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Research Article

On generalized one-step derivative-free iterative family for evaluating multiple roots

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Abstract

In this study, we propose a family of iterative procedures with no derivatives for calculating multiple roots of one-variable nonlinear equations. We also present an iterative technique to approximate the multiplicity of the roots. The new class is optimal since it fits the Kung–Traub hypothesis and has second-order convergence. Derivative-free methods for calculating multiple roots are rarely found in literature, especially in the case of one-step

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methods, which are the simplest ones in terms of their structure. Moreover, this new family contains almost all the existing single-step derivative-free iterative schemes as its special cases, with an additional degree of freedom. Several results are used to confirm its theoretical order of convergence. Through the complex discrete dynamics analysis, the stability of the suggested class is illustrated, and the most stable methods are found. Several test problems are included to check the performance of the proposed methods, whether the multiplicity of the roots is estimated or known, comparing the numerical results with those obtained by other methods.

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Keywords: Nonlinear equations; Derivative–free iterative method; Multiple roots; Order of convergence; Stability.

1 Introduction

One well-known derivative-free scheme for obtaining a multiple root x^* of a nonlinear complex-valued function $f(x)$, $x \in \mathbb{C}$ is the modified Traub–Steffensen's approach [19]. The structure of this scheme is as follows:

$$x_{t+1} = x_t - m\frac{f(x_t)}{f[\eta_t, x_t]}, \quad t = 0, 1, 2, \ldots, \tag{1}$$

where m denotes the known multiplicity of x^* and $f[\eta_t, x_t] = \frac{f(\eta_t)-f(x_t)}{\eta_t - x_t}$ is the first-order divided difference at point x_t and $\eta_t = x_t + \beta f(x_t)$, $\beta \neq 0$, $\beta \in \mathbb{R}$. For $m \geq 2$, the order of convergence of this scheme is quadratic. Scheme (1) is the derivative-free version of the familiar modified Newton's method [15] as it is obtained from this method by estimating the derivative $f'(x_t)$ by $f[\eta_t, x_t]$. Centered on these two schemes, many methods [6, 13, 16, 18, 22, 11, 17, 21, 7, 10] have been derived in earlier studies. Among these, the second-order schemes presented by Kumar, Sharma, and Argyros [9] and Kansal et al. [8], are the nicest ones as they possess the following features: (i) the schemes are derivative-free, (ii) they are optimal in terms of the Kung–Traub conjecture (see [10]), and (iii) the schemes are single-step, which make them easy to implement. The general type of optimal derivative-free procedure for multiple roots defined by Kumar, Sharma, and Argyros [9] is given by

$$x_{t+1} = x_t - H(\Phi_t), \tag{2}$$

where $\Phi_t = \frac{f(x_t)}{f[\eta_t, x_t]}$ and $H(\Phi_t)$ is a single variable weight function. Other main aspect of this scheme is that it holds Traub–Steffensen's method (1) as a special case.

The scheme from Kansal et al. [8] has the following structure:

$$x_{t+1} = x_t - m\frac{(1-a)f(\eta_t) + af(x_t)}{f[\eta_t, x_t]}, \tag{3}$$

where $a \in \mathbb{R}$ is a free parameter. For $a = 1$, this scheme reduces to Traub–Stefensen's method (1).

Motivated by these multiple root-finding schemes, we aim to design a new derivative-free iterative family that is more general than class (2). Therefore, the new class is a single-step optimal scheme with second-order convergence, and it has an additional degree of freedom. In addition, we include an iterative process that allows us to estimate the multiplicity when it is not known.

In order to choose the most stable members of the proposed class, we now introduce some dynamic terms used in this manuscript (see, for instance, [2]). To achieve this aim, the key concept is the rational function. Usually, a rational function T is found by applying the proposed class on a low-degree polynomial $p(z)$ with multiple roots. The properties of this rational function, the existence of fixed points or periodic orbits, and their repulse or attracting character, among others, allow us to detect the best elements of the class or to compare known schemes (see, for example, [4, 5]).

Let us consider a rational function $T : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$, being $\widehat{\mathbb{C}}$ the Riemann sphere. Then, we define the orbit of a point z as follows:

$$\{z, T(z), T^2(z), \ldots, T^n(z), \ldots\},$$

where T^k denotes the kth composition of T with itself.

We analyze T by characterizing the starting points from the asymptotical performance of their orbits. In these terms, a point z is a fixed point of T if $T(z) = z$; it is a p-periodic point, $p > 1$, if $T^p(z) = z$ and $T^k(z) \neq z$ for $k < p$.

The basin of an attractor ξ is defined as

$$\mathcal{A}(\xi) = \{z \in \widehat{\mathbb{C}} : T^n(z) \to \xi, n \to \infty\}. \tag{4}$$

The Fatou set of T, is the set of points whose orbits tend to an attractor (fixed point, periodic point, or infinity). It's complementary in $\widehat{\mathbb{C}}$ is the Julia set. So, the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

On the other hand, a point z is called a critical point of T if $T'(z) = 0$. The asymptotic behavior of the critical points is a key fact for analyzing the stability of the method: A classical result from Fatou and Julia (see, for instance, [1]). Each immediate basin of attraction holds at least one critical point; that is, in the connected component of the basin of attraction holding the attractor, there is also a critical point.

Moreover, a fixed point of T, z, is called attracting if $|T'(z)| < 1$, or super attracting if $|T'(z)| = 0$; it is repulsive if $|T'(z)| > 1$; and parabolic if $|T'(z)| = 1$.

Let us also remark that when fixed and critical points are not equivalent to the roots of $p(z)$, then they are called strange fixed and free critical points, respectively. The asymptotical performance of free critical points is a key element in this study, as when they do not belong to the basin of attraction of the searched zeros. Then, they lay in the basin of an attracting element (a periodic orbit or a strange fixed point) that should be avoided in practice.

Indeed, when T also depends on one or several parameters r_i, $i = 1, 2, \dots, k$, and the stability of a fixed point z is analyzed, $|T'(z, r_1, \dots, r_k)|$ is not a scalar but a function of r_i for $i = 1, 2, \dots, k$. In this case, $|T'(z, r_1, \dots, r_k)|$ is called a stability function of the fixed point, and it gives us the character of the fixed point in terms of the value of r_i, $i = 1, 2, \dots, k$.

Let us summarize the rest of the paper: Section 2 includes the construction and convergence analysis of the new family. Also, in Section 2, special cases of the proposed class are found. To examine the stability of the new methods, we present the analysis of the discrete dynamical system associated with the different sub-classes defined in Section 3, selecting the most stable ones. Section 4 is devoted to the application of proposed methods on a number of numerical problems to back up the theoretical conclusions. In Section 5, we conclude with some final observations.

2 Establishment of second-order class

Let us define a new derivative-free Traub–Steffensen-type class of iterative schemes for finding multiple roots as

$$x_{t+1} = x_t - L(\Theta_t), \tag{5}$$

where $L(\Theta_t)$ is a single-variable weight function, $\Theta_t = \frac{f(x_t)}{f[\eta_t, x_t] + \gamma f(x_t)}$, $\gamma \in \mathbb{R}$, and $\eta_t = x_t + \beta f(x_t),\ \beta \neq 0,\ \beta \in \mathbb{R}$.

We show in the following result that the proposed scheme (5) achieves maximum second-order convergence for every $\beta \neq 0$, without using any derivative evaluation. Moreover, we show that the conditions to be imposed on the weight function are directly related to the multiplicity of the root.

Theorem 1. Let $f : \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ be an analytic function in the open set $\mathbb{D}$, that holds a multiple root x^*, with multiplicity m. Then, the class of iterative schemes (5) has second-order of convergence, if

$$L(0) = 0, \quad L'(0) = m, \quad |L''(0)| < \infty, \tag{6}$$

and therefore it satisfies the error equation

$$e_{t+1} = \frac{1}{2m^2} \left(2m(\gamma + B_1) - L''(0)\right) e_t^2 + O(e_t^3),$$

where $e_t = x_t - x^*$ and $B_1 = \dfrac{1}{m+1} \dfrac{f^{(m+1)}(x^*)}{f^{(m)}(x^*)}$.

Proof. Along Taylor's series expansion of $f(x_t)$ and $f(\eta_t)$ around x^* that has multiplicity m, it is assumed that $f(x^*) = f'(x^*) = f''(x^*) = \cdots = f^{(m-1)}(x^*) = 0$ and $f^{(m)}(x^*) \neq 0$. Also, let us denote B_i by

$$B_i = \frac{m!}{(m+i)!} \frac{f^{(m+i)}(x^*)}{f^{(m)}(x^*)},$$

for $i = 1, 2, 3, \ldots$. Then,

$$f(x_t) = \frac{f^{(m)}(x^*)}{m!} e_t^m \Big(1 + B_1 e_t + B_2 e_t^2 + B_3 e_t^3 + O(e_t^4)\Big), \tag{7}$$

$$f(\eta_t) = \frac{f^{(m)}(x^*)}{m!} e_{\eta_t}^m \Big(1 + B_1 e_{\eta_t} + B_2 e_{\eta_t}^2 + B_3 e_{\eta_t}^3 + O(e_{\eta_t}^4)\Big), \tag{8}$$

where

$$e_{\eta_t} = \eta_t - x^* = e_t + \beta \frac{f^{(m)}(x^*)}{m!} e_t^m \Big(1 + B_1 e_t + B_2 e_t^2 + B_3 e_t^3 + O(e_t^4)\Big).$$

By replacing (7)–(8) in the definition of Θ_t, we obtain

$$\Theta_t = \frac{e_t}{m} - \frac{1}{m^2}(\gamma + B_1)e_t^2 + O(e_t^3). \tag{9}$$

From (9), it can be inferred that Θ_t tends to zero when t tends to infinity. Then, by using MacLaurin's theorem on the weight function $L(\Theta_t)$, we obtain

$$L(\Theta_t) = L(0) + L'(0)\Theta_t + \frac{1}{2!}L''(0)\Theta_t^2 + O(\Theta_t^3). \tag{10}$$

By imposing (9) and (10) in (5), we get

$$e_{t+1} = -L(0) + \left(1 - \frac{L^{(1)}(0)}{m}\right)e_t + \frac{1}{2m^2}\Big(2(\gamma + B_1)L^{(1)}(0) - L^{(2)}(0)\Big)e_t^2 + O(e_t^3). \tag{11}$$

From this error equation, we deduce that schemes belonging to class (5) possess, at least, second-order of convergence, provided that $L(0) = 0$ and $1 - \frac{L'(0)}{m} = 0$, that is, $L'(0) = m$.

Finally, we find the error equation

$$e_{t+1} = \frac{1}{2m^2}\left(2m(\gamma + B_1) - L^{(2)}(0)\right)e_t^2 + O(e_t^3), \tag{12}$$

provided $|L''(0)| < \infty$. Hence, the proposed class (5) converges to the m-multiple root x^* with second-order convergence. □

Remark 1. It is known that $E = \rho^{\frac{1}{\psi}}$ is the computational efficiency in which ρ is the order of convergence of the method under consideration and ψ is the number of functional evaluations per iteration; see [14]. The E-value corresponding to the newly proposed scheme is $E = 2^{\frac{1}{2}} \approx 1.414$, which is the same as that of modified Newton's method and modified Traub–Steffensen's method.

Remark 2. For $\gamma = 0$, the proposed class (5) reduces to scheme (2), and in addition, if we also have $L(\Theta_t) = m\Theta_t$, then we get the modified Traub–Steffensen's method (1).

2.1 Special cases

In this section, we show that the number of special cases for the new presented family is equal to the particular forms of the function $L(\Theta_t)$, fulfilling the convergence conditions of Theorem 1. A few of them are listed in Table 1.

Table 1: Some special cases of class (5)

Method	Weight function	Subcases
TM-1	$L(\Theta_t) = m\Theta_t(1 + a\Theta_t)$	$a = \frac{1}{10}$, $x_{t+1} = x_t - m\Theta_t(1 + \frac{\Theta_t}{10})$
TM-2	$L(\Theta_t) = \frac{m\Theta_t}{1+a\Theta_t}$	$a = \frac{-1}{2}$, $x_{t+1} = x_t - \frac{2m\Theta_t}{2-\Theta_t}$
TM-3	$L(\Theta_t) = \frac{m\Theta_t}{1+am\Theta_t}$	$a = \frac{-1}{2}$, $x_{t+1} = x_t - \frac{2m\Theta_t}{2-m\Theta_t}$
TM-4	$L(\Theta_t) = \frac{m\Theta_t}{(1+a\sqrt{m\Theta_t})^2}$	$a = \frac{-1}{2}$, $x_{t+1} = x_t - \frac{4m\Theta_t}{(2-\sqrt{m\Theta_t})^2}$
TM-5	$L(\Theta_t) = \frac{m(\Theta_t^2+\Theta_t)}{1+am\Theta_t}$	$a = \frac{1}{5}$, $x_{t+1} = x_t - \frac{5m(\Theta_t^2+\Theta_t)}{5-m\Theta_t}$

3 Dynamical analysis

In order to arrange this analysis, we apply our proposed sub-families TM-1 to TM-5 on the nonlinear function $p(z) = (z-1)^m(z+1)$, with $z = 1$ as a multiple root of multiplicity m and a simple root at $z = -1$. In all cases, $\gamma = 1$ and $\beta = \dfrac{1}{10}$; qualitatively, similar results are found for other values of γ and β, but the values of β close to zero make the schemes more stable, due to the estimation of the derivative provided by the first-order divided difference is better. This is the most simple nonlinear function containing two roots, one simple and one m-multiple. Although the results cannot be directly extrapolated to any nonlinear function, several analyses on different nonlinear problems confirm, in the numerical section, these results.

In what follows, we analyze the rational functions related to the particular cases of (5) presented: TM-1 to TM-5.

3.1 Fixed points and stability of rational TM cases

When the classes TM-1, TM-2, TM-3, TM-4, and TM-5 are applied on the polynomial $p(z)$ with multiplicity $m = 2$, rational functions $T_i(z,a)$, $i = 1,2,\dots,5$ are, respectively, obtained. They all depend on the parameter of the class a, considered a free complex value. These rational operators are

$$T_1(z,a) = \frac{N_1(z,a)}{\left(z^5 - z^4 + 28z^3 + 92z^2 + 271z + 9\right)^2},$$

being $N_1(z,a) = -20000a\left(z^2-1\right)^2 + z^{11} - 2z^{10} + 57z^9 + 128z^8 + 942z^7 + 4828z^6 + 18222z^5 + 31768z^4 + 26497z^3 + 21478z^2 + 54281z + 1800$,

$$T_2(z,a) = \frac{4(25a+23)z^3 + (9-100a)z + z^6 - z^5 + 28z^4 + 71z^2 + 200}{100a\left(z^2-1\right) + z^5 - z^4 + 28z^3 + 92z^2 + 271z + 9},$$

$$T_3(z,a) = \frac{N_3(z,a)}{\left(z^5 - z^4 + 28z^3 + 92z^2 + 271z + 9\right)^2},$$

where $N_3(z,a) = -40000a\left(z^2-1\right)^2 + z^{11} - 2z^{10} + 57z^9 + 128z^8 + 942z^7 + 4828z^6 + 18222z^5 + 31768z^4 + 26497z^3 + 21478z^2 + 54281z + 1800$,

$$T_4(z,a) = \frac{N_4(z,a)}{\left(100\sqrt{2}a\left(x^2-1\right) + x^5 - x^4 + 28x^3 + 92x^2 + 271x + 9\right)^2},$$

being $N_4(z,a) = x^{11} - 2x^{10} + 57x^9 + 8(25\sqrt{2}a+16)x^8 + (942-200\sqrt{2}a)x^7 + (5400\sqrt{2}a+4828)x^6 + 2(10000a^2+9300\sqrt{2}a+9111)x^5 + 8(6075\sqrt{2}a+3971)x^4 + (-40000a^2 - 16600\sqrt{2}a + 26497)x^3 + (21478 - 54200\sqrt{2}a)x^2 + (20000a^2 - 1800\sqrt{2}a + 54281)x + 1800$, and

$$T_5(z,a) = \frac{N_5(z,a)}{D_5(z,a)},$$

where $N_5(z,a) = -18200 + (54281-1800a)z + (61478-54200a)z^2 + (26497-16600a)z^3 + 8(1471+6075a)z^4 + 6(3037+3100a)z^5 + (4828+5400a)z^6 + (942-$

$200a)z^7+8(16+25a)z^8+57z^9-2z^{10}+z^{11}$ and $D_5(z,a)=(9+271z+92z^2+28z^3-z^4+z^5)(9+271z+92z^2+28z^3-z^4+z^5+200a(-1+z^2))$.

By solving each equation $T_i(z,a)=z$, $i=1,2,\ldots,5$, the fixed points of the ith rational function are obtained. Two of them are the roots of polynomial $p(z)$; the rest of them, if they exist, are strange fixed points. The asymptotical behavior of all the fixed points (both multiple and simple, strange or not) plays a key role in the stability of the iterative methods involved, as the convergence to fixed points different from the roots means an important drawback for an iterative method; so, we proceed below with this analysis.

In order to study the stability of the fixed points z^F of $T_i(z,a)$, we calculate its stability function, $\left|T_i'(z^F,a)\right|$. This function gives us information about the asymptotical behavior of the point in terms of the value of a. In general, the stability of other fixed points than the multiple roots of $p(z)$ depends on the value of parameter a. From these rational functions, the following result can be stated.

Theorem 2. Rational function $T_i(z,a)$ has $z=1$ and $z=-1$ as superattracting and parabolic fixed points, respectively. There can also exist strange fixed points, denoted by $r_i^j(a)$, where the index $i=1,2,\ldots,5$ corresponds to the rational operator $T_i(z,a)$ and the index $j=1,2,\ldots,5$ corresponds to the fixed point as the root of fifth-degree polynomial $q_i(t)$ described below:

$i=1$: Roots of polynomial $q_1(t)=9-100a+271t+(92+100a)t^2+28t^3-t^4+t^5$ are the strange fixed points of $T_1(z,a)$.

$i=2$: There are no strange fixed points of $T_2(z,a)$.

$i=3$: When $T_3(z,a)$ is analyzed, the strange fixed points are the roots of $q_3(t)=9-200a+271t+(92+200a)t^2+28t^3-t^4+t^5$.

$i=4$: $T_4(z,a)$ has as strange fixed points the roots of $q_4(t)=9+271t+92t^2+28t^3-t^4+t^5$.

$i=5$: Strange fixed points of $T_5(z,a)$ are the roots of $q_5(t)=-91+271t+192t^2+28t^3-t^4+t^5$.

Proof. By solving the equation $T_i(z,a) = z$, $i = 1,2,3,5$, it is found that, in the case when $i = 1$,

$$T_1(z,a) = z \Leftrightarrow N_1(z,a) = z\left(z^5 - z^4 + 28z^3 + 92z^2 + 271z + 9\right)^2,$$

or, equivalently,

$$(z^2-1)(9+271z+92z^2+28z^3-z^4+z^5+100a1(z^2-1)) = 0 \Leftrightarrow (z^2-1)q_1(z) = 0.$$

For $i = 2$, it can be checked that

$$T_2(z,a) = z \Leftrightarrow z^1 - 1 = 0.$$

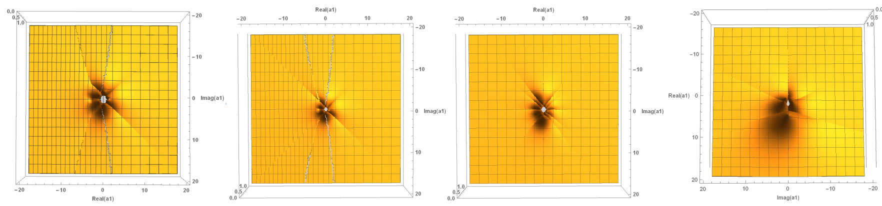
In a similar way, the statements for $i = 3$, $i = 4$, and $i = 5$ are obtained. □

So, it has been proven that the multiple root is always a superattracting fixed point, whereas the simple root is parabolic. Hence, it always lays in the Julia set and would never be reached exactly by the iterative method (although it can be very close). For $i = 1,3,4,5$, there exist strange fixed points of their respective operators, $T_i(z,a)$. Regarding the stability of these fixed points, the following remark can be checked.

Remark 3. Operators $T_i(z,a)$ for $i = 1,3,4,5$, have attracting strange fixed point at any complex value of a, except in a disk $|a| < r$ with

- $r \approx 1.12$ for $T_1(z,a)$;
- $r \approx 0.56$ for $T_3(z,a)$;
- $r \approx 0.75$ for $T_4(z,a)$;
- $r \approx 0.11$ for $T_5(z,a)$.

These small regions, where all the strange fixed points are repulsive, can be observed in Figure 1. In it, the code color is as follows: One point of the mesh is represented in orange color if there exists at least one strange fixed point that is attracting this value of the parameter. So, if one point of the mesh is not colored, it means that for this value of a, all the strange fixed points are repulsive.



(a) $|T_1'(r_1^j, a)|$ (b) $|T_3'(r_3^j, a)|$ (c) $|T_4'(r_4^j, a)|$ (d) $|T_5'(r_5^j, a)|$

Figure 1: Union of stability functions for rational operators on strange fixed points $r_i^j(a)$, $i = 1, 3, 4, 5$ and $j = 1, 2, \ldots, 5$.

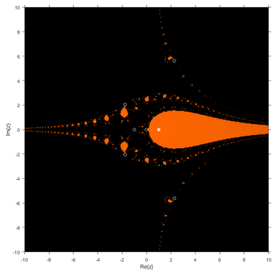
3.2 Dynamical planes

Each value of the parameter corresponds with a member of each subclass of iterative methods. When a is fixed in a rational function $T_i(z, a)$, $i = 1, 2, \ldots, 5$, the performance of the iterative process can be visualized in a dynamical plane. It is obtained by iterating the chosen element of the family under study and by using each point of the complex plane as an initial estimation. In this section, we have used a mesh of 1000×1000 points.

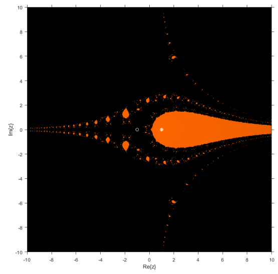
We represent in blue color those points whose orbit converges to infinity, in orange the points converging to the multiple root $z = 1$ (with a tolerance of 10^{-3}), and in other colors (green, red, etc.) those points whose orbit converges to one of the fixed points (all fixed points appear marked as a white star in the figures if they are attractive or by a white circle if they are repulsive or parabolic). Moreover, a point appears in black if it reaches the maximum number of 500 iterations without converging to any of the fixed points. The routines used appear in [3].

Firstly, we plot the dynamical planes related to rational functions $T_i(z, a)$, $i = 1, 2, 3, 5$ for $a = 0.05$. These planes can be visualized in Figure 2. From Theorem 2 and Remark 3, only the multiple root of $p(z)$ is an attracting fixed point.

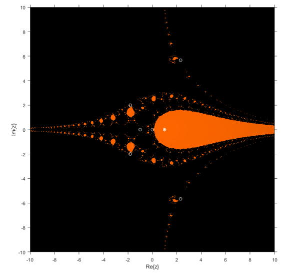
From the dynamical planes of Figure 2, it is again deduced that the only basin of attraction is that of the multiple roots; moreover, if there exist strange fixed points, they lay in the Julia set, as per their repulsive character. Black areas correspond to divergent performance. The wideness of the basins



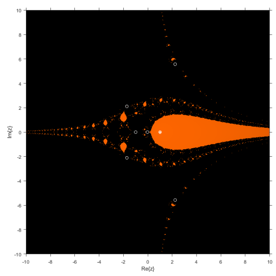
(a) $T_1(z, 0.05)$



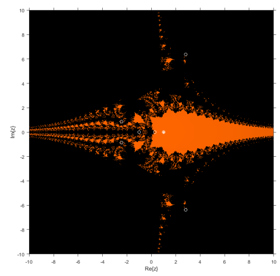
(b) $T_2(z, 0.05)$



(c) $T_3(z, 0.05)$



(d) $T_4(z, 0.05)$



(e) $T_5(z, 0.05)$

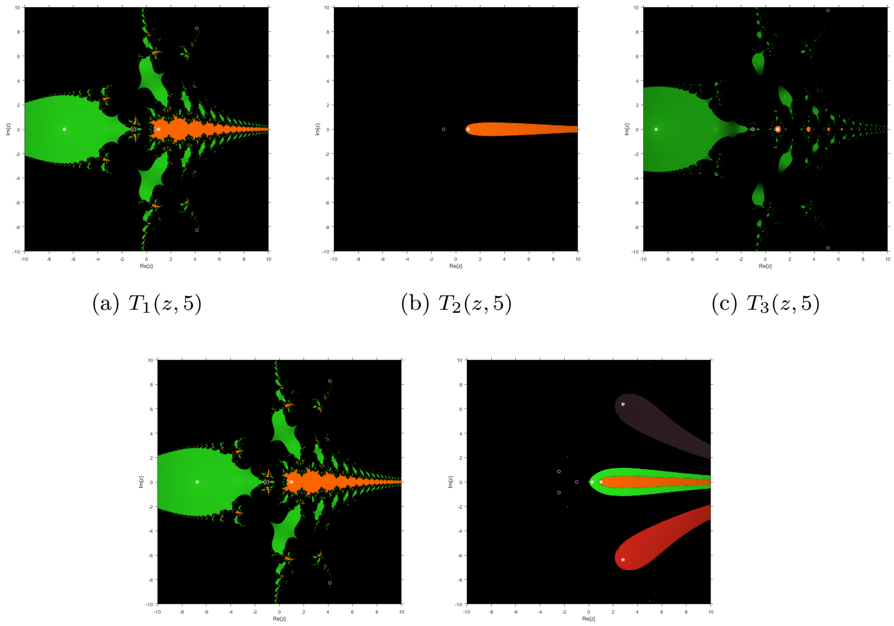
Figure 2: Dynamical planes corresponding to rational operators under study for $a = 0.05$.

of attraction composing the Fatou set is similar in all cases but slightly better in the case of $T_5(z, 0.05)$.

In Figure 3, the dynamical planes corresponding to $a = 5$ are presented. For this value of the parameter, only T_2 operator does not attract strange fixed points, from Theorem 2 and Remark 3.

We observe in Figure 3e that $T_5(z, 5)$ has three attracting strange fixed points, with their corresponding basins of attraction in green, red, and dark aubergine color, respectively. Also, $T_1(z, 5)$, $T_4(z, 5)$, and $T_3(z, 5)$ show green basins of attraction of strange fixed points. The latter case is especially inefficient, as the basin of attraction of the multiple roots is really small. The basin of attraction of the multiple roots in the case of the operator $T_2(z, 5)$ is the only one in the dynamical plane, although it is narrower than in the case of $a = 0.05$.

Figure 4 corresponds to the dynamical planes of rational functions $T_i(z, -1)$, $i = 1, 2, \ldots, 5$. Cases $i = 3$, $i = 4$, and $i = 5$ show basins of attraction of



(a) $T_1(z,5)$ (b) $T_2(z,5)$ (c) $T_3(z,5)$

(d) $T_4(z,5)$ (e) $T_5(z,5)$

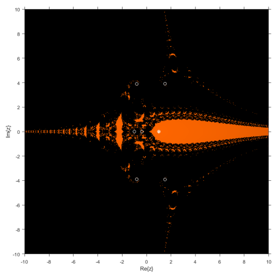
Figure 3: Dynamical planes corresponding to rational operators under study for $a = 5$.

strange fixed points. Meanwhile cases $i = 1$ and $i = 2$ only converge to the multiple roots, being the last one the widest.

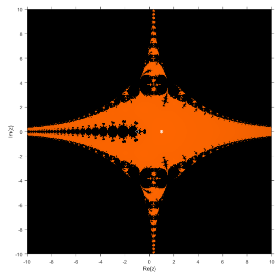
We conclude from the analysis made that the most subclass is $T_2(z, a)$, although the rest of the operators perform adequately for small absolute values of a.

4 Numerical results

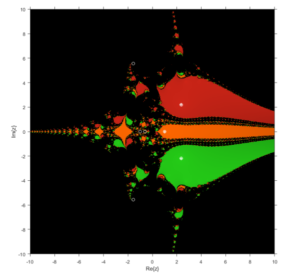
In this section, we use numerical examples to confirm the performance of the newly created iterative approaches as well as to validate the theoretical results presented in the previous sections. Furthermore, we compare our new proposed methods TM-1–TM-5 with the second-order methods given by equations (1) and (3). We denote these methods by MNM and KM, respectively. For the scheme (3), we choose the parameter $a = 6/7$. All the



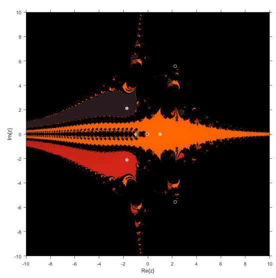
(a) $T_1(z, -1)$



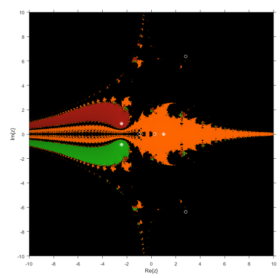
(b) $T_2(z, -1)$



(c) $T_3(z, -1)$



(d) $T_4(z, -1)$



(e) $T_5(z, -1)$

Figure 4: Dynamical planes corresponding to rational operators under study for $a = -1$.

computations for the above-mentioned methods are performed using software Mathematica with 4096 significant digits as multiple precision arithmetic. In numerical results, we have calculated the following terms:

- The number of iterations t required to obtain the desired root fulfilling the condition that $|x_{t+1} - x_t| + |f(x_{t+1})| < 10^{-50}$.
- Estimated error $e_{t+1} = |x_{t+1} - x_t|$ in the last three iterations.
- Residual error for the considered function, that is, $|f(x_{t+1})|$.
- The computational order of convergence COC [20], which is given by

$$COC = \frac{\log|\frac{x_{t+2}-x^*}{x_{t+1}-x^*}|}{\log|\frac{x_{t+1}-x^*}{x_t-x^*}|}, \quad t = 1, 2, \dots. \tag{13}$$

The problems used for validating the theoretical results and comparison are mentioned in Table 2. The numerical results obtained for these problems have been presented in Tables 3–6. From them, we see that the new proposed methods are, at least, as stable as those existing methods used for comparison. Also, the presented results have verified the theoretical results proven so far. Moreover, from Tables 3–6, it is observed that methods TM-2, TM-4, and TM-5 have performed rightly well in all the problems. Furthermore, the performance of the methods TM-1 and TM-3 is either at par or better than the results obtained for the existing methods.

Table 2: Test functions considered, their corresponding roots and multiplicity

Test Functions	x_0	*Root*	m
$f_1(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675$	1.8	$1.75000\ldots$	2
$f_2(x) = \left(\tan^{-1}(\frac{1}{x} - 1) + e^{x^2} - 3\right)^2$	1	$1.05653\ldots$	2
$f_3(x) = (\sin x \cos x - x^3 + 1)^9$	1.4	$1.11707\ldots$	9
$f_4(x) = ((x-1)^3 - 1)^{50}$	2.1	$2.00000\ldots$	50

Table 3: Numerical results for $f_1(x)$

Methods	t	$\|e_{t-1}\|$	$\|e_t\|$	$\|e_{t+1}\|$	$\|f(x_{t+1})\|$	COC
TM-1 ($\gamma = 0$)	8	5.45×10^{-15}	4.93×10^{-28}	4.04×10^{-54}	2.21×10^{-213}	2.00000
($\gamma = -1$)	8	1.45×10^{-15}	3.38×10^{-29}	1.84×10^{-56}	8.94×10^{-223}	2.00000
TM-2 ($\gamma = 0$)	8	3.25×10^{-15}	1.73×10^{-28}	4.92×10^{-55}	4.76×10^{-217}	2.00000
($\gamma = -1$)	8	8.25×10^{-16}	1.08×10^{-29}	1.87×10^{-57}	9.27×10^{-227}	2.00000
TM-3 ($\gamma = 0$)	8	1.66×10^{-15}	4.46×10^{-29}	3.22×10^{-56}	8.40×10^{-222}	2.00000
($\gamma = -1$)	8	3.97×10^{-16}	2.47×10^{-30}	9.55×10^{-59}	6.11×10^{-232}	2.00000
TM-4 ($\gamma = 0$)	8	9.41×10^{-16}	1.41×10^{-29}	3.19×10^{-57}	7.90×10^{-226}	2.00000
($\gamma = -1$)	8	2.13×10^{-16}	7.02×10^{-31}	7.63×10^{-60}	2.42×10^{-236}	2.00000
TM-5 ($\gamma = 0$)	8	9.76×10^{-16}	1.52×10^{-29}	3.69×10^{-57}	1.42×10^{-225}	2.00000
($\gamma = -1$)	8	2.22×10^{-16}	7.62×10^{-31}	8.98×10^{-60}	4.67×10^{-236}	2.00000
MNM	8	6.18×10^{-15}	6.37×10^{-28}	6.76×10^{-54}	1.74×10^{-212}	2.00000
KM	8	6.18×10^{-15}	6.37×10^{-28}	6.76×10^{-54}	1.74×10^{-212}	2.00000

Table 4: Numerical results for $f_2(x)$

Methods	t	$\lvert e_{t-1}\rvert$	$\lvert e_t\rvert$	$\lvert e_{t+1}\rvert$	$\lvert f(x_{t+1})\rvert$	COC
TM-1 ($\gamma=0$)	7	3.54×10^{-16}	2.36×10^{-31}	1.05×10^{-61}	1.35×10^{-242}	2.00000
($\gamma=-1$)	7	2.80×10^{-18}	1.09×10^{-35}	1.63×10^{-70}	4.24×10^{-278}	2.00000
TM-2 ($\gamma=0$)	7	6.03×10^{-17}	6.14×10^{-33}	6.36×10^{-65}	1.44×10^{-255}	2.00000
($\gamma=-1$)	7	2.48×10^{-19}	7.30×10^{-38}	6.33×10^{-75}	6.98×10^{-296}	2.00000
TM-3 ($\gamma=0$)	7	4.87×10^{-18}	3.41×10^{-35}	1.67×10^{-69}	4.95×10^{-274}	2.00000
($\gamma=-1$)	7	6.40×10^{-21}	3.84×10^{-41}	1.38×10^{-81}	9.92×10^{-323}	2.00000
TM-4 ($\gamma=0$)	7	4.08×10^{-19}	2.05×10^{-37}	5.15×10^{-74}	3.29×10^{-292}	2.00000
($\gamma=-1$)	7	1.27×10^{-22}	1.17×10^{-44}	1.01×10^{-88}	1.68×10^{-351}	2.00000
TM-5 ($\gamma=0$)	7	4.03×10^{-19}	2.01×10^{-37}	4.99×10^{-74}	2.93×10^{-292}	2.00000
($\gamma=-1$)	7	1.25×10^{-22}	1.15×10^{-44}	9.74×10^{-89}	1.51×10^{-351}	2.00000
MNM	7	5.36×10^{-16}	5.56×10^{-31}	5.99×10^{-61}	1.49×10^{-239}	2.00000
KM	7	4.94×10^{-16}	4.69×10^{-31}	4.23×10^{-61}	3.66×10^{-240}	2.00000

Table 5: Numerical results for $f_3(x)$

Methods	t	$\lvert e_{t-1}\rvert$	$\lvert e_t\rvert$	$\lvert e_{t+1}\rvert$	$\lvert f(x_{t+1})\rvert$	COC
TM-1 ($\gamma=0$)	7	8.49×10^{-14}	6.77×10^{-27}	4.30×10^{-53}	8.02×10^{-938}	2.00000
($\gamma=-1$)	7	2.05×10^{-15}	3.46×10^{-30}	9.93×10^{-60}	9.12×10^{-1058}	2.00000
TM-2 ($\gamma=0$)	7	2.23×10^{-14}	4.45×10^{-28}	1.77×10^{-55}	6.06×10^{-981}	2.00000
($\gamma=-1$)	7	2.85×10^{-16}	6.37×10^{-32}	3.17×10^{-63}	6.68×10^{-1121}	2.00000
TM-3 ($\gamma=0$)	7	1.91×10^{-18}	1.64×10^{-36}	1.21×10^{-72}	1.30×10^{-1292}	2.00000
($\gamma=-1$)	7	3.56×10^{-17}	4.28×10^{-34}	6.21×10^{-68}	6.26×10^{-1209}	2.00000
TM-4 ($\gamma=0$)	7	2.23×10^{-23}	3.07×10^{-46}	5.81×10^{-92}	4.14×10^{-1639}	2.00000
($\gamma=-1$)	7	3.83×10^{-20}	7.40×10^{-40}	2.77×10^{-79}	1.10×10^{-1411}	2.00000
TM-5 ($\gamma=0$)	7	3.45×10^{-21}	7.58×10^{-42}	3.66×10^{-83}	1.42×10^{-1480}	2.00000
($\gamma=-1$)	7	3.19×10^{-21}	5.36×10^{-42}	1.52×10^{-83}	3.18×10^{-1488}	2.00000
MNM	7	1.16×10^{-13}	1.27×10^{-26}	1.53×10^{-52}	7.46×10^{-928}	2.00000
KM	8	7.64×10^{-25}	5.54×10^{-49}	2.91×10^{-97}	7.99×10^{-1733}	2.00000

4.1 Unknown multiplicity

Now, it is clear that the proposed schemes are able to find efficiently the multiple roots of nonlinear equations with a known multiplicity. Then, the question about unknown multiplicity arises: Is it possible to modify these methods in order to simultaneously estimate the roots and their multiplicity? McNamee [12] presented different techniques due to several authors that estimated with unequal results the searched multiplicity. We propose a

Table 6: Numerical results for $f_4(x)$

Methods	t	$\lvert e_{t-1}\rvert$	$\lvert e_t\rvert$	$\lvert e_{t+1}\rvert$	$\lvert f(x_{t+1})\rvert$	COC
TM-1 ($\gamma = 0$)	7	3.38×10^{-17}	1.14×10^{-33}	1.30×10^{-66}	1.36×10^{-6565}	2.00000
($\gamma = -1$)	7	2.52×10^{-17}	6.22×10^{-34}	3.78×10^{-67}	1.25×10^{-6619}	2.00000
TM-2 ($\gamma = 0$)	7	3.01×10^{-17}	8.96×10^{-34}	7.95×10^{-67}	4.56×10^{-6587}	2.00000
($\gamma = -1$)	7	2.24×10^{-17}	4.86×10^{-34}	2.29×10^{-67}	1.42×10^{-6641}	2.00000
TM-3 ($\gamma = 0$)	7	1.20×10^{-21}	7.16×10^{-43}	2.57×10^{-85}	5.21×10^{-8451}	2.00000
($\gamma = -1$)	7	6.44×10^{-22}	1.99×10^{-43}	1.90×10^{-86}	6.21×10^{-8565}	2.00000
TM-4 ($\gamma = 0$)	7	3.80×10^{-18}	1.24×10^{-35}	1.31×10^{-70}	2.63×10^{-6968}	2.00000
($\gamma = -1$)	7	2.69×10^{-18}	6.06×10^{-36}	3.08×10^{-71}	7.54×10^{-7032}	2.00000
TM-5 ($\gamma = 0$)	7	9.27×10^{-19}	6.70×10^{-37}	3.51×10^{-73}	8.88×10^{-7228}	2.00000
($\gamma = -1$)	7	6.32×10^{-19}	3.04×10^{-37}	7.01×10^{-74}	3.04×10^{-7298}	2.00000
MNM	7	3.48×10^{-17}	1.21×10^{-33}	1.47×10^{-66}	2.98×10^{-6560}	2.00000
KM	7	3.48×10^{-17}	1.21×10^{-33}	1.47×10^{-66}	2.98×10^{-6560}	2.00000

modification of our schemes based on the idea of Traub (see [19]),

$$m = \lim_{t\to+\infty} \frac{\ln|f(x_t)|}{\ln\left|\frac{f(x_t)}{f'(x_t)}\right|},$$

for estimating the multiplicity in the Schöder scheme

$$x_{t+1} = x_t - m\frac{f(x_t)}{f'(x_t)}.$$

Expanding this idea to our class of iterative methods, we estimate the multiplicity m in expression (5) as

$$m_t \approx \frac{\ln|f(x_t)|}{\ln\left|\frac{L(\Theta)}{m_{t-1}}\right|}, \qquad t = 1, 2, \dots,$$

starting with an initial $m_0 = \frac{\ln|f(x_t)|}{\ln|\Theta|}$. This provides different expressions of m_t for each special case of the presented family. These estimators of the multiplicity appear described in Table 7. For known methods, MNM and KM, the estimation of the multiplicity has been made by the original idea of Traub,

$$m_t = \frac{\ln|f(x_t)|}{\ln\left|\frac{f(x_t)}{f[\eta_t, x_t]}\right|}, \qquad t = 1, 2, \dots,$$

and

$$m_t = \frac{\ln|f(x_t)|}{\ln\left|\frac{(1-a)f(\eta_t)+af(x_t)}{f[\eta_t,x_t]}\right|}, \qquad t = 1, 2, \dots,$$

respectively.

Table 7: Estimators of the multiplicity for the special cases of class (5)

Method	m_t
TM-1	$\Theta_t(1+a\Theta_t)$
TM-2	$\frac{\Theta_t}{1+a\Theta_t}$
TM-3	$\frac{\Theta_t}{1+am_{t-1}\Theta_t}$
TM-4	$\frac{\Theta_t}{(1+a\sqrt{m_{t-1}}\Theta_t)^2}$
TM-5	$\frac{(\Theta_t^2+\Theta_t)}{1+am_{t-1}\Theta_t}$

We have made several tests on the same nonlinear equations described in Table 2, with the same initial estimations or other more distant estimates, using the same stopping criterium and number of digits. The obtained results are presented in Tables 8–11. In them, the estimated order of convergence, COC, does not appear as convergence is linear in all cases for proposed and existing methods. We consider that it is due to the natural instability provided by divided differences when the estimation of the multiplicity involves more uncertainty in data. We show the last error estimations, $|e_{t+1}|$, $|f(x_{t+1})|$, the number of iterations, iter, and the last approximation of the multiplicity provided by the iterative scheme, m_{t+1}.

Tables 8–11 show how the number of iterations has increased for all methods, possibly due to the instability generated by the estimation of the root multiplicity. All the schemes converge with a similar number of iterations, and all of them achieve, in all cases, a reasonable estimate of the multiplicity. Let us remark that the precision of the estimation of the multiplicity in the cases $f_1(x)$ to $f_3(x)$ is excellent. In the case of $f_4(x)$, the relative error of this estimation is around 4%.

Finally, we test our methods with a more challenging function $f_5(x) = 4x^2+8\sin(x)-4\pi x+\pi^2-8$, whose multiplicity is not obvious. Moreover, there exists a wide neighborhood around the multiple roots $\frac{\pi}{2}$, where the function is almost null. Due to the complexity of the problem, the stopping criterium

Table 8: Numerical results for $f_1(x)$, with $x_0 = 1.8$ and estimation of the multiplicity

Methods	$iter$	$\lvert e_{t+1}\rvert$	$\lvert f(x_{t+1})\rvert$	m_{t+1}
TM-1 ($\gamma = 0$)	32	2.48×10^{-51}	1.49×10^{-107}	2.0200
($\gamma = -1$)	32	4.89×10^{-51}	5.84×10^{-107}	2.0175
TM-2 ($\gamma = 0$)	32	3.25×10^{-51}	2.56×10^{-107}	2.0174
($\gamma = -1$)	32	5.66×10^{-51}	7.83×10^{-107}	2.0175
TM-3 ($\gamma = 0$)	32	4.49×10^{-51}	4.91×10^{-107}	2.0175
($\gamma = -1$)	32	6.71×10^{-51}	1.10×10^{-106}	2.0175
TM-4 ($\gamma = 0$)	32	5.24×10^{-51}	6.69×10^{-106}	2.0175
($\gamma = -1$)	32	7.21×10^{-51}	1.28×10^{-106}	2.0175
TM-5 ($\gamma = 0$)	32	5.12×10^{-51}	6.41×10^{-107}	2.0175
($\gamma = -1$)	32	7.13×10^{-51}	1.25×10^{-106}	2.0175
MNM	32	2.30×10^{-51}	1.28×10^{-107}	2.0174
KM	32	2.31×10^{-51}	1.29×10^{-107}	2.0174

Table 9: Numerical results for $f_2(x)$, with $x_0 = 1$ and estimation of the multiplicity

Methods	$iter$	$\lvert e_{t+1}\rvert$	$\lvert f(x_{t+1})\rvert$	m_{t+1}
TM-1 ($\gamma = 0$)	45	1.16×10^{-51}	1.79×10^{-104}	1.9605
($\gamma = -1$)	45	1.76×10^{-51}	4.17×10^{-104}	1.9603
TM-2 ($\gamma = 0$)	45	1.42×10^{-51}	2.72×10^{-104}	1.9604
($\gamma = -1$)	45	2.15×10^{-51}	6.27×10^{-104}	1.9603
TM-3 ($\gamma = 0$)	45	1.35×10^{-51}	2.47×10^{-104}	1.9604
($\gamma = -1$)	45	2.05×10^{-51}	5.71×10^{-104}	1.9603
TM-4 ($\gamma = 0$)	45	1.63×10^{-51}	3.59×10^{-104}	1.9604
($\gamma = -1$)	45	2.47×10^{-51}	8.27×10^{-104}	1.9602
TM-5 ($\gamma = 0$)	45	1.80×10^{-51}	4.38×10^{-104}	1.9603
($\gamma = -1$)	45	2.72×10^{-51}	1.01×10^{-103}	1.9602
MNM	45	1.15×10^{-51}	1.78×10^{-104}	1.9605
KM	45	1.94×10^{-51}	5.11×10^{-104}	1.9603

have changed to $|f(x_{t+1})| < 10^{-15}$. The numerical results are presented in Tables 12 and 13.

Table 10: Numerical results for $f_3(x)$, with $x_0 = 0.8$ and estimation of the multiplicity

Methods	*iter*	$\lvert e_{t+1}\rvert$	$\lvert f(x_{t+1})\rvert$	m_{t+1}
TM-1 ($\gamma = 0$)	66	1.01×10^{-51}	1.92×10^{-467}	8.7317
($\gamma = -1$)	67	4.68×10^{-52}	1.83×10^{-470}	8.7333
TM-2 ($\gamma = 0$)	66	3.77×10^{-51}	3.07×10^{-462}	8.7287
($\gamma = -1$)	67	1.77×10^{-51}	3.24×10^{-465}	8.7304
TM-3 ($\gamma = 0$)	66	1.92×10^{-51}	6.68×10^{-465}	8.7302
($\gamma = -1$)	67	8.92×10^{-52}	6.40×10^{-468}	8.7319
TM-4 ($\gamma = 0$)	66	4.75×10^{-51}	2.51×10^{-461}	8.7282
($\gamma = -1$)	67	2.22×10^{-51}	2.58×10^{-464}	8.7299
TM-5 ($\gamma = 0$)	67	6.41×10^{-52}	3.19×10^{-469}	8.7327
($\gamma = -1$)	67	9.67×10^{-51}	1.60×10^{-458}	8.7266
MNM	66	9.79×10^{-52}	1.50×10^{-467}	8.7317
KM	67	7.90×10^{-52}	2.13×10^{-468}	8.7322

Table 11: Numerical results for $f_4(x)$, with $x_0 = 1.5$ and estimation of the multiplicity

Methods	*iter*	$\lvert e_{t+1}\rvert$	$\lvert f(x_{t+1})\rvert$	m_{t+1}
TM-1 ($\gamma = 0$)	72	3.09×10^{-51}	2.07×10^{-2570}	47.9690
($\gamma = -1$)	72	4.97×10^{-51}	4.75×10^{-2560}	47.9610
TM-2 ($\gamma = 0$)	72	3.90×10^{-51}	2.48×10^{-2565}	47.9650
($\gamma = -1$)	72	6.26×10^{-51}	5.57×10^{-2555}	47.9570
TM-3 ($\gamma = 0$)	72	7.98×10^{-51}	1.15×10^{-2549}	47.9530
($\gamma = -1$)	73	5.40×10^{-52}	1.17×10^{-2608}	47.9980
TM-4 ($\gamma = 0$)	72	7.46×10^{-51}	3.98×10^{-2551}	47.9540
($\gamma = -1$)	73	5.05×10^{-52}	4.06×10^{-2610}	47.9999
TM-5 ($\gamma = 0$)	72	7.16×10^{-51}	4.96×10^{-2552}	47.9555
($\gamma = -1$)	73	4.84×10^{-52}	5.02×10^{-2611}	47.9999
MNM	72	3.08×10^{-51}	1.63×10^{-2570}	47.9699
KM	72	3.09×10^{-51}	1.96×10^{-2570}	47.9699

All the methods perform in a similar way, but schemes TM-1 to TM-5 get better results than MNM and KM methods with $x_0 = 1.8$, being approximately equal in the case where $x_0 = 1$. In all cases, we conclude that the

Table 12: Numerical results for $f_5(x)$, with $x_0 = 1.8$ and estimation of the multiplicity

Methods	*iter*	$\|f(x_{t+1})\|$	m_{t+1}
TM-1 ($\gamma = 0$)	5	5.27×10^{-16}	3.4652
($\gamma = -1$)	5	6.03×10^{-16}	3.4857
TM-2 ($\gamma = 0$)	5	5.78×10^{-16}	3.4755
($\gamma = -1$)	5	6.13×10^{-16}	3.4939
TM-3 ($\gamma = 0$)	5	6.11×10^{-16}	3.4910
($\gamma = -1$)	5	6.15×10^{-16}	3.4951
TM-4 ($\gamma = 0$)	5	6.14×10^{-16}	3.4952
($\gamma = -1$)	5	6.08×10^{-16}	3.4839
TM-5 ($\gamma = 0$)	5	6.13×10^{-16}	3.4939
($\gamma = -1$)	5	6.12×10^{-16}	3.4909
MNM	5	5.25×10^{-16}	3.4649
KM	5	5.26×10^{-16}	3.4650

Table 13: Numerical results for $f_5(x)$, with $x_0 = 1$ and estimation of the multiplicity

Methods	*iter*	$\|f(x_{t+1})\|$	m_{t+1}
TM-1 ($\gamma = 0$)	7	6.05×10^{-16}	3.4804
($\gamma = -1$)	7	5.65×10^{-16}	3.4722
TM-2 ($\gamma = 0$)	7	6.14×10^{-16}	3.4950
($\gamma = -1$)	7	2.18×10^{-16}	3.4440
TM-3 ($\gamma = 0$)	7	5.64×10^{-16}	3.4719
($\gamma = -1$)	8	5.93×10^{-16}	3.4635
TM-4 ($\gamma = 0$)	7	2.86×10^{-16}	3.4468
($\gamma = -1$)	8	6.16×10^{-16}	3.4979
TM-5 ($\gamma = 0$)	7	1.35×10^{-16}	3.4411
($\gamma = -1$)	8	6.16×10^{-16}	3.4970
MNM	7	6.04×10^{-16}	3.4788
KM	7	6.06×10^{-16}	3.4870

multiplicity of the root is 4. Regarding the precision of the estimation of multiplicity, the relative error is, in this case, between 12% and 13%.

5 Conclusions

In this paper, a novel second-order scheme free from derivatives for locating the multiple roots of univariate nonlinear functions was developed. A weight function $L(\Theta_t)$ was used during the development of the new scheme. The greatest benefit of using weight functions is that we can produce as many special cases as possible and various choices of the weight functions subject to some conditions. The error equation in the main result corroborated the second-order convergence of the proposed scheme. It has been shown that the new family of methods is a generalized one as it contains almost all the existing methods of the same type as its special cases. Not only the basins of attraction, but also a deep stability analysis in the Riemann sphere have also been performed for the new schemes, showing the good performance of all the proposed subclasses for small values of the parameter, especially the good performance of the TM-2 class. On the other hand, we have assigned to each method, parallel to the iterative process to obtain the root, an iterative scheme to approximate the multiplicity when it is not known.

The numerical results for different examples showed that our new method is efficient, especially in those cases where the derivative of the nonlinear function is not available.

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