \\ &= (\Psi - \Psi^{n-1},\ \Psi)^\hbar + (\Gamma - \Gamma^{n-1},\ \Gamma)^\hbar + \alpha\Delta t(\nabla\Psi,\ \nabla\Psi) \\ &\quad + \frac{\Delta t}{\gamma_q}(\Gamma,\ \Gamma) - s\alpha\Delta t(1,\ \Psi) - \frac{\Delta t}{\gamma_q}(\Psi,\ \Gamma) \\ &\geq \frac{1}{2}|\Psi|_\hbar^2 + \frac{1}{2}|\Psi^{n-1}|_\hbar^2 + \frac{1}{2}\|\Psi\|_0^2 + \frac{1}{2}\|\Gamma^{n-1}\|_0^2 + \alpha\Delta t|\Psi|_1^2 \\ &\quad + \frac{\Delta t}{\gamma_q}\|\Gamma\|_0^2 - \frac{\Delta t}{2\gamma_q}\|\Psi\|_0^2 - s\alpha\Delta t|\Psi|_\hbar^2 \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2}|\Psi|_{\hbar}^2 + \frac{1}{2}|\Psi^{n-1}|_{\hbar}^2 + \frac{1}{2}|\Gamma|_{\hbar}^2 + \frac{1}{2}|\Gamma^{n-1}|_{\hbar}^2 + \alpha\Delta t|\Psi|_1^2 \\
&\quad + \frac{\Delta t}{\gamma_q}|\Gamma|_{\hbar}^2 - \frac{\Delta t}{2\gamma_q}|\Psi|_{\hbar}^2 - s\alpha\Delta t|\Psi|_{\hbar}^2 \\
&\geq (\frac{1}{2} - \frac{\Delta t}{2\gamma_q} - s\alpha\Delta t)|\Psi|_{\hbar}^2 + (\frac{1}{2} + \frac{\Delta t}{\gamma_q})|\Gamma|_{\hbar}^2 + \alpha\Delta t|\Psi|_1^2 + \frac{1}{2}|\Psi^{n-1}|_{\hbar}^2 + \frac{1}{2}|\Gamma^{n-1}|_{\hbar}^2 \\
&\geq \min\{(\frac{1}{2} - \frac{\Delta t}{2\gamma_q} - s\alpha\Delta t), (\frac{1}{2} + \frac{\Delta t}{\gamma_q})\}R^2 + \alpha\Delta t|\Psi|_1^2 + C(\Psi^{n-1}, \Gamma^{n-1}) \\
&\geq 0. \qquad (73)
\end{aligned}
$$

Observing that Ψ, Γ is a fixed point of the function $\mathtt{B}$, and taking into account (69) and (73), it follows, for R sufficiently large, that

$$
(\Psi,\ \Psi)^{\hbar} + (\Gamma,\ \Gamma) = \frac{-R[(A_{\psi}(\Psi,\Gamma),\Psi)^{\hbar} + (A_{\gamma}(\Psi,\Gamma),\Gamma)]}{|(A_{\psi}(\Psi,\Gamma), A_{\gamma}(\Psi,\Gamma)|_{S^{\hbar}\times S^{\hbar}}} < 0. \qquad (74)
$$

It comes from (70) that

$$
(\Psi,\ \Psi)^{\hbar} + (\Gamma,\ \Gamma) = |\Psi|_{\hbar}^2 + \|\Gamma\|_0^2 \geq C[|\Psi|_{\hbar}^2 + |\Gamma|_{\hbar}^2] \geq R^2 \geq 0,
$$

which contradicts (74). The contradiction establishes the existence of $\Psi^n, \Gamma^n \in S^{\hbar} \times S^h$ such that $A_{\psi}(\Psi^n, \Gamma^n) = A_{\gamma}(\Psi^n, \Gamma^n) = 0$. In other words, it confirms the existence of a solution Ψ^n, Γ^n for the nth step of $(\Lambda^{h,\Delta t})$. □

4.2 Uniqueness of approximation

By selecting $\Upsilon = \Psi = \Psi_1^n - \Psi_2^n$ in (56) and $\Upsilon = \Gamma = \Gamma_1^n - \Gamma_2^n$ in (57), and subsequently subtracting the corresponding fully-discrete approximations for both Ψ and Γ, we arrive at the following equations:

$$
\frac{1}{\Delta t\alpha}(\Psi,\Psi)^{\hbar} + (\nabla\Psi,\ \nabla\Psi) = 0, \qquad (75)
$$

$$
\frac{\gamma_q}{\Delta t}(\Gamma,\Gamma) + (\Gamma,\Gamma) = (\Psi,\Gamma). \qquad (76)
$$

Multiplying equation (75) by $\Delta t\alpha$ and combining this result with equation (76) results in

$$
|\Psi|_{\hbar}^2 + \Delta t\alpha|\Psi|_1^2 + \frac{\gamma_q}{\Delta t}\|\Gamma\|_0^2 + \|\Gamma\|_0^2 = (\Psi,\Gamma). \qquad (77)
$$

By using Young's inequality and (11), we have

$$\frac{1}{2}|\Psi|_\hbar^2 + \frac{\gamma_q}{\Delta t}\|\Gamma\|_0^2 \leq 0.$$

This leads to the conclusion that $\Psi_1^n = \Psi_2^n$ and $\Gamma_1^n - \Gamma_2^n$ for all $n \geq 1$, as needed.

4.3 Existence of weak solution

Firstly, let us introduce the following definitions:

$$\Psi(t) := (\frac{t-t_{n-1}}{\Delta t})\Psi^n + (\frac{t_n - t}{\Delta t})\Psi^{n-1}, \quad t \in [t_{n-1}, t_n] \quad n \geq 1, \tag{78}$$

$$\Gamma(t) := (\frac{t-t_{n-1}}{\Delta t})\Gamma^n + (\frac{t_n - t}{\Delta t})\Gamma^{n-1}, \quad t \in [t_{n-1}, t_n] \quad n \geq 1, \tag{79}$$

and

$$\Psi^+(t) := \Psi^n, \quad \Psi^-(t) := \Psi^{n-1}, \quad t \in (t_{n-1}, t_n], \quad n \geq 1, \tag{80}$$

$$\Gamma^+(t) := \Gamma^n, \quad \Gamma^-(t) := \Gamma^{n-1}, \quad t \in (t_{n-1}, t_n], \quad n \geq 1. \tag{81}$$

Considering (78), (79), along with (80) and (81), we derive that

$$\frac{\partial \Psi}{\partial t} = \frac{\Psi^+ - \Psi^-}{\Delta t} = \frac{\Psi^+ - \Psi}{t_n - t} = \frac{\Psi - \Psi^-}{t - t_{n-1}}, \quad t \in (t_{n-1}, t_n) \quad n \geq 1, \tag{82}$$

$$\frac{\partial \Gamma}{\partial t} = \frac{\Gamma^+ - \Gamma^-}{\Delta t} = \frac{\Gamma^+ - \Gamma}{t_n - t} = \frac{\Gamma - \Gamma^-}{t - t_{n-1}}, \quad t \in (t_{n-1}, t_n) \quad n \geq 1. \tag{83}$$

Leveraging the aforementioned information, we can reformulate the problem $(\Lambda^{h,\Delta t})$ in the following manner:

$$\int_0^{\Re} (\frac{\partial \Psi}{\partial t},\ X)^\hbar\, dt + \alpha \int_0^{\Re} (\nabla \Psi^+,\ \nabla X)\, dt = \alpha s \int_0^{\Re} (1,\ X)^\hbar\, dt, \tag{84}$$

$$\int_0^{\Re} (\frac{\partial \Gamma}{\partial t},\ X)\, dt + \frac{1}{\gamma_q} \int_0^{\Re} (\Gamma^+,\ X)\, dt = \frac{1}{\gamma_q} \int_0^{\Re} (\Psi^+,\ X)\, dt. \tag{85}$$

Theorem 5. Assuming that the conditions of Theorem 3 are met, there exists a subsequence of $\Psi^\pm, \Gamma^\pm$ that solves (84) and (85) as $\hbar \to 0$ such that

$$\Psi, \Psi^\pm \rightharpoonup^* \psi \qquad \textit{and} \qquad \Gamma, \Gamma^\pm \rightharpoonup^* \gamma \qquad \textit{in} \quad L^\infty(0, \Re; L^2(\Im)), \tag{86}$$

$$\begin{aligned} \Psi, \Psi^{\pm} &\rightharpoonup^{*} \psi \qquad in \quad L^{\infty}(0,\Re; H^1(\Im)), \\ \Psi, \Psi^{\pm} &\rightharpoonup \psi \qquad in \quad L^{2}(0,\Re; H^1(\Im)), \\ \Gamma, \Gamma^{\pm} &\to \gamma \qquad in \quad L^2(\Im_{\Re}) \end{aligned} \tag{87}$$

$$\frac{\partial \Psi}{\partial t} \to \frac{\partial \psi}{\partial t} \qquad and \qquad \frac{\partial \Gamma}{\partial t} \to \frac{\partial \gamma}{\partial t} \qquad in \quad L^2(\Im_{\Re}). \tag{88}$$

Proof. The aforementioned convergence results stem from the bounds provided in (58), taking into account that the spaces $L^2(0,\Re;H^1(\Im))$ and $L^2(\Im_{\Re})$ are reflexive Banach spaces. Additionally, the spaces $L^{\infty}(0,\Re;H^1(\Im))$ and $L^{\infty}(0,\Re;L^2(\Im))$ serve as the dual spaces of $L^1(0,\Re;(H^1(\Im))')$ and $L^1(0,\Re;L^2(\Im))$, respectively. While these dual spaces are separable Banach spaces, they are not reflexive. □

Theorem 6. Assuming that the conditions of Theorem 5 are satisfied, the functions Ψ, Γ constitute a global weak solution in the following sense, for all $\Upsilon \in L^2(0,\Re;H^1(\Im))$:

$$\int_0^{\Re} (\frac{\partial \psi}{\partial t},\ \Upsilon)^{\hbar}\, dt + \alpha \int_0^{\Re} (\nabla \psi,\ \nabla \Upsilon)\, dt = \alpha s \int_0^{\Re} (1,\ \Upsilon)^{\hbar}\, dt, \tag{89}$$

$$\int_0^{\Re} (\frac{\partial \gamma}{\partial t},\ \Upsilon)\, dt + \frac{1}{\gamma_q} \int_0^{\Re} (\gamma,\ \Upsilon)\, dt = \frac{1}{\gamma_q} \int_0^{\Re} (\psi,\ \Upsilon)\, dt. \tag{90}$$

Proof. If we substitute $X \equiv \Pi^{\hbar}\Upsilon$ into (84), then we obtain that

$$\int_0^{\Re} (\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}\Upsilon)^{\hbar}\, dt + \alpha \int_0^{\Re} (\nabla \Psi^{+},\ \nabla \Pi^{\hbar}\Upsilon)\, dt = \alpha s \int_0^{\Re} (1,\ \Pi^{\hbar}\Upsilon)^{\hbar}\, dt.$$

For any $\Upsilon \in L^2(0,\Re;H^1(\Im))$ and $\widetilde{\Upsilon} \in H^1(0,\Re;H^1(\Im))$, we have

$$\begin{aligned} \int_0^{\Re} (\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}\Upsilon)^{\hbar} dt = & \int_0^{\Re} \left[\left(\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}(\Upsilon - \widetilde{\Upsilon}) \right)^{\hbar} - \left(\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}(\Upsilon - \widetilde{\Upsilon}) \right) \right] dt \\ & + \int_0^{\Re} \left[\left(\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}\widetilde{\Upsilon} \right)^{\hbar} - \left(\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}\widetilde{\Upsilon} \right) \right] dt \\ & + \int_0^{\Re} \left(\frac{\partial \Psi}{\partial t},\ (\Pi^{\hbar} - I)\Upsilon \right) dt \\ & + \int_0^{\Re} \left(\frac{\partial \Psi}{\partial t},\ \Upsilon \right) dt \end{aligned}$$

$$:= K_{1,1} + K_{1,2} + K_{1,3} + K_{1,4}. \tag{91}$$

Utilizing (15), (13), (14), Hölder's inequality, and (58), provides that

$$\begin{aligned}
|K_{1,1}| &\equiv \left| \int_0^{\Re} \left[\left(\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}(\Upsilon - \widetilde{\Upsilon}) \right)^{\hbar} - \left(\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}(\Upsilon - \widetilde{\Upsilon}) \right) \right] dt \right| \\
&\leq \int_0^{\Re} \left| \left[\left(\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}(\Upsilon - \widetilde{\Upsilon}) \right)^{\hbar} - \left(\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}(\Upsilon - \widetilde{\Upsilon}) \right) \right] \right| dt \\
&\leq Ch \int_0^{\Re} \| \frac{\partial \Psi}{\partial t} \|_0 \quad |\Pi^{\hbar}(\Upsilon - \widetilde{\Upsilon})|_1 \, dt \\
&\leq C\hbar \| \frac{\partial \Psi}{\partial t} \|_{L^2(\Im_{\Re})} \| \Upsilon - \widetilde{\Upsilon} \|_{L^2(0,\Re;H^1(\Im))} \\
&\leq C\hbar \| \Upsilon - \widetilde{\Upsilon} \|_{L^2(0,\Re;H^1(\Im))} \quad \to 0 \ \ as \ \ \hbar \to 0.
\end{aligned} \tag{92}$$

It can be deduced from (15), (13), (14), Hölder's inequality, and (86) that

$$\begin{aligned}
|K_{1,2}| &\equiv \left| \int_0^{\Re} \left[\left(\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar} \widetilde{\Upsilon} \right)^{\hbar} - \left(\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar} \widetilde{\Upsilon} \right) \right] dt \right| \\
&\leq \int_0^{\Re} \left| \left(\Psi,\ \frac{\partial (\Pi^{\hbar} \widetilde{\Upsilon})}{\partial t} \right)^{\hbar} - \left(\Psi,\ \frac{\partial (\Pi^{\hbar} \widetilde{\Upsilon})}{\partial t} \right) \right| dt \\
&\quad + \left| \left(\Psi(\cdot, \Re),\ \Pi^{\hbar} \widetilde{\Upsilon}(\cdot, \Re) \right)^{\hbar} - \left(\Psi(\cdot,\ \Re), \Pi^{\hbar} \widetilde{\Upsilon}(\cdot,\ \Re) \right) \right| \\
&\quad + \left| \left(\Psi(\cdot, 0),\ \Pi^{\hbar} \widetilde{\Upsilon}(\cdot, 0) \right)^{\hbar} - \left(\Psi(\cdot,\ 0), \Pi^{\hbar} \widetilde{\Upsilon}(\cdot,\ 0) \right) \right| \\
&\leq C\hbar \int_0^{\Re} \|\Psi\|_0 | \frac{\partial (\Pi^{\hbar} \widetilde{\Upsilon})}{\partial t} |_1 \, dt + Ch \|\Psi(\cdot, \Re)\|_0 |\Pi^{\hbar} \widetilde{\Upsilon}(\cdot, \Re)|_1 \\
&\quad + Ch \|\Psi(\cdot, 0)\|_0 |\Pi^{\hbar} \widetilde{\Upsilon}(\cdot, 0)|_1 \\
&\leq Ch \|\Psi\|_{L^\infty(0,\Re;L^2(\Im))} \|\widetilde{\Upsilon}\|_{H^1(0,\Re;H^1(\Im))} \\
&\leq Ch \|\widetilde{\Upsilon}\|_{H^1(0,\Re;H^1(\Im))} \quad \to \quad 0 \quad as \quad \hbar \to 0.
\end{aligned} \tag{93}$$

To analyze the term $K_{1,3}$, we proceed by utilizing (16), applying Hölder's inequality, and considering (88), resulting in

$$\begin{aligned}
|K_{1,3}| \equiv | \int_0^{\Re} (\frac{\partial \Psi}{\partial t},\ (\Pi^{\hbar} - I)\Upsilon) \, dt | = | \int_0^{\Re} \langle \frac{\partial \Psi}{\partial t},\ (\Pi^{\hbar} - I)\Upsilon \rangle \, dt | \\
\leq \| \frac{\partial \Psi}{\partial t} \|_{L^2(\Im_{\Re})} \| (\Pi^{\hbar} - I)\Upsilon \|_{L^2(0,\Re;H^1(\Im))}
\end{aligned}$$

$$\leq C\|(\Pi^{\hbar}-I)\Upsilon\|_{L^2(0,\Re;H^1(\Im))}. \tag{94}$$

From (16) and the outcome of weak convergence as stated in (88), it can be concluded for any $\Upsilon \in L^2(0,\Re;H^1(\Im))$ that

$$K_{1,4} \equiv \int_0^{\Re} (\frac{\partial \Psi}{\partial t},\ \Upsilon)\, dt = \int_0^{\Re} \langle \frac{\partial \Psi}{\partial t},\ \Upsilon\rangle dt \to \int_0^{\Re} \langle \frac{\partial \psi}{\partial t},\ \Upsilon\rangle\, dt \quad as \quad \hbar \to 0. \tag{95}$$

Combining (91)–(95) and the denseness of $H^1(0,\Re;H^1(\Im))$ in $L^2(0,\Re;H^1(\Im))$ yields for all $\Upsilon \in L^2(0,\Re;H^1(\Im))$, that

$$\int_0^{\Re} (\frac{\partial \Psi}{\partial t},\ \Pi^{\hbar}\Upsilon)^{\hbar} dt \to \int_0^{\Re} \langle \frac{\partial \psi}{\partial t},\ \Upsilon\rangle\, dt \quad as \quad \hbar \to 0.$$

By applying Hölder's inequality, considering (14), and taking into account (87), we obtain the following outcome for any $\Upsilon \in L^2(0,\Re;H^1(\Im))$

$$\begin{aligned}
\int_0^{\Re} (\nabla\Psi^+,\ \nabla\Pi^{\hbar}\Upsilon)\, dt &= \int_0^{\Re} (\nabla\Psi^+,\ \nabla(\Pi^{\hbar}-I)\Upsilon) dt + \int_0^{\Re} (\nabla\Psi^+,\ \nabla\Upsilon) dt \\
&\leq \left| \int_0^{\Re} (\nabla\Psi^+,\ \nabla(\Pi^{\hbar}-I)\Upsilon) dt + \int_0^{\Re} (\nabla\Psi^+,\ \nabla\Upsilon) dt \right| \\
&\leq \int_0^{\Re} |(\nabla\Psi^+,\ \nabla(\Pi^{\hbar}-I)\Upsilon)| dt + \int_0^{\Re} |(\nabla\Psi^+,\ \nabla\Upsilon)| dt \\
&\leq \int_0^{\Re} |\Psi^+|_1 |(\Pi^{\hbar}-I)\Upsilon|_1\, dt + \int_0^{\Re} |(\nabla\Psi^+,\ \nabla\Upsilon)| dt \\
&\leq \|\Psi^+\|_{L^2(0,\Re;H^1(\Im))} \|(\Pi^{\hbar}-I)\Upsilon\|_{L^2(0,\Re;H^1(\Im))} \\
&\quad + \int_0^{\Re} |(\nabla\Psi^+,\ \nabla\Upsilon)| dt \\
&\to \int_0^{\Re} (\nabla\psi,\ \nabla\Upsilon)\, dt \quad as \quad \hbar \to 0.
\end{aligned}$$

Now, if we set $X \equiv \Pi^{\hbar}\Upsilon$ in (85), then we can determine that

$$\int_0^{\Re} (\frac{\partial \Gamma}{\partial t},\ \Pi^{\hbar}\Upsilon)\, dt + \frac{1}{\gamma_q} \int_0^{\Re} (\Gamma^+,\ \Pi^{\hbar}\Upsilon)\, dt = \frac{1}{\gamma_q} \int_0^{\Re} (\Psi^+,\ \Pi^{\hbar}\Upsilon)\, dt.$$

For any $\Upsilon \in L^2(0,\Re;H^1(\Im))$, it holds that

$$\begin{aligned}
\int_0^{\Re} (\frac{\partial \Gamma}{\partial t},\ \Pi^{\hbar}\Upsilon)^{\hbar} dt &= \int_0^{\Re} \left(\frac{\partial \Gamma}{\partial t},\ (\Pi^{\hbar}-I)\Upsilon \right) dt + \int_0^{\Re} \left(\frac{\partial \Gamma}{\partial t},\ \Upsilon \right) dt \\
&:= K_{2,1} + K_{2,2}.
\end{aligned}$$

By employing (16), Hölder's inequality, and taking into account (88), it can be deduced that

$$|K_{2,1}| \equiv |\int_0^{\Re}(\frac{\partial\Gamma}{\partial t},\ (\Pi^{\hbar}-I)\Upsilon)\,dt| = |\int_0^{\Re}\langle\frac{\partial\Gamma}{\partial t},\ (\Pi^{\hbar}-I)\Upsilon\rangle\,dt|$$
$$\leq \|\frac{\partial\Gamma}{\partial t}\|_{L^2(\Im_{\Re})}\|(\Pi^{\hbar}-I)\Upsilon\|_{L^2(0,\Re;H^1(\Im))}$$
$$\leq C\|(\Pi^{\hbar}-I)\Upsilon\|_{L^2(0,\Re;H^1(\Im))}.$$

Derived from (16) and the outcome of weak convergence as stated in (88), it can be concluded for any $\Upsilon \in L^2(0,\Re;H^1(\Im))$ that

$$K_{2,2} \equiv \int_0^{\Re}(\frac{\partial\Gamma}{\partial t},\ \Pi^{\hbar}\Upsilon)dt \to \int_0^{\Re}\langle\frac{\partial\gamma}{\partial t},\ \Upsilon\rangle\,dt \quad as \quad \hbar\to 0.$$

Now, we have for all $\Upsilon \in L^2(0,\Re;H^1(\Im))$, that

$$\int_0^{\Re}(\Gamma^+,\ \Pi^{\hbar}\Upsilon)^{\hbar}dt = \int_0^{\Re}(\Gamma^+,\ (\Pi^{\hbar}-I)\Upsilon)dt + \int_0^{\Re}(\Gamma^+,\ \Upsilon)dt := K_{3,1}+K_{3,2}. \tag{96}$$

Through the application of the Hölder inequality, considering (13), (58), and taking into consideration (11), we can establish that

$$|K_{3,1}| = \left|\int_0^{\Re}(\Gamma^+,\ (\Pi^{\hbar}-I)\Upsilon)dt\right| \leq C\hbar\|\Gamma^+\|_{L^2(\Im_{\Re})}\|\Upsilon\|_{L^2(0,\Re;H^1(\Im))}$$
$$\leq C\hbar\|\Upsilon\|_{L^2(0,\Re;H^1(\Im))} \quad \to \quad 0 \quad as \quad \hbar\to 0. \tag{97}$$

Consolidating (96)–(97) results in the following expression for all $\Upsilon \in L^2(0,\Re;H^1(\Im))$:

$$\int_0^{\Re}(\Gamma^+,\ \Upsilon)dt \to \int_0^{\Re}(\gamma,\ \Upsilon)dt \quad as \quad \hbar\to 0.$$

Similarly to (96), we can show that

$$\int_0^{\Re}(\Psi^+,\ \Upsilon)dt \to \int_0^{\Re}(\psi,\ \Upsilon)dt \quad as \quad \hbar\to 0.$$

Thus, the proof of (89) and (90) has been completed. □

4.4 Error estimate of the approximation

Initially, we can rephrase (56) and (57) in the following form:
Find $\Psi^n(\cdot,\Re), \Gamma^n(\cdot,\Re) \in H^1(0,\Re;S^{\hbar})$ such that $\Psi^0 := \mathbb{P}^{\hbar}\psi^0$ and $\Gamma^0 := \mathbb{P}^{\hbar}\gamma_1^0$,

$$\left(\frac{\partial\Psi}{\partial t},\ \Upsilon\right)^{\hbar} + \alpha(\nabla\Psi^+,\ \nabla\Upsilon) = \alpha s(1,\Upsilon)^{\hbar} \qquad \text{for all } \Upsilon \in S^{\hbar}, \tag{98}$$

$$\left(\frac{\partial\Gamma}{\partial t},\ \Upsilon\right) + \frac{1}{\gamma_q}(\Gamma^+,\ \Upsilon) = \frac{1}{\gamma_q}(\Psi^+,\ \Upsilon) \qquad \text{for all } \Upsilon \in S^{\hbar}. \tag{99}$$

Theorem 7. Assuming the validity of the results from Theorem 2, it follows that

$$\|\psi - \Psi^+\|_{L^\infty(0,\Re;L^2(\Im))} + \|\gamma - \Gamma^+\|_{L^\infty(0,\Re;L^2(\Im))} + \|\psi - \Psi^+\|_{L^2(0,\Re;H^1(\Im))}$$
$$+ \|\gamma - \Gamma^+\|_{L^2(\Im_\Re)} \leq C\Delta t.$$

Proof. Let

$$\begin{aligned} E_1 &= \psi^{\hbar} - \Psi, \qquad E_1^+ = \psi^{\hbar} - \Psi^+, \\ E_2 &= \gamma^{\hbar} - \Gamma, \qquad \Gamma_1^+ = \gamma^{\hbar} - \Gamma^+. \end{aligned} \tag{100}$$

Then, based on (100), it follows that

$$\begin{aligned} E_1^+ - E_1 = \Psi - \Psi^+ &= (t - t_n)\frac{\partial\Psi}{\partial t}, \\ E_1^- - E_1 = \Psi - \Psi^- &= (t - t_{n-1})\frac{\partial\Psi}{\partial t}, \end{aligned} \tag{101}$$

$$\begin{aligned} E_2^+ - E_2 = \Gamma - \Gamma^+ &= (t - t_n)\frac{\partial\Gamma}{\partial t}, \\ E_2^- - E_2 = \Gamma - \Gamma^- &= (t - t_{n-1})\frac{\partial\Gamma}{\partial t}. \end{aligned} \tag{102}$$

By employing (100), (101), and (102), we obtain the ensuing set of inequalities:

$$|E_1^+|_{\hbar} \leq |E_1|_{\hbar} + |\Psi^+ - \Psi^-|_{\hbar}, \tag{103}$$

$$|E_1|_{\hbar} \leq |E_1^+|_{\hbar} + |\Psi^+ - \Psi^-|_{\hbar},$$

$$|E_2^+|_{\hbar} \leq |E_2|_{\hbar} + |\Gamma^+ - \Gamma^-|_{\hbar}, \tag{104}$$

$$|E_2|_\hbar \le |E_2^+|_\hbar + |\Gamma^+ - \Gamma^-|_\hbar.$$

Upon substituting $\Upsilon^\hbar = X$ into both (17) and (18) and subsequently subtracting (98) from (17) and (99) from (18), we obtain the following result:

$$\left(\frac{\partial E_1}{\partial t},\ X\right)^\hbar + \alpha(\nabla {E_1}^+,\ \nabla X) = 0, \tag{105}$$

$$\left(\frac{\partial E_2}{\partial t},\ X\right) + \frac{1}{\gamma_q}(E_2^+,\ X) = \frac{1}{\gamma_q}(E_1^+,\ X). \tag{106}$$

By choosing $X = E_1^+$ in (105) and $X = E_2^+$ in (106), we reach the following expressions:

$$\left(\frac{\partial E_1}{\partial t},\ E_1^+\right)^\hbar + \alpha(\nabla {E_1}^+,\ \nabla E_1^+) = 0,$$

$$\left(\frac{\partial E_2}{\partial t},\ E_2^+\right) + \frac{1}{\gamma_q}(E_2^+,\ E_2^+) = \frac{1}{\gamma_q}(E_1^+,\ E_2^+).$$

Upon utilizing (101), (102), applying Young's inequality, considering (11), (103), and taking into account (104), we can conclude that

$$\begin{aligned}
&\frac{d}{2dt}[|E_1|_\hbar^2 + \|E_2\|_0^2] + \alpha|E_1|_1^2 + \frac{1}{\gamma_q}\|E_2\|_0^2 \\
&\le \frac{1}{2\gamma_q}\|E_1^+\|_0^2 + \frac{1}{2\gamma_q}\|E_2^+\|_0^2 + (\frac{\partial E_1}{\partial t}, \Psi^+ - \Psi)^\hbar + (\frac{\partial E_2}{\partial t}, \Gamma^+ - \Gamma) \\
&\le C|E_1^+|_\hbar^2 + \frac{1}{2\gamma_q}\|E_2^+\|_0^2 + (\frac{\partial E_1}{\partial t}, \Psi^+ - \Psi)^\hbar + (\frac{\partial E_2}{\partial t}, \Gamma^+ - \Gamma) \\
&\le C|E_1|_\hbar^2 + C|\Psi^+ - \Psi^-|_\hbar^2 + \frac{1}{2\gamma_q}\|E_2\|_0^2 + C|\Gamma^+ - \Gamma^-|_\hbar^2 \\
&\quad + (\frac{\partial E_1}{\partial t}, \Psi^+ - \Psi)^\hbar + (\frac{\partial E_2}{\partial t}, \Gamma^+ - \Gamma).
\end{aligned} \tag{107}$$

Subsequently, through the utilization of (12), (100), and (11), we derive the following expression:

$$(\frac{\partial E_1}{\partial t}, \Psi^+ - \Psi)^\hbar \le |\frac{\partial \psi^\hbar}{\partial t}|_\hbar |\Psi^+ - \Psi^-|_\hbar + \frac{1}{\Delta t}|\Psi^+ - \Psi^-|_\hbar^2, \tag{108}$$

$$\begin{aligned}
&(\frac{\partial E_2}{\partial t}, \Gamma^+ - \Gamma) \le \|\frac{\partial \gamma^\hbar}{\partial t}\|_0 \|\Gamma^+ - \Gamma^-\|_0 + \frac{1}{\Delta t}\|\Gamma^+ - \Gamma^-\|_0^2 \\
&\quad \le |\frac{\partial \gamma^\hbar}{\partial t}|_\hbar |\Gamma^+ - \Gamma^-|_\hbar + \frac{1}{\Delta t}|\Gamma^+ - \Gamma^-|_\hbar^2.
\end{aligned} \tag{109}$$

By substituting (108) and (109) into (107), we arrive at the ensuing inequality:

$$\frac{d}{2dt}[|E_1|_\hbar^2 + \|E_2\|_0^2] + \alpha|E_1|_1^2 + \frac{1}{\gamma_q}\|E_2\|_0^2$$
$$\leq C|E_1|_\hbar^2 + \frac{1}{2\gamma_q}\|E_2\|_0^2 + C|\Psi^+ - \Psi^-|_\hbar^2 + C|\Gamma^+ - \Gamma^-|_\hbar^2 + |\frac{\partial\psi^\hbar}{\partial t}|_\hbar|\Psi^+ - \Psi^-|_\hbar$$
$$+ \frac{1}{\Delta t}|\Psi^+ - \Psi^-|_\hbar^2 + |\frac{\partial\gamma^\hbar}{\partial t}|_\hbar|\Gamma^+ - \Gamma^-|_\hbar + \frac{1}{\Delta t}|\Gamma^+ - \Gamma^-|_\hbar^2.$$

Multiplying the above outcomes by 2 and employing the Grönwall lemma, while taking into account that $E_1(0) = E_2(0) = 0$, yield the following result:

$$|E_1|_\hbar^2 + \|E_2\|_0^2 + 2\alpha\int_0^{\Re}|E_1|_1^2 dt + \frac{2}{\gamma_q}\int_0^{\Re}\|E_2\|_0^2 dt$$
$$\leq e^{C\Re}\int_0^{\Re}[|\Psi^+ - \Psi^-|_\hbar^2 + |\Gamma^+ - \Gamma^-|_\hbar^2 + \frac{1}{\Delta t}|\Psi^+ - \Psi^-|_\hbar^2 + \frac{1}{\Delta t}|\Gamma^+ - \Gamma^-|_\hbar^2$$
$$+ |\frac{\partial\psi^\hbar}{\partial t}|_\hbar|\Psi^+ - \Psi^-|_\hbar + |\frac{\partial\gamma^\hbar}{\partial t}|_\hbar|\Gamma^+ - \Gamma^-|_\hbar]dt. \tag{110}$$

The terms on the right-hand side of (110) can be bounded by applying Theorem 3, leading to the determination that

$$\int_0^{\Re}[|\Psi^+ - \Psi^-|_\hbar^2 + |\Gamma^+ - \Gamma^-|_\hbar^2]dt$$
$$= \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}[|\Psi^n - \Psi^{n-1}|_\hbar^2 + |\Gamma^n - \Gamma^{n-1}|_\hbar^2]dt \leq C\Delta t,$$

$$\int_0^{\Re}\frac{1}{\Delta t}[|\Psi^+ - \Psi^-|_\hbar^2 + |\Gamma^+ - \Gamma^-|_\hbar^2]dt$$
$$= \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}[|\Psi^n - \Psi^{n-1}|_\hbar^2 + |\Gamma^n - \Gamma^{n-1}|_\hbar^2]dt \leq C.$$

To bound the fifth and sixth terms on the right-hand side of (110), we employ the Cauchy–Schwarz inequality and utilize the results of Theorems 3 and 2, to find that

$$\int_0^{\Re}[|\frac{\partial\psi^\hbar}{\partial t}|_\hbar|\Psi^+ - \Psi^-|_\hbar \leq C(\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}|\Psi^+ - \Psi^-|_\hbar^2)^{\frac{1}{2}} \leq C\Delta t,$$

$$\int_0^{\Re}[|\frac{\partial \gamma^{\hbar}}{\partial t}|_{\hbar}|\Gamma^{+}-\Gamma^{-}|_{\hbar} \leq C(\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}|\Gamma^{+}-\Gamma^{-}|_{\hbar}^{2})^{\frac{1}{2}} \leq C\Delta t.$$

This pertains to the subsequent outcome

$$|E_1|_{\hbar}^2+\|E_2\|_0^2+2\alpha\int_0^{\Re}|E_1|_1^2 dt+\frac{2}{\gamma_q}\int_0^{\Re}\|E_2\|_0^2 dt \leq C\Delta t. \tag{111}$$

Therefore, based on (111), it follows that

$$\|E_1\|_{L^\infty(0,\Re;L^2(\Im))}+\|E_2\|_{L^\infty(0,\Re;L^2(\Im))}+\|E_1\|_{L^2(0,\Re;H^1(\Im))}+\|E_2\|_{L^2(\Im_{\Re})} \leq C\Delta t.$$

5 Numerical results

This section delves into the numerical analysis of the microscale heat equation. An iterative method has been developed to solve the nonlinear equation system presented by the problem $(\Lambda^{h,\Delta t})$. Additionally, it presents and examines the numerical error outcomes for the Dirichlet non-homogeneous boundary conditions applied to the microscale heat equation in one, two, and three dimensions.

5.1 Numerical algorithm

We begin by presenting a practical algorithm designed to solve the nonlinear algebraic system generated by the approximate problem $(\Lambda^{h,\Delta t})$ at each time level: $(\Lambda_k^{h,\Delta t})$: Given $\{\Psi^{n,0},\Gamma^{n,0}\} \in [S^{\hbar}]^2$, then for $k \geq 1$ find $\{\Psi^{n,k},\Gamma^{n,k}\} \in [S^{\hbar}]^2$ such that for all $\Upsilon \in S^{\hbar}$

$$\left(\frac{\Psi^{n,k}-\Psi^{n-1}}{\Delta t},\ \Upsilon\right)^{\hbar}+\alpha(\nabla\Psi^{n,k},\ \nabla\Upsilon)=\alpha s(1,\Upsilon)^{\hbar}, \tag{112}$$

$$\left(\frac{\Gamma^{n,k}-\Gamma^{n-1}}{\Delta t},\ \Upsilon\right)+\frac{1}{\gamma_q}(\Gamma^{n,k},\ \Upsilon)=\frac{1}{\gamma_q}(\Psi^{n,k},\ \Upsilon). \tag{113}$$

Initiating with $\Psi^0 \equiv \Pi^{\hbar}\psi^0$ and $\Gamma^0 \equiv \Pi^{\hbar}\gamma^0$, and for $n \geq 1$, we initialize $\Psi^{n,0} \equiv \Psi^{n-1}$ and $\Gamma^{n,0} \equiv \Gamma^{n-1}$. Equations (112) and (113) can be reformulated as a system of $2 \times (J+1)^d$ linear equations by testing (112) and

(113) with basis functions $\varphi_j, j = 0, \ldots, J$. For the numerical experiments, we choose a tolerance $TOL = 10^{-7}$ and define the stopping criteria based on this tolerance. Now,

$$\max \left\{ |\Psi^{n,k} - \Psi^{n,k-1}|_{0,\infty}, |\Gamma^{n,k} - \Gamma^{n,k-1}|_{0,\infty} \right\} < TOL, \tag{114}$$

that is, for k satisfying (114), we set $\Psi^n \equiv \Psi^{n,k}, \Gamma^n \equiv \Gamma^{n,k}$.

The programs were developed in MATLAB, and the resulting linear systems were addressed using the Gauss–Seidel iteration method. Although a formal proof of the convergence of $\Psi^{n,k}, \Gamma^{n,k}$ towards Ψ^n, Γ^n for a fixed n has not been established, practical observations have shown promising convergence characteristics. It was observed that the iterative method consistently achieved good convergence (requiring only a few iterations to meet the stopping criteria at each time step).

5.2 Error computations

To evaluate the error, we introduce a slight alteration to Problem (Λ) by incorporating source terms $f(\mathbf{x}, t)$ and $h(\mathbf{x}, t)$. This modification transforms the system denoted by (4)–(5) into the following configuration: (Λ) Find $\{\psi, \gamma\}$ such that

$$\partial_t \psi = \alpha \Delta \psi + \alpha s + f(\mathbf{x}, t),$$
$$\partial_t \gamma = \frac{1}{\gamma_q} \psi - \frac{1}{\gamma_q} \gamma + h(\mathbf{x}, t).$$

Hence, we propose the following fully-discrete finite element approximation for $(\Lambda_k^{h,\Delta t})$:

$(\Lambda_k^{h\,\Delta t})$: Given $\{\Psi^{n,0}, \Gamma^{n,0}\} \in [S^\hbar]^2$, then for $k \geq 1$ find $\{\Psi^{n,k}, \Gamma^{n,k}\} \in [S^\hbar]^2$, such that for all $\Upsilon \in S^\hbar$

$$\left(\frac{\Psi^{n,k} - \Psi^{n-1}}{\Delta t},\ \Upsilon \right)^\hbar + \alpha(\nabla \Psi^{n,k},\ \nabla \Upsilon) - \alpha s(1, \Upsilon)^\hbar = (f(\mathbf{x}, t_n),\ \Upsilon), \tag{115}$$

$$\left(\frac{\Gamma^{n,k} - \Gamma^{n-1}}{\Delta t},\ \Upsilon \right) + \frac{1}{\gamma_q}(\Gamma^{n,k},\ \Upsilon) - \frac{1}{\gamma_q}(\Psi^{n,k},\ \Upsilon) = (h(\mathbf{x}, t_n),\ \Upsilon). \tag{116}$$

5.2.1 One-dimensional error

In this section, we present two numerical examples that solve the system denoted by (115) and (116) under homogeneous Dirichlet boundary conditions and given initial conditions. For all examples, for the sake of simplicity, we set $\alpha = 1, \gamma_q = 1, s = 0$, with the spatial domain $\Im = [0, 1]$ and the time duration $\Re = 1$. The initial and boundary conditions, along with the source terms $f(x, t)$ and $h(x, t)$, are determined based on the specific analytical solution for each example. Initially, we divided the domain $\Im = [0, 1]$ uniformly into J intervals to create a square mesh. Let $h = 1/J$ represent the mesh size for the element, and let $\Delta t = 10^{-6}$ be the time step. The analytical solutions are defined as follows:

(i) $\psi = \exp(t + x^3)$,

(ii) $\psi = 1 + \exp(0.5 * t + x^2)$.

For this example, the errors in the L^1, L^2, and L^∞-norms are detailed in Tables 1–2.

Table 1: Discrete L^1, L^2, L^∞-norms error for Example (i)

	$\|\psi - \Psi\|$			$\|\gamma - \Gamma\|$		
J	L^1	L^2	L^∞	L^1	L^2	L^∞
10	2.3E-02	8.1E-03	3.4E-02	5.4E-02	1.9E-02	8.0E-02
20	5.5E-03	1.4E-03	8.6E-03	1.3E-02	3.3E-03	2.0E-02
25	3.5E-03	7.9E-04	5.5E-03	8.4E-03	1.9E-03	1.3E-02
40	1.3E-03	2.4E-04	2.2E-03	3.2E-03	5.7E-04	5.1E-03
50	8.6E-04	1.4E-04	1.4E-03	2.0E-03	3.2E-04	3.3E-03
100	2.1E-04	2.4E-05	3.4E-04	5.1E-04	5.7E-05	8.2E-04

5.2.2 Two-dimensional error

For a two-dimensional example involving the system described by (115) and (116) with Dirichlet boundary conditions and an initial condition, the spatial

Table 2: Discrete L^1, L^2, L^∞-norms error for Example (ii)

	$\|\psi - \Psi\|$			$\|\gamma - \Gamma\|$		
J	L^1	L^2	L^∞	L^1	L^2	L^∞
10	3.9E-03	1.4E-03	5.4E-03	8.0E-03	2.8E-03	1.1E-02
20	9.4E-04	2.3E-04	1.4E-03	1.9E-03	4.7E-04	2.8E-03
25	6.0E-04	1.3E-04	8.8E-04	1.2E-03	2.7E-04	1.8E-03
40	2.3E-04	4.0E-05	3.4E-04	4.7E-04	8.1E-05	7.0E-04
50	1.5E-04	2.3E-05	2.2E-04	2.9E-04	4.6E-05	4.4E-04
100	1.0E-05	1.3E-06	2.0E-05	2.1E-05	2.5E-06	4.0E-05

domain is simplified to $\Im = [0, 1] \times [0, 1]$. The time interval is set to $[0, \Re] = [0, 1]$. For ease of analysis, we select $\alpha = 1$, $\gamma_q = 1$, and $s = 0$. The initial and boundary conditions, along with the source terms $f(x, y, t)$ and $h(x, y, t)$, should be chosen in line with the specific analytical solution pertinent to each case. The time step selected for the computations is $\Delta t = 10^{-6}$. The analytical solution be

(i) $\psi = \exp(t + x + y)$,

(ii) $\psi = \exp(t + x^2 + y^2)$.

The errors in the L^1, L^2, and L^∞-norms for the corresponding simulations are provided in Tables 3–4.

5.2.3 Three-dimensional error

In this section, we introduce a three-dimensional example, considering the system denoted by (115) and (116) with Dirichlet boundary and initial conditions. For the purposes of this numerical example, we simplify the spatial domain to $\Im = [0, 1] \times [0, 1] \times [0, 1]$, with the time interval set to $[0, \Re] = [0, 1]$. We set the parameters $\alpha = 1$, $\gamma_q = 1$, and $s = 0$. The initial and boundary conditions, along with the source terms $f(x, y, z, t)$ and $h(x, y, z, t)$, are to be established based on the specific analytical solution for each example. The time step is chosen as $\Delta t = 10^{-5}$. The analytical solution will be

Table 3: Discrete L^1, L^2, L^∞-norms error for Example (i)

	$\|\psi - \Psi\|$			$\|\gamma - \Gamma\|$		
J	L^1	L^2	L^∞	L^1	L^2	L^∞
10	4.4E-04	5.4E-05	7.8E-04	1.0E-03	1.3E-04	1.8E-03
20	1.0E-04	6.1E-06	2.0E-04	2.4E-04	1.4E-05	4.6E-04
25	6.4E-05	3.0E-06	1.3E-04	1.5E-04	7.0E-06	2.9E-04
40	2.5E-05	7.4E-07	5.0E-05	5.7E-05	1.7E-06	1.2E-04
50	1.7E-05	3.9E-07	3.3E-05	3.6E-05	8.5E-07	7.4E-05
100	7.4E-06	9.1E-08	1.7E-05	2.1E-05	2.5E-07	4.4E-05

Table 4: Discrete L^1, L^2, L^∞-norms error for Example (ii)

	$\|\psi - \Psi\|$			$\|\gamma - \Gamma\|$		
J	L^1	L^2	L^∞	L^1	L^2	L^∞
10	9.6E-03	1.2E-03	1.8E-02	2.3E-02	2.8E-03	4.2E-02
20	2.2E-03	1.3E-04	4.6E-03	5.3E-03	3.2E-04	1.1E-02
25	1.4E-03	6.7E-05	2.9E-03	3.3E-03	1.6E-04	6.9E-03
40	5.4E-04	1.6E-05	1.1E-03	1.3E-03	3.8E-05	2.7E-03
50	3.4E-04	8.1E-06	7.4E-04	8.0E-04	1.9E-05	1.7E-03
100	7.6E-05	9.0E-07	1.7E-04	1.8E-04	2.1E-06	3.9E-04

$$\psi = \exp(t + x^2 + y^2 + z^2).$$

The errors in the L^1, L^2, and L^∞-norms for the corresponding simulations are detailed in Table 5.

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Table 5: Discrete L^1, L^2, L^∞-norms error

	$\|\psi - \Psi\|$			$\|\gamma - \Gamma\|$		
J	L^1	L^2	L^∞	L^1	L^2	L^∞
10	1.3E-02	5.4E-04	3.2E-02	3.0E-02	1.3E-03	7.5E-02
20	2.9E-03	4.1E-05	8.2E-03	6.7E-03	9.7E-05	1.9E-02
25	1.8E-03	1.8E-05	5.2E-03	4.1E-03	4.3E-05	1.2E-02
40	6.2E-04	3.2E-06	2.0E-03	1.4E-03	7.5E-06	4.6E-03
50	3.7E-04	1.4E-06	1.2E-03	8.5E-04	3.2E-06	2.9E-03

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