



A new method for exact product form and approximation solutions of a parabolic equation with nonlocal initial condition using Ritz method

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Abstract

Many phenomena in various fields of physics are simulated by parabolic partial differential equations with the nonlocal initial conditions, while there are few numerical methods for solving these problems. In this paper, the Ritz–Galerkin method with a new approach is proposed to give the exact and approximate product solution of a parabolic equation with the nonstandard initial conditions. For this purpose, at first, we introduce a function called satisfier function, which satisfies all the initial and boundary conditions. The uniqueness of the satisfier function and its relation to the exact solution are discussed. Then the Ritz–Galerkin method with satisfier function is used to simplify the parabolic partial differential equations to the solution of algebraic equations. Error analysis is worked by using the property of interpolation. The comparisons of the obtained results with the results of other methods show more accuracy in the presented technique.

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1 Introduction

Various problems arising in geology [19], heat conduction [6, 7, 20], plasma physics [16], chemical engineering [12], thermostatic [30], and hydrodynamics [11], can be reduced to nonclassical initial-boundary value problems. The investigation of a nonclassical problem with nonlocal initial condition is considered in this article. Nonclassical problems with nonlocal initial conditions,

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which are generalizations of the classical or time-periodic problems, can be used in science with better results than the classical initial condition. Nonlocal initial conditions are useful in the modeling of radionuclides propagation in Stokes fluid, sewage causing pollution processes in rivers and seas, and diffusion in porous media; see [17, 25, 29].

The following parabolic equation with nonlocal initial condition is considered in this paper:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial \gamma^2} + \phi(\gamma, t), \quad 0 < \gamma < 1, \quad 0 < t \leq T, \quad (1)$$

with boundary conditions

$$v(0, t) = h_0(t), \quad 0 \leq t \leq T, \quad (2)$$

$$v(1, t) = h_1(t), \quad 0 \leq t \leq T, \quad (3)$$

and the nonlocal initial condition

$$v(\gamma, 0) = \sum_{j=1}^N \lambda_j(\gamma) v(\gamma, T_j) + \chi(\gamma),$$

$$0 \leq \gamma \leq 1, \quad 0 < T_1 < T_2 < \dots < T_N = T, \quad (4)$$

The function v should be found, but ϕ, h_0, h_1, χ , and λ_j are known functions.

The multiplier method, semi-group theory, the maximum principle, and potential theoretical representation of the solution are applied to prove the existence and uniqueness of the above problem; see [8, 10, 3, 4, 5, 9, 24]. There are many publications about numerical methods for parabolic partial differential equations with classical initial condition and with nonlocal boundary conditions, but there are few numerical methods for nonstandard initial condition problem presented here [13, 14, 15, 22]. Therefore, it is significant to propose numerical methods for this problem. Dehghan [13, 14] concentrated on finite difference schemes to solve problem (1)–(4). The implicit collocation technique is applied for solving the above problem in [15]. Recently, in [22], the authors analyzed finite difference schemes for solving (1)–(4).

Several partial differential equations are numerically solved by Ritz and Galerkin method [31, 32]. In the Ritz–Galerkin method, applying the appropriate satisfier function has been important; see, for instance, [33, 34, 35, 2, 21, 26, 1]. Utilizing satisfier function that fulfills all the problem conditions helps to satisfy the problem conditions quickly, in addition, it leads to a smaller system of algebraic equations and reduces the time of computation.

In this paper, we concentrate on satisfier function of (2)–(4). The uniqueness of the satisfier function and its relation to the exact solution are discussed in some theorems. Moreover, the Ritz–Galerkin method in the Bernstein polynomial basis applying the satisfier function is utilized to give an ap-

proximate solution of (1)–(4). The comparison of the obtained results with those results obtained by [13, 14, 22] shows more efficiency of the presented technique.

This paper is separated into the following sections: In Section 2, the satisfier functions for (2)–(4) are introduced. Theorems on the uniqueness of product satisfier functions are proven and used to obtain the exact solution of the problem. Moreover, the Ritz–Galerkin method with satisfier function is applied for solving the problem numerically. Some results concerning the error analysis are obtained in Section 3. In Section 4, we apply these results to solve three nonlocal initial condition problems. Section 5 is dedicated to the conclusion.

2 Satisfier function for Ritz–Galerkin method

The solution $v(\gamma, t)$ is approximated with the following series in the Ritz method:

$$\hat{v}(\gamma, t) = \sum_{i=0}^n \sum_{j=0}^m b_{ij} \sigma_{ij}(\gamma, t) + \eta(\gamma, t), \quad (\gamma, t) \in [0, 1] \times [0, T), \quad (5)$$

in which $\sigma_{ij}(\gamma, t) = \gamma(\gamma - 1) \prod_{j=1}^N (t - T_j) \rho_i(\gamma) \rho_j(t)$, where $\rho_i(\gamma) \rho_j(t)$ are basis functions. The function $\eta(\gamma, t)$ is called the satisfier function. In this study, we take $\rho_i(\gamma)$ the Bernstein polynomials in $[0, 1]$. What is important in the Ritz method is to find a proper satisfier function. The satisfier function is an arbitrary function that satisfies all problem conditions [35]. Satisfier functions are not unique, and the construction of these functions is often based on interpolation techniques. Here we focus on the construction of satisfier function for one-dimensional parabolic equation with the nonlocal initial condition.

We assume that the compatibility conditions

$$h_0(0) = \chi(0) + \sum_{j=1}^N \lambda_j(0) h_0(T_j), \quad (6)$$

$$h_1(0) = \chi(1) + \sum_{j=1}^N \lambda_j(1) h_1(T_j), \quad (7)$$

are satisfied and define the satisfier function $\eta(\gamma, t)$ that satisfies the boundary conditions (2), (3), and the nonlocal time weighting initial condition (4) when

$$h_0(0) - \sum_{j=1}^N \lambda_j(\gamma) h_0(T_j) \neq 0, \quad 0 \leq \gamma \leq 1,$$

as

$$\eta(\gamma, t) = a(\gamma)h_0(t) + \gamma h_1(t),$$

when

$$h_1(0) - \sum_{j=1}^N \lambda_j(\gamma)h_1(T_j) \neq 0, \quad 0 \leq \gamma \leq 1,$$

as

$$\eta(\gamma, t) = (1 - \gamma)h_0(t) + b(\gamma)h_1(t)$$

and when

$$h_0(0) - \sum_{j=1}^N \lambda_j(\gamma)h_0(T_j) = h_1(0) - \sum_{j=1}^N \lambda_j(\gamma)h_1(T_j) = 0, \quad 0 \leq \gamma \leq 1,$$

as

$$\eta(\gamma, t) = (1 - \gamma)h_0(t) + \gamma h_1(t) + c(t)\chi(\gamma),$$

where

$$a(\gamma) = \frac{\chi(\gamma) - xg_1(0) + \sum_{j=1}^N \lambda_j(\gamma)h_1(T_j)\gamma}{h_0(0) - \sum_{j=1}^N \lambda_j(\gamma)h_0(T_j)},$$

$$b(\gamma) = \frac{\chi(\gamma) - (1 - \gamma)h_0(0) + \sum_{j=1}^N \lambda_j(\gamma)(1 - \gamma)h_0(T_j)}{h_1(0) - \sum_{j=1}^N \lambda_j(\gamma)h_1(T_j)},$$

$$c(t) = \prod_{j=1}^N \frac{(-1)^N(t - T_j)}{T_j}.$$

Theorem 1. If at least one of the values $h_0(0) - \sum_{j=1}^N \lambda_j(\gamma)h_0(T_j)$ and $h_1(0) - \sum_{j=1}^N \lambda_j(\gamma)h_1(T_j)$ are not equal to zero in $[0, 1]$, then the product satisfier function for (2) – (4) is unique.

Proof. Suppose that

$$h_1(0) - \sum_{j=1}^N \lambda_j(\gamma)h_1(T_j) \neq 0 \quad (8)$$

and that the product functions $A_1(\gamma)A_2(t)$ and $F_1(\gamma)F_2(t)$ satisfy (2) – (4). Then

$$\begin{aligned} A_1(0)A_2(t) &= F_1(0)F_2(t), \\ A_1(1)A_2(t) &= F_1(1)F_2(t), \end{aligned} \quad (9)$$

$$A_1(\gamma)A_2(0) - F_1(\gamma)F_2(0) = \sum_{j=1}^N \lambda_j(\gamma)(A_1(\gamma)A_2(T_j) - F_1(\gamma)F_2(T_j)). \quad (10)$$

From (9), in particular, we have

$$A_1(1)A_2(T_j) = F_1(1)F_2(T_j), \quad A_1(1)A_2(0) = F_1(1)F_2(0). \quad (11)$$

By multiplying (9) by (10) and using (9) and (11), we have

$$(A_1(\gamma)A_2(t) - F_1(\gamma)F_2(t))h_1(0) = (A_1(\gamma)A_2(t) - F_1(\gamma)F_2(t))\left(\sum_{j=1}^N \lambda_j(\gamma)h_1(T_j)\right).$$

Thus

$$(A_1(\gamma)A_2(t) - F_1(\gamma)F_2(t))(h_1(0) - \left(\sum_{j=1}^N \lambda_j(\gamma)h_1(T_j)\right)) = 0;$$

therefore from (8), we get

$$A_1(\gamma)A_2(t) = F_1(\gamma)F_2(t).$$

In a similar way, the other case $h_0(0) - \sum_{j=1}^N \lambda_j(\gamma)h_0(T_j) \neq 0$ is proved. \square

Theorem 2. Suppose that $h_1(0) - \sum_{j=1}^N \lambda_j(\gamma)h_1(T_j) \neq 0$, $0 \leq \gamma \leq 1$ and that

$$[h_1(0) - \sum_{j=1}^N \lambda_j(0)h_1(T_j)]h_0(t) = [h_0(0) - \sum_{j=1}^N \lambda_j(0)h_0(T_j)]h_1(t). \quad (12)$$

Then the unique separable satisfier function for (2) – (4) is given by

$$\eta(\gamma, t) = \frac{\chi(\gamma)h_1(t)}{h_1(0) - \sum_{j=1}^N \lambda_j(\gamma)h_1(T_j)}.$$

Proof. From compatibility condition (7), the function $\eta(\gamma, t)$ satisfies the boundary condition (3) obviously. Now let $h_0(0) - \sum_{j=1}^N \lambda_j(0)h_0(T_j) \neq 0$. Then by using (12) and the compatibility condition (6), $\eta(\gamma, t)$ satisfies the boundary condition (2) and when $h_0(0) - \sum_{j=1}^N \lambda_j(0)h_0(T_j) = 0$, from (12), we have $h_0(t) = 0$. Thus from the compatibility condition (6), we obtain $\chi(0) = 0$ and condition (2) also is fulfilled. In addition, $\eta(\gamma, t)$ satisfies the initial condition (4), because

$$\begin{aligned} \eta(\gamma, 0) &= \frac{\chi(\gamma)h_1(0)}{h_1(0) - \sum_{j=1}^N \lambda_j(\gamma)h_1(T_j)} \\ &= \frac{\chi(\gamma)h_1(0) - \chi(\gamma)\sum_{j=1}^N \lambda_j(\gamma)h_1(T_j) + \chi(\gamma)\sum_{j=1}^N \lambda_j(\gamma)h_1(T_j)}{h_1(0) - \sum_{j=1}^N \lambda_j(\gamma)h_1(T_j)}. \end{aligned}$$

Hence

$$\eta(\gamma, 0) = \chi(\gamma) + \sum_{j=1}^N \lambda_j(\gamma) \left(\frac{\chi(\gamma) h_1(t_j)}{h_1(0) - \sum_{j=1}^N \lambda_j(\gamma) h_1(T_j)} \right) = \chi(\gamma) + \sum_{j=1}^N \lambda_j(\gamma) \eta(\gamma, T_j).$$

□

Theorem 3. Suppose that $h_0(0) - \sum_{j=1}^N \lambda_j(\gamma) h_0(T_j) \neq 0$ and that

$$[h_1(0) - \sum_{j=1}^N \lambda_j(0) h_1(T_j)] h_0(t) = [h_0(0) - \sum_{j=1}^N \lambda_j(0) h_0(T_j)] h_1(t).$$

Then the unique separable satisfier function for (2) – (3) and (4) is given by

$$\eta(\gamma, t) = \frac{\chi(\gamma) h_0(t)}{h_0(0) - \sum_{j=1}^N \lambda_j(\gamma) h_0(T_j)}.$$

Proof. The proof is similar to the previous theorem. □

Now we consider the approximation (5) and apply the following Galerkin equations to obtaining coefficients b_{ij} :

$$\left\langle \frac{\partial \hat{v}}{\partial t} - \frac{\partial^2 \hat{v}}{\partial \gamma^2} - \phi(\gamma, t), \rho_{i,n}(\gamma) \rho_{j,n}(t) \right\rangle = 0, \quad (i = 0, \dots, n), \quad (j = 0, \dots, m), \quad (13)$$

where \langle, \rangle is defined as

$$\begin{aligned} & \left\langle \frac{\partial \hat{v}}{\partial t} - \frac{\partial^2 \hat{v}}{\partial \gamma^2} - \phi(\gamma, t), \rho_{i,n}(\gamma) \rho_{j,m}(t) \right\rangle \\ &= \int_0^1 \int_0^T \left(\frac{\partial \hat{v}}{\partial t} - \frac{\partial^2 \hat{v}}{\partial \gamma^2} - \phi(\gamma, t) \right) \rho_{i,n}(\gamma) \rho_{j,m}(t) dt d\gamma. \end{aligned}$$

The coefficients b_{ij} ($i = 0, \dots, n$) ($j = 0, \dots, m$) are obtained with solving algebraic equations system (13).

Now let $h_0(t) = h_1(t) = 0$ and let a product form for the satisfier function be taken. Then it can be denoted as

$$\eta(\gamma, t) = f(\gamma)g(t). \quad (14)$$

Substituting the function (14) in to conditions (2)–(3), gives the following conditions on $f(\gamma)$:

$$f(0) = 0, \quad f(1) = 0.$$

On the other hand, from the compatibility conditions (6) and (7), we have

$$\chi(0) = 0, \quad \chi(1) = 0.$$

Therefore, we can conclude that one of the options for the function $f(\gamma)$ can be $\chi(\gamma)$, and for fulfilling the initial condition (4) we choose $g(t)$ such that

$$g(0) = 1 + \sum_{j=1}^N \lambda_j(\gamma)g(T_j). \quad (15)$$

Therefore

$$\eta(\gamma, t) = \chi(\gamma)g(t), \quad (16)$$

with condition (15) is a satisfier function for (2)–(4).

It is noteworthy that if $h_0(t) = h_1(t) = 0$ and $\chi(\gamma) \neq 0$, then we can approximate the product solution as

$$\hat{v}(\gamma, t) = \chi(\gamma)h(t) = \chi(\gamma) \sum_{i=0}^n b_i \rho_{i,n}(t). \quad (17)$$

This approximation supplies higher adaptability in the nonlinear initial and boundary conditions. Then, with the Galerkin equations and

$$\sum_{i=0}^n b_i \rho_{i,n}(0) - (1 + \sum_{j=1}^N \lambda_j(\gamma) \sum_{i=0}^n b_i \rho_{i,n}(T_j)) = 0, \quad (18)$$

the coefficients b_i are specified. If we choose Bernstein polynomials for basis functions $\rho_{i,n}(t)$, then from properties of Bernstein polynomials, equation (18) is equal to the following equation:

$$b_0 - (1 + b_n \sum_{j=1}^N \lambda_j(\gamma)) = 0.$$

On the other hand, when $\frac{\dot{\chi}(\gamma)}{\chi(\gamma)}$ is a constant, the separable satisfier function is also the solution of the problem(1)–(4) and the exact solution will also be obtained from (16) with

$$g(t) = \exp\left(\frac{\dot{\chi}(\gamma)}{\chi(\gamma)}t\right) \left(\int_0^t \frac{\phi(\gamma, s) \exp\left(-\frac{\dot{\chi}(\gamma)}{\chi(\gamma)}s\right)}{\chi(\gamma)} ds + c \right),$$

such that c can be obtained from the following condition:

$$g(0) = 1 + \sum_{j=1}^N \lambda_j(\gamma)g(T_j). \quad (19)$$

Also if $h_0(t) = h_1(t) = \chi(\gamma) = 0$, then we can try to find the following approximation for the solution of the problem:

$$\hat{v}(\gamma, t) = A(\gamma)g(t) = A(\gamma) \sum_{i=0}^n b_i \rho_{i,n}(t),$$

and $A(\gamma)$ is selected such that $A(0) = A(1) = 0$. For example, $\gamma(\gamma - 1)$ and $\sin(\pi\gamma)$ are two selections for $A(\gamma)$. Note that some times $\phi(\gamma, t)$ in (1) can help us to obtain an appropriate selection for $A(\gamma)$. Then, the expansion coefficients b_i are specified with the Galerkin equations and the following equation:

$$b_0 - b_n \sum_{j=1}^N \lambda_j(\gamma) = 0.$$

3 Error analysis

In the current section, we analyze the error of the proposed method. We obtain an estimate of the error norm of the best approximation of a smooth function of two variables, on $[0, 1] \times [0, 1]$, by Bernstein polynomials.

Lemma 1. Let $f(\gamma, t)$ be a sufficiently smooth function in $[a, b] \times [c, d]$ and let $p_{n-1, m-1}(\gamma, t)$ be the interpolating polynomial to $f(x, t)$ at points (γ_i, t_j) , where γ_i , $i = 0, 1, \dots, n-1$, are the roots of the $n-1$ -degree shifted Chebyshev polynomial in $[a, b]$ and t_j , $j = 0, 1, \dots, m-1$ are the roots of the $m-1$ -degree shifted Chebyshev polynomial in $[c, d]$. Then [18, 23]

$$\begin{aligned} |f(\gamma, t) - p_{n-1, m-1}(\gamma, t)| &\leq \frac{(b-a)^n}{n!2^{2n-1}} \max_{(\gamma, t) \in [a, b] \times [c, d]} \left| \frac{\partial^n f(\gamma, t)}{\partial x^n} \right| \\ &\quad + \frac{(d-c)^m}{m!2^{2m-1}} \max_{(\gamma, t) \in [a, b] \times [c, d]} \left| \frac{\partial^m f(\gamma, t)}{\partial t^m} \right| \\ &\quad + \frac{(b-a)^n (d-c)^m}{m!n!2^{2(n+m)-2}} \max_{(\gamma, t) \in [a, b] \times [c, d]} \left| \frac{\partial^{n+m} f(\gamma, t)}{\partial t^m \partial \gamma^n} \right|. \end{aligned}$$

Theorem 4. Suppose that $f_{n,m}(\gamma, t) = \sum_{i=0}^n \sum_{j=0}^m b_{ij} \sigma_{ij}(\gamma, t) = C^T \Psi(\gamma, t)$ is the Bernstein expansion of the real sufficiently smooth function $f(\gamma, t) \in [0, 1] \times [0, 1]$. Then, there exist real numbers c_1, c_2 , and c_3 such that

$$\|f - f_{n,m}(\gamma, t)\|_2 \leq \frac{c_1}{(n+1)!2^{2n+1}} + \frac{c_2}{(m+1)!2^{2m+1}} + \frac{c_3}{(m+1)!(n+1)!2^{2(n+m)+2}}, \quad (20)$$

where

$$\begin{aligned} \max_{(\gamma, t) \in [0, 1] \times [0, 1]} \left| \frac{\partial^{n+1} f(\gamma, t)}{\partial \gamma^{n+1}} \right| &\leq c_1, \\ \max_{(\gamma, t) \in [0, 1] \times [0, 1]} \left| \frac{\partial^{m+1} f(\gamma, t)}{\partial t^{m+1}} \right| &\leq c_2, \\ \max_{(\gamma, t) \in [0, 1] \times [0, 1]} \left| \frac{\partial^{n+m+2} f(\gamma, t)}{\partial t^{m+1} \partial \gamma^{n+1}} \right| &\leq c_3. \end{aligned}$$

Proof. Let $p_{n,m}(\gamma, t)$ be the interpolating polynomial to f at the roots of the Chebyshev polynomial. Then from the definition of the best approximation, we have

$$\|f(\gamma, t) - f_{n,m}(\gamma, t)\|_2 \leq \|f(\gamma, t) - p_{n,m}(\gamma, t)\|_2.$$

Therefore by Lemma 1, we obtain

$$\begin{aligned} & \int_0^1 \int_0^1 [f(\gamma, t) - C^T \Psi(\gamma, t)]^2 d\gamma dt \\ & \leq \int_0^1 \int_0^1 [f(\gamma, t) - p_{n,m}(\gamma, t)]^2 d\gamma dt \\ & \leq \int_0^1 \int_0^1 \left[\frac{1}{(n+1)!2^{2n+1}} \max_{(\gamma,t) \in [0,1] \times [0,1]} \left| \frac{\partial^{n+1} f(\gamma, t)}{\partial \gamma^{n+1}} \right| \right. \\ & \quad \left. + \frac{1}{(m+1)!2^{2m+1}} \max_{(\gamma,t) \in [0,1] \times [0,1]} \left| \frac{\partial^{m+1} f(\gamma, t)}{\partial t^{m+1}} \right| \right. \\ & \quad \left. + \frac{1}{(m+1)!(n+1)!2^{2(n+m)+2}} \max_{(\gamma,t) \in [0,1] \times [0,1]} \left| \frac{\partial^{n+m+2} f(\gamma, t)}{\partial t^{m+1} \partial \gamma^{n+1}} \right| \right]^2 d\gamma dt \\ & \leq \left[\frac{c_1}{(n+1)!2^{2n+1}} + \frac{c_2}{(m+1)!2^{2m+1}} + \frac{c_3}{(m+1)!(n+1)!2^{2(n+m)+2}} \right]^2 \end{aligned}$$

and this completes the proof. \square

Remark 1. In the special case when $n = m$, we have

$$\|f(\gamma, t) - f_{n,n}(\gamma, t)\|_2 \leq (c_1 + c_2 + \frac{c_3}{(n+1)!2^{2n+1}}) \frac{1}{(n+1)!2^{2n+1}}.$$

Therefore when considering the upper bound of the error, given by (20), the term containing c_3 can be neglected and we have

$$\|f(\gamma, t) - f_{n,n}(\gamma, t)\|_2 = O\left(\frac{1}{(n+1)!2^{2n+1}}\right).$$

Remark 2. Let $f(t) \in C^{n+1}[0, 1]$ and let $f_n(t) = \sum_{i=0}^n b_i \rho_{i,n}(t)$ be the Bernstein expansion of $f(t)$. Then

$$\|f(t) - f_n(t)\|_2 = O\left(\frac{1}{(n+1)!2^{2n+1}}\right).$$

Now let $Y_{n,m}$ be the space of bivariate polynomials of degree less than or equal to n on γ and degree less than or equal to m on t such that the Ritz–Galerkin approximation belongs to it. The error norm of the numerical results obtained from the Ritz–Galerkin method tends to zero with the same

convergence rate as the error norm of the best approximation of the exact solution in $Y_{n,m}$ [27, 28], which is confirmed by the obtained results in the next section. Hence we present the following theorem.

Theorem 5. Suppose that $v(\gamma, t)$ is the exact solution of (1)–(4) and that $\hat{v}(\gamma, t)$ is the approximate solution (5) with $(m = n)$. Then

$$\|v(\gamma, t) - \hat{v}(\gamma, t)\|_2 = O\left(\frac{1}{(n+1)!2^{2n+1}}\right).$$

4 Numerical examples

In this section, we report some results of our numerical calculations using the methods presented in the previous section for finding the exact and approximation solutions of (1) – (4). We select Examples 1 and 2 of [13, 14, 15, 22] and compare our approximation results. We use the package of Mathematica version 11, with the following hardware configuration: Intel(R) Core(TM) i3 – 2100 CPU, 8 GB of RAM, 64-bit Operating System (Windows 7) for numerical calculations in all examples. The approximate L^2 norm of absolute error and the maximum norm of the absolute error are, respectively, calculated by

$$\|e(\gamma, t)\|_{L^2}^2 = \int_0^1 \int_0^T e^2(\gamma, t) dt d\gamma = \int_0^1 \int_0^T (v(\gamma, t) - \hat{v}(\gamma, t))^2 dt d\gamma$$

and

$$E_\infty = \max_{0 < \gamma < 1, 0 < t < T} |e(\gamma, t)| = \max_{0 < \gamma < 1, 0 < t < T} |v(\gamma, t) - \hat{v}(\gamma, t)|.$$

Moreover we introduce the notation $\lambda_n = \frac{1}{(n+1)!2^{2n+1}}$. According to the previous section, λ_n has the same asymptotic behavior as the error norm of the numerical results, which is confirmed by the results in the tables.

Example 1. Consider (1)–(4) with $N = 2$, and let

$$\phi(\gamma, t) = (\pi^2 - 1) \sin(\pi\gamma) \exp(-t),$$

$$v(\gamma, 0) = v(\gamma, T_1) - v(\gamma, T_2) + \chi(\gamma),$$

$$\chi(\gamma) = \sin(\pi\gamma)(1 - e^{-T_1} - e^{-T_2}),$$

$$h_0(t) = 0, \quad h_1(t) = 0,$$

$$T_1 = 0.5, \quad T_2 = 1.0,$$

$$\lambda_1(\gamma) = 1, \lambda_2(\gamma) = -1.$$

for which the exact solution is [13, 14, 15, 22]:

$$v(\gamma, t) = \sin(\pi\gamma) \exp(-t).$$

The results of errors evaluated for different values of n by using the method presented in this article with the truncated series (17), together with CPU time are shown in Table 1. In Figure 1, the absolute difference between the exact and approximate solution for $n = 5$ is shown.

In [22], Martín-Vaquero and Sajavičius employed the two-level finite difference schemes to solve this problem for different values of h and τ , where h and τ are space and time step sizes, respectively. The best absolute error of the approximate solutions at some different points reported in [22] is shown in Table 2 and compared with presented method with $n = 7$.

The best errors in L^2 norm obtained in [13] and [14] for solving this problem using finite difference schemes with several choices of h , are shown in Table 3.

Table 1: Absolute errors and L^2 norm of errors for Example 1

$\gamma=t$	n=1	n=3	n=5	n=7
0	0	0	0	0
0.1	8.77648×10^{-2}	5.50078×10^{-5}	8.63336×10^{-8}	1.22592×10^{-9}
0.2	1.54731×10^{-1}	2.9701×10^{-5}	2.10798×10^{-7}	3.87638×10^{-10}
0.3	1.89538×10^{-1}	1.10471×10^{-4}	9.06801×10^{-8}	5.27107×10^{-10}
0.4	1.88218×10^{-1}	2.3564×10^{-4}	4.16751×10^{-7}	1.47424×10^{-9}
0.5	1.54818×10^{-1}	2.21861×10^{-4}	5.25682×10^{-7}	1.29762×10^{-9}
0.6	1.0049×10^{-1}	6.15521×10^{-5}	8.80609×10^{-9}	9.66877×10^{-10}
0.7	4.12702×10^{-2}	1.27031×10^{-4}	4.84358×10^{-7}	1.08301×10^{-9}
0.8	5.05864×10^{-3}	1.89116×10^{-4}	2.55435×10^{-7}	6.65846×10^{-10}
0.9	2.24724×10^{-2}	7.97571×10^{-5}	1.52571×10^{-7}	2.07613×10^{-10}
1	0	0	0	0
λ_n	6.25×10^{-2}	3.25521×10^{-4}	6.78168×10^{-7}	7.56884×10^{-10}
errors in L^2 norm	1.3015×10^{-1}	1.35747×10^{-4}	3.08435×10^{-7}	9.2949×10^{-9}
CPU Time	0.374 s	0.39 s	0.39 s	0.405 s

Table 2: Absolute errors for ($t = 1$) and different values of γ for Example 1

γ	Scheme 1 in [22]	Scheme 2 in [22]	Scheme 3 in [22]	Presented method
0.25	2.89742×10^{-6}	2.32464×10^{-6}	9.99528×10^{-11}	5.12468×10^{-11}
0.5	4.09758×10^{-6}	3.28755×10^{-6}	1.41368×10^{-10}	7.24995×10^{-11}
0.75	2.89742×10^{-6}	2.32464×10^{-6}	9.99524×10^{-11}	5.12648×10^{-11}

It is worth pointing out that since the exact solution for $v(\gamma, t)$ is separable, $\chi(\gamma) \neq 0$, and $\frac{\dot{\chi}(\gamma)}{\chi(\gamma)}$ is a constant, from (2) and (19), we have

$$g(t) = \frac{\exp(-t)}{(1 - e^{-T_1} - e^{-T_2})}.$$

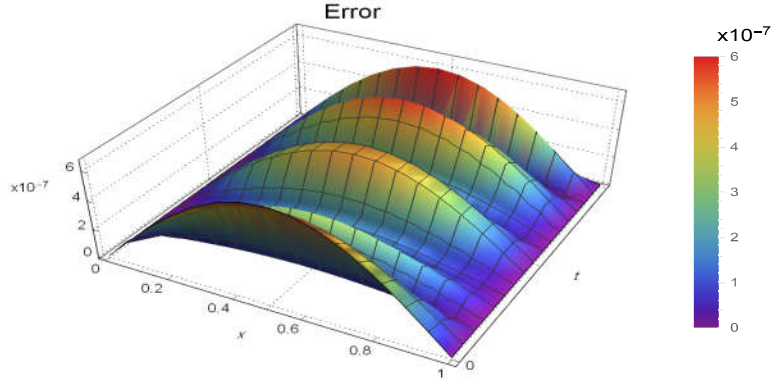


Figure 1: The graph of the absolute error with $n = 5$ for Example 1.

Table 3: The best errors in L^2 norm obtained from different methods and presented method for Example 1

Presented method with $n = 6$	FTCP[13]	BTCS[13]	Crank–Nicolson[13]	Crandall’s[13]
8.9×10^{-9}	5.0×10^{-5}	6.0×10^{-5}	6.1×10^{-5}	1.1×10^{-7}
Saulyeve I[13]	Saulyeve II[13]	Three-point FTCS [14]	Dufort–Frankel [14]	(1, 3, 1) Scheme[14]
2.0×10^{-4}	3.0×10^{-4}	2.5×10^{-5}	4.0×10^{-5}	1.2×10^{-6}

Therefor we can obtain the exact solution $v(\gamma, t) = \chi(\gamma)g(t) = \sin(\pi\gamma) \exp(-t)$.

Example 2. In this example, the functions $\phi(\gamma, t), h_0(t), h_1(t)$, and $\chi(\gamma)$ are chosen such that

$$v(\gamma, t) = (\sin(\pi\gamma) + \cos(\pi\gamma) + 2\gamma - 1)e^{-t},$$

with initial condition

$$v(\gamma, 0) = \lambda_1 v(\gamma, T_1) + \chi(\gamma),$$

is the exact solution of the problem; see [22].

The maximum norms of the absolute error (E_∞) obtained with $T = 1, T_1 = 0.9$, and $\lambda_1 = -10$, computed for various values of n using the truncated series (17) and the method in [22] are compared in Table 4. In addition, CPU time for different values of n is listed in this table. In Figure 2, the exact and approximate solutions of $v(\gamma, t)$ in $t = 0.5$ for $n = 6$ are shown.

Example 3. In the last example, we solve equations (1)–(4) with $N = 1$ and

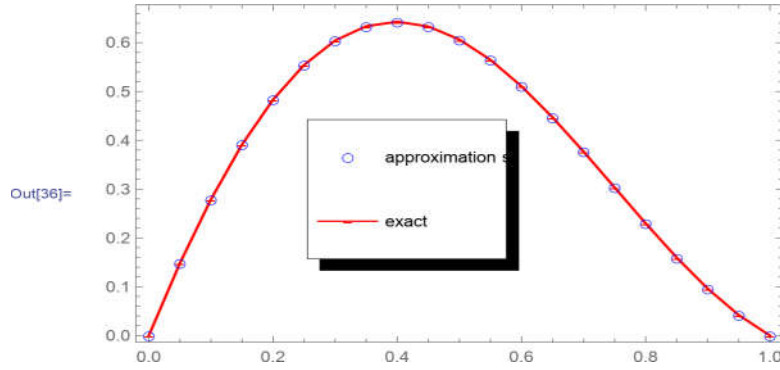


Figure 2: Exact and approximate solutions of $v(\gamma, t)$ in $t = 0.5$ for Example 2.

Table 4: Maximum norms of the absolute error for Example 2

	Methods in [22]	Our method	
	The best reported E_∞	E_∞	CPU Time
$\tau = 1/200, h = 1/400$	5.946431×10^{-4}	$n = 3$ 1.88002×10^{-4}	0.297 s
$\tau = 1/1000, h = 1/400$	2.404325×10^{-5}	$n = 5$ 2.25173×10^{-7}	0.343 s
$\tau = 1/2000, h = 1/400$	2.424209×10^{-5}	$n = 7$ 2.27617×10^{-9}	0.375 s

$$\phi(\gamma, t) = \frac{(\pi^2 + 1) \sin(\pi\gamma) \exp(t)}{e^T},$$

$$v(\gamma, 0) = \lambda_1 v(\gamma, T),$$

$$\chi(\gamma) = 0,$$

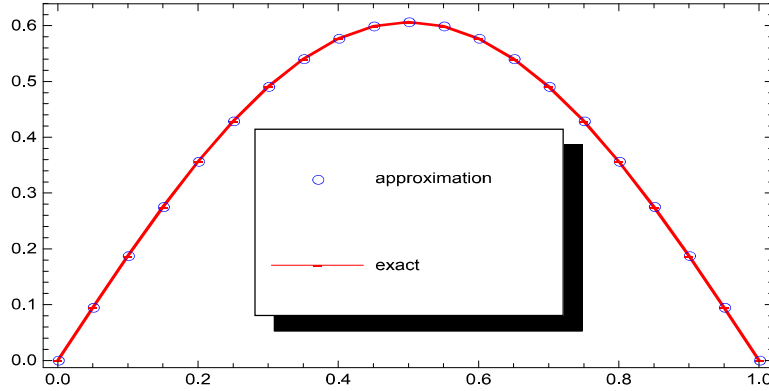
$$h_0(t) = 0,$$

$$h_1(t) = 0,$$

for which the exact solution is[10]:

$$v(\gamma, t) = \frac{\sin(\pi\gamma) \exp(t)}{e^T}.$$

The results of errors obtained with $T = 1$ and $\lambda_1 = e^{-1}$ computed for various values of n together with CPU time using the method presented in this article with the truncated series (2), with $A(\gamma) = \sin(\pi\gamma)$ are shown in Table 5. In Figure 3, the exact and approximate solutions of $v(\gamma, t)$ in $t = 0.5$ for $n = 7$ are shown. The absolute difference between the exact and approximate solution for $n = 5$ is shown in Figure 4.

Figure 3: Exact and approximate solutions with $n = 7$ for Example 3.Table 5: Absolute errors and L^2 norm of errors for Example 3

$\gamma=t$	n=1	n=3	n=5	n=7
0	0	0	0	0
0.1	4.9854×10^{-3}	2.71955×10^{-5}	1.24108×10^{-7}	7.29989×10^{-10}
0.2	9.64955×10^{-4}	2.70053×10^{-5}	2.00455×10^{-7}	3.9988×10^{-10}
0.3	6.75748×10^{-3}	7.2157×10^{-5}	7.28962×10^{-9}	4.50094×10^{-10}
0.4	1.27224×10^{-2}	1.07862×10^{-5}	2.67932×10^{-7}	1.05798×10^{-9}
0.5	1.29088×10^{-2}	6.40166×10^{-5}	1.47392×10^{-7}	2.27621×10^{-10}
0.6	6.05834×10^{-3}	5.57487×10^{-5}	2.89113×10^{-7}	9.60188×10^{-10}
0.7	5.56389×10^{-3}	1.56428×10^{-4}	3.7217×10^{-7}	5.04738×10^{-11}
0.8	1.61871×10^{-2}	1.24564×10^{-4}	1.17941×10^{-7}	4.33895×10^{-10}
0.9	1.7427×10^{-2}	2.72464×10^{-5}	2.59873×10^{-7}	3.82577×10^{-11}
1	0	0	0	0
λ_n	6.25×10^{-2}	3.25521×10^{-4}	6.78168×10^{-7}	7.56884×10^{-10}
errors in L^2 norm	2.11753×10^{-2}	1.50581×10^{-4}	3.55517×10^{-7}	9.2949×10^{-10}
CPU Time	0.359 s	0.406s	0.437 s	0.499 s

5 Conclusion

In this article, the satisfier function in the Ritz–Galerkin method for a parabolic equation with the nonlocal initial condition has been investigated. Theorems on the uniqueness of product satisfier functions have been proved. Satisfier functions enable high adaptability to satisfy, boundary, and non-standard initial conditions. In addition, the use of satisfier function in the Ritz–Galerkin method has reduced the number of basis functions needed to find an accurate approximation of the problem. It is indicated that the satisfier function in the special separable case eventuated in an exact solution of the problem. Obtained results in numerical examples are compared with the results of other published methods and demonstrated the more efficiency of the proposed scheme.

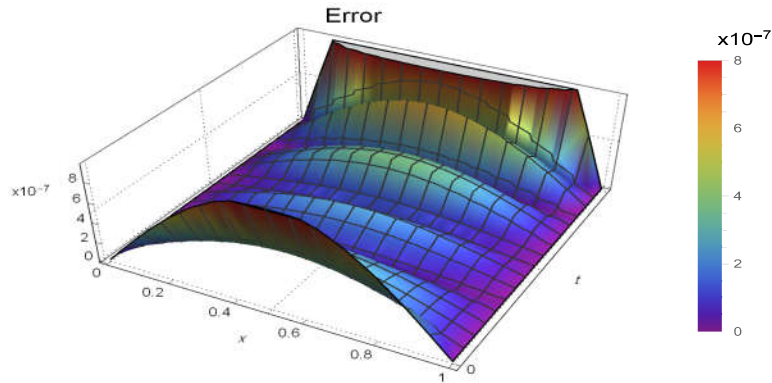


Figure 4: The graph of the absolute error with $n = 5$ for Example 3.

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