Iranian Journal of Numerical Analysis and Optimization
Vol. 13, No. 2, 2023, pp 157–169
https://doi.org/10.22067/ijnao.2022.74901.1095
https://ijnao.um.ac.ir/

Research Article

A three-free-parameter class of power series based iterative method for approximation of nonlinear equations solution

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Abstract

In this manuscript, for approximation of solutions to equations that are nonlinear, a new class of two-point iterative structure that is based on a weight function involving two converging power series, is developed. For any method constructed from the developed class of methods, it requires three separate functions evaluation in a complete iteration cycle that is of order four convergence. Also, some well-known existing methods are typical members of the new class of methods. The numerical test on some concrete methods derived from the class of methods indicates that they are effective and competitive when employed in solving a nonlinear equation.

AMS subject classifications (2020): Primary 65H05; Secondary 41A25.

Keywords: Newton method, Iterative method, Power series, Weight function.

1 Introduction

One famous iterative method (IM) that is effective when used for the approximation of the solution s^* of the nonlinear equation (NE) $f(s) = 0$ is the

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Received 23 January 2022; revised 12 March 2022; accepted 2 July 2022

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Newton Iterative Method (NIM) [18], expressed as

$$w_{k+1} = s_k - u\left(s_k\right), \tag{1}$$

where $u\left(s_k\right) = \frac{f(s_k)}{f'(s_k)}$, $k = 0, 1, 2, \ldots$, and $\left|f'\left(s_k\right)\right| \neq 0$. The convergence order (CO) of the NIM is 2 with efficiency index (EI) 1.4142. Since the emergence of the NIM, the plethora of techniques have been employed to modify it with the motivation of enhancing its CO or EI when utilized to solve an NE. One of such techniques that have been used is the weight function, which is a power series based on the form:

$$w\left(\theta(s)\right) = 1 + \sum_{i=1}^{m} \alpha_i (\theta(s))^i \tag{2}$$

or

$$w\left(\theta(s)\right) = 1 + \sum_{i=1}^{m} \alpha_i (\mu(s) - 1)^i, \tag{3}$$

where $\alpha_i \in R$, $1 \leq m \leq \infty$ and $\theta\left(s\right), \mu\left(s\right)$ are often quotients of two real functions involving the evaluations of $f\left(\cdot\right)$ or $f'\left(\cdot\right)$.

In order to acknowledge some good literature in this direction, we note that, in the work of Khattri and Abbabandy [9], the NIM was adjusted and used as a predictor and a corrector iterative function that involves the series (2) to construct a one-parameter family of IM with CO four presented as

$$\begin{cases} y(s_k) = s_k - \frac{2}{3}u(s_k), \\ s_{k+1} = s_k - u(s_k)\left[1 + \sum_{i=1}^{3} \tau_i (v(s))^i\right], \\ \tau_1 = \frac{21}{8} - \alpha_4, \quad \tau_2 = \left(-\frac{9}{2} - 3\alpha_4\right), \quad \tau_3 = \left(\frac{15}{8} - 3\alpha_4\right), \end{cases} \tag{4}$$

where $v(s_k) = \frac{f'(y_k)}{f'(s_k)}$ and $\alpha_4$is a real parameter that is free.

In a similar manner, the power series (2) was utilized in Chun [4] to suggest the two-point IM

$$\begin{aligned} w_k &= s_k - u(s_k), \\ s_{k+1} &= s_k - \frac{f(w_k)}{f'(s_k)}\left[1 + 2\frac{f(w_k)}{f(s_k)} + \left(\frac{f(w_k)}{f(s_k)}\right)^2\right]. \end{aligned} \tag{5}$$

Ghanbari [6] used a power series like the one of (2) given as

$$s_{k+1} = w_k - \left[\frac{Af(s_k)^2 + Bf(w_k)f(s_k) + Cf(w_k)^2}{Df(s_k)^2 + Ef(w_k)f(s_k) + Ff(w_k)^2}\right]\frac{f(w_k)}{f'(s_k)}. \tag{6}$$

To develop a class of three-free-parameter CO three, the IM is obtained as

$$s_{k+1} = w_k - \left[\frac{1+(2+\alpha)\,t_k+\sigma t_k^2}{1+\alpha t_k+\beta t_k^2}\right]\frac{f\,(w_k)}{f'\,(s_k)}, \tag{7}$$

where $t = \frac{(w_k)}{f(s_k)}$ and $\alpha, \beta, \sigma \in R$. It was also shown in [6] that the class of methods (7) has the methods presented by Ostrowski [14], Chun [4, 3], Kou, Li, and Wang [10], and Chun and Ham [5] as concrete members.

In the work of Sharma and Bahl [16], the combination and power series of the form (2) were utilized to construct a CO four method as

$$s_{k+1} = s_k - \left[-\frac{1}{2}+\frac{9f'(s_k)}{8f'(y_k)}+\frac{3f'(y_k)}{8f'(s_k)}\right]\frac{f(s_k)}{f'(s_k)}. \tag{8}$$

Ogbereyivwe and Ojo-Orobosa [13] employed a kind of the power series (2) and an additional other weight function to put forward an iterative structure of the form:

$$s_{k+1} = w_k - \left[1+\sum_{i=1}^{m}\alpha_i\Phi\,(t,u)^i\right]G\,(u)\,, \tag{9}$$

where $t = \frac{f(w_k)}{f(x_k)}$ and $\Phi\,(t,u) = \frac{t}{1+u}$. This iterative structure was utilized in developing a three-parameter family of IM of CO four.

In the separate work of Ahmad [1] and Babajee [2], the power series (3) was used as a weight function to develop IMs with CO four presented as

$$s_{k+1} = s_k - u(s_k)\left[1-\frac{3\,(\mu(s_k)-1)}{4}+\frac{9\,(\mu(s_k)-1)^2}{8}\right], \tag{10}$$

where $\mu(s_k) = \frac{f'(y_k)}{f'(s_k)}$.

Also, Mahdu [11] used (3) to develop methods that utilize the same function evaluations to achieve order four presented as

$$s_{k+1} = s_k - u(s_k)\frac{1}{2}\,(3-\mu(s_k))\left[1-\frac{(\mu(s_k)-1)}{4}+(\mu(s_k)-1)^2\right]. \tag{11}$$

Motivated by the above literature that uses the power series of the kinds (2) or (3) as weight functions in developing some good IMs for obtaining s^* of NE, we put forward a new class of power series-based IM that has some existing class of IMs that are also power series based, as concrete members. The method uses the quotient of two power series in the form of (3) as weight functions as against one power series used in Ahmad [1] and Babajee [2]. This technique led to the development of a class of three real and free parameter IM for solving NE with CO four. The manuscript is organized in the following sequence. Section 1 discusses some literature background, while Section 2 puts forward the methodology. In Section 3, the convergence test on the method is carried out. Section 4 presents the results of a numerical experiment on the developed method. The final section of the work includes the conclusion and suggested areas for further research.

2 The method

Consider the IM (iterative method) that is based on the terminal and convergent power series put forward as:

$$s_{k+1} = s_k - u(s_k)\left[\left(1+\sum_{i=1}^{m}\alpha_i(\mu(s_k)-1)^i\right)/\left(1+\sum_{i=1}^{m}\beta_i(\mu(s_k)-1)^i\right)\right], \tag{12}$$

where $u(s_k) = \frac{f(s_k)}{f'(s_k)}$, $\mu(s_k) = \frac{f'(y_k)}{f'(s_k)}$, α_i and $\beta_i, i = 1, 2, 3, \ldots$, are free and real parameters to be established. To illustrate the contribution of the IM (12), the case of $m = 3$ is considered for investigation and its convergence conditions. First, the following definitions are acknowledged.

Definition 1 (Asymptotic error, Asymptotic constant, and CO)**.** Let $A_j = s_j - s^*$ be the IM error at the jth iteration, and suppose that the equation $A_{j+1} = \eta A_k^{\rho} + O(A_j^{\rho+1})$ can be obtained from an IM by the use of the Taylor expansions on the functions $f(\cdot)$. Then A_{j+1} is called the Asymptotic error equation, η is Asymptotic constant, and ρ is CO.

Definition 2 (Efficiency)**.** If the equation $A_{j+1} = \eta A_j^{\rho} + O(A_j^{\rho+1})$ is derived from an IM as described in Definition 1, then the value $E_{eff} = \rho^{\frac{1}{\tau}}$ (where τ is total number of distinct functions $f(\cdot)$ in one IM cycle) is called the efficiency index of the IM.

3 Method convergence investigation

The investigation on the convergence of the method put forward in (12) for the case $m = 3$ is presented in this section. Firstly, the theorem that enables us the establishment of the method's convergence is stated and then followed by its proof.

Theorem 1. Define $f : D \subset R \to R$ as a function that is real valued such that it is differentiable for at least three times in the domain D. Suppose that $s^* \subset D$ and that $|f'(\cdot)| \neq 0$ in D. Then for an initial guess s_0 close to s^* in D, the sequence $\{s_j\}_{j\geq 0}, (s_j \in D)$ of approximations, generated by the class of IM in (12), where $m = 3$, converges to s^* with CO four whenever the free parameters α_i and β_i satisfy the following conditions: $\beta_1 - \alpha_1 + \frac{3}{4} = 0$, $\beta_2 - \frac{1}{16}(16\alpha_2 + 12\alpha_1 - 19) = 0$ and $\beta_3 - \frac{1}{64}(64\alpha_3 + 48\alpha_2 - 36\alpha_1 + 54) = 0$.

Proof. Suppose in the Taylor expansion of $f(s)$ and $f'(s)$ about s^* that we set $s{=}s_k$. Then the following expression can be obtained:

$$f(s_j) = f'(s^*)\,(A_j + \sum_{n=2}^{4} c_n A_j^n + O\left(A_j^5\right)), \tag{13}$$

and

$$f'(s_j) = f'(s^*)\,(1 + \sum_{n=2}^{5} n c_n A_j^{n-1} + O\left(A_j^5\right)), \qquad j{=}0,1,2,\ldots, \tag{14}$$

where $c_n = \frac{1}{n!}\frac{f^n(s^*)}{f'(s^*)}$, $n \geq 2$.

Using (13) and (14), the series expansions of $u(s_k)$ and $\mu(s_k)$ are, respectively,

$$u(s_k) = A_k - c_2 A_k^2 + \left(2c_2^2 - 2c_3\right) A_k^3 + \left(-4c_2^3 + 7c_2c_3 - 3c_4\right) A_k^4 + O(A_k^5), \tag{15}$$

and

$$\begin{aligned}\mu(s_k) =& 1 + \frac{4}{3}c_2A_k + \left(4c_2^2 - \frac{8}{3}c_3\right) A_k^2 + \left(36c_2^3 - 45c_2c_3 + 13c_4\right) A_k^3 \\ &+ \left(\frac{80}{3}c_2^4 - \frac{148}{3}c_2^2c_3 + \frac{48}{27}c_4 - 5c_5\right) A_k^4 + O(A_k^5).\end{aligned} \tag{16}$$

From (15), we have

$$\begin{aligned}y_k &= A_k - \frac{2}{3}u(s_k) \\ &= \frac{1}{3}A_k + \frac{2}{3}c_2A_k^2 - \frac{4}{3}(c_2^2 - c_3)A_k^3 + \frac{2}{3}\left(4c_2^3 - 7c_2c_3 + 3c_4\right) A_k^4 + O(A_k^5).\end{aligned} \tag{17}$$

Using (17), the expansion of the function $f(y_k)$ is obtained as

$$\begin{aligned}f(y_k) = &\frac{1}{3}A_k + \frac{7}{9}c_2A_k^2 \\ &+ \left(\frac{4c_2^2}{9} - \frac{4(c_2^2 - c_3)}{3} + \frac{c_3}{27}\right) A_k^3 \\ &+ \left(c_2\left(\frac{4}{9}c_2^2 - \frac{8}{9}(c_2^2 - c_3)\right) + \frac{2}{9}c_2c_3 + \frac{2}{3}\left(4c_2^3 - 7c_2c_3 + 3c_4\right)\right) A_k^4 \\ &+ O(A_k^5),\end{aligned} \tag{18}$$

and using (18), its derivative $f'(y_k)$ is

$$
\begin{aligned}
f'(y_k) =& 1+\frac{2}{3}c_2A_k \\
&+\left(\frac{4}{3}c_2^2+\frac{1}{3}c_3\right)A_k^2+\left(-\frac{8}{3}c_2\left(c_2^2-c_3\right)+\frac{4}{3}c_2c_3+\frac{4}{27}c_4\right)A_k^3 \\
&+\left(\left(\frac{4}{3}c_2^2-\frac{8}{3}\left(c_2^2-c_3\right)\right)c_3+\frac{8}{9}c_2c_4+\frac{4}{3}c_2\left(4c_2^3-7c_2c_3+3c_4\right)\right)A_k^4 \\
&+O(A_k^5).
\end{aligned}
\tag{19}
$$

From the expressions in (14) and (19), the Taylor expansion of the quotient $\frac{f'(y_k)}{f'(s_k)}$ is

$$
\begin{aligned}
\frac{f'(y_k)}{f'(s_k)} =& 1+\frac{4}{3}c_2A_k \\
&+\left(4c_2^2+\frac{8}{3}c_3\right)A_k^2-\frac{8}{27}\left(36c_2^3-45c_2c_3+13c_4\right)A_k^3 \\
&+\left(\frac{80}{3}c_2^4-\frac{148}{3}c_2^2c_3+\frac{32}{3}c_3^2+\frac{484}{27}c_2c_4-5c_5\right)A_k^4 \\
&+O(A_k^5).
\end{aligned}
\tag{20}
$$

Using (16), the expansion of the quotient of the two power series in (12) is obtained as

$$
\begin{aligned}
&\left(1+\sum_{i=1}^{3}\alpha_i(\mu(s_k)-1)^i\right)\Big/\left(1+\sum_{i=1}^{3}\beta_i(\mu(s_k)-1)^i\right) \\
&= 1-\frac{4}{3}\left(\alpha_1-\beta_1\right)c_2A_k \\
&\quad+\frac{4}{9}\left(4\alpha_2c_2^2-9\beta_1c_2^2+\beta_1^2c_2^2-4\beta_2c_2^2+\alpha_1\left(\left(9-4\beta_1\right)c_2^2-6c_3\right)+6\beta_1c_3\right)A_k^2 \\
&\quad+\frac{8}{27}(8\alpha_3c_2^3-36\beta_1c_2^3+36\beta_1^2c_2^3-8\beta_1^3c_2^3+\cdots \\
&\quad+\alpha_1\left(4\left(9-9\beta_1+2\beta_1^2-2\beta_2\right)c_2^3+3\left(-15+8\beta_1\right)c_2c_3+13c_4\right))A_k^3 \\
&\quad+\frac{1}{81}(1728\alpha_3c_2^4+256\alpha_4c_2^4-2160\beta_1c_2^4-256\alpha_3\beta_1c_2^4+3600\beta_1^2c_2^4+\cdots \\
&\quad+4\left(-363+208\beta_1\right)c_2c_4+9\left(32\left(-3+2\beta_1\right)c_3^2+45c_3\right))A_k^4+O\left(A_k^5\right).
\end{aligned}
\tag{21}
$$

Now, substituting (15) and (21) into the iterative structure in (12), the following error equation is obtained:

$$
\begin{aligned}
s_{k+1} =& s^* + \left(c_2 + \frac{4}{3}\left(\alpha_1 - \beta_1\right) c_2\right) A_k^2 \\
&+ \frac{2}{9}(\left(-9 - 8\alpha_2 + 8\alpha_1\left(\beta_1 - 3\right) + 24\beta_1 - 8\left(\beta_1^2 - \beta_2\right)\right) c_2^2 \\
&+ 3\left(3 + 4\alpha_1 - 4\beta_1\right) c_3) A_k^3 \\
&+ \frac{1}{27}((4(27 + 84\alpha_2 + 16\alpha_3 - 117\beta_3 - 16\alpha_2\beta_1 + 84\beta_1^2 - 16\beta_1^3 \\
&+ \alpha_1\left(117 - 84\beta_1 + 16\beta_1^2 - 16\beta_2\right) - 84\beta_2 + 32\beta_1\beta_2 - 16\beta_3) c_2^3 \\
&+ 3\left(-63 - 64\alpha_2 + 168\beta_1 - 64\beta_1^2 + 8\alpha_1\left(8\beta_1 - 21\right) + 64\beta_1\right) c_2 c_3 \\
&+ \left(81 + 104\alpha_1 - 104\beta_1\right) c_4)) A_k^4 + O\left(A_k^5\right).
\end{aligned}
\tag{22}
$$

For the method (12) to attain order four convergence, the second and third terms of (22) must be annihilated. This can be achieved when the following set of equations are satisfied:

$$
\begin{aligned}
\beta_1 &= \alpha_1 + \frac{3}{4} \\
\beta_2 &= \frac{1}{16}\left(16\alpha_2 + 12\alpha_1 - 9\right) \\
\beta_3 &= \frac{1}{64}\left(64\alpha_3 + 48\alpha_2 - 36\alpha_1 + 54\right).
\end{aligned}
\tag{23}
$$

When the conditions in (23) are substituted in (22), the error equation is obtained as

$$
A_{k+1} = s^* - \left(\frac{-9c_2c_3 + c_4}{9}\right) A_k^4 + O\left(A_k^5\right). \tag{24}
$$

Following Definition 1, the asymptotic error equation for the method (12) is the expression on (24), and consequently, the CO is four. □

Remark 1. By substituting the conditions (23) into (12), the class of three-free-parameter IM for solving NE is obtained as

$$
\begin{cases}
s_{k+1} = s_k - u\left(s_k\right)\left[\frac{1+\sum_{i=1}^{3}\alpha_i(\mu(s_k)-1)^i}{1+\sum_{i=1}^{3}\beta_i(\mu(s_k)-1)^i}\right], \\
\beta_1 = \alpha_1 + \frac{3}{4}, \quad \beta_2 = \frac{16\alpha_2+12\alpha_1-9}{16}, \\
\beta_3 = \frac{64\alpha_3+48\alpha_2-36\alpha_1+54}{64}.
\end{cases}
\tag{25}
$$

We emphasis here that, for any free choice of $\alpha_i \in R$ in (25), a CO four method with $E_{eff} = 1.5867$ can be constructed.

Remark 2. Obtaining suitable replacement function(s) for the weight function(s) used in [17, 13, 12, 15], and many other existing methods that were developed via the weight function(s) technique, requires subjecting the replacement function(s) to satisfy certain conditions that involve its evaluation and many times differentiation at some fixed points. These procedures are tedious and cumbersome compared to the proposed three-parameter-based class of methods (25) that requires the arbitrary substitution of values for the parameters. In fact, for any real values for the parameters, the method works.

Remark 3. It is important to note that for $m = 2$ in (12), a subclass of method (25) is obtained as

$$s_{k+1} = s_k - u\left(s_k\right)\left(\frac{1+\alpha_1\left(\mu(s_k)-1\right)+\alpha_2\left(\mu(s_k)-1\right)^2}{1+\left(\alpha_1+\frac{3}{4}\right)\left(\mu(s_k)-1\right)+\left(\frac{16\alpha_2+12\alpha_1-9}{16}\right)\left(\mu(s_k)-1\right)^2}\right). \tag{26}$$

Some existing methods and their variants are particular members of the class of methods in (26).

For instance, the method developed by Chun [4] is a concrete method of (26).

Furthermore, when $\alpha_1 = -\frac{3}{4}$ and $\alpha_2 = \frac{9}{8}$ in (26), the CO four methods by Ahmad [1] and Babajee [2] are rediscovered as

$$s_{k+1} = s_k - u\left(s_k\right)\left(1-\frac{3}{4}\left(\mu(s_k)-1\right)+\frac{9}{8}\left(\mu(s_k)-1\right)^2\right). \tag{27}$$

Again, for $\alpha_1 = \frac{3}{4}$ and $\alpha_2 = 0$ in (26), the well-known Jarratt method in [8] is rediscovered as

$$s_{k+1} = s_k - u\left(s_k\right)\left(\frac{1+\frac{3}{4}\left(\mu(s_k)-1\right)}{1+\frac{3}{2}\left(\mu(s_k)-1\right)}\right). \tag{28}$$

For some new IMs of the kind (26), the following cases are presented:

Case 1

If $\alpha_1 = -\frac{3}{4}$ and $\alpha_2 = 0$, then the following iterative structure is discovered:

$$M_1: \qquad s_{k+1} = s_k - u\left(s_k\right)\left(\frac{1-\frac{3}{4}\left(\mu(s_k)-1\right)}{1-\frac{9}{8}\left(\mu(s_k)-1\right)^2}\right).$$

Case 2

The substitution of $\alpha_1 = \frac{11}{6}$ and $\alpha_2 = \frac{1}{4}$ in (26) yields the method

$$M_2: \qquad s_{k+1} = s_k - u\left(s_k\right)\left(\frac{1+\frac{11}{6}\left(\mu(s_k)-1\right)+\frac{1}{4}\left(\mu(s_k)-1\right)^2}{1+\frac{31}{12}\left(\mu(s_k)-1\right)+\frac{17}{16}\left(\mu(s_k)-1\right)^2}\right).$$

Case 3

Consider $\alpha_1 = \frac{4}{3}$ and $\alpha_2 = 0$. Then a new method is obtained as

$$M_3: \qquad s_{k+1} = s_k - u\left(s_k\right)\left(\frac{1+\frac{9}{16}\left(\mu(s_k)-1\right)^2}{1+\frac{3}{4}\left(\mu(s_k)-1\right)}\right).$$

Remark 4. In order to illustrate some new methods of kind (25), the following IMs are constructed for some real values of α_i:

Case 4

When $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then the method M_4 is constructed as

$$s_{k+1} = s_k - u(s_k)\left(\frac{1}{1+\frac{3}{4}(\mu(s_k)-1)-\frac{9}{16}(\mu(s_k)-1)^2+\frac{27}{32}(\mu(s_k)-1)^3}\right).$$

Case 5

Consider the case $\alpha_1 = \alpha_2 = \alpha_3 = \frac{3}{2}$. Then the IM M_5 is obtained as

$$s_{k+1} = s_k - u(s_k)\left[\frac{1+\frac{3}{2}\left((\mu(s_k)-1)+(\mu(s_k)-1)^2+(\mu(s_k)-1)^3\right)}{1+\frac{9}{4}(\mu(s_k)-1)+\frac{33}{16}(\mu(s_k)-1)^2+\frac{21}{8}(\mu(s_k)-1)^3}\right].$$

4 Numerical test

In this section, the methods derived from the developed class of methods are subjected to numerical tests in order to verify their effectiveness when used to obtain a solution to NE. The numerical results of the new methods are compared with the results of some existing methods that are of CO four. These include the method developed by Sharma and Bahl [16] (presented in (8)) denoted by SBM, method developed by Ogbereyivwe and Ojo-Orobosa in [13] (presented in (9)) with the parameters $\beta = 2, \eta = \sigma = 0$, denoted by OOM, the method of Hafiz and Khirallah [7], denoted as HKM, the bi-parameter method in Shams et al. [15] (given in (28)) with $\beta = 1$ and $\alpha = 2$ and denoted as SEM, the methods in Ghanbari [6] (presented in (7)) obtained by taking $\alpha = -2, \beta = 1, \theta = 4$ and $\alpha = -2, \beta = 1, \sigma = -1$, and denoted as GM1 and GM2, respectively. The metric used for comparison includes the computational CO ρ_{coc} is given by

$$\rho_{coc} = \frac{\log_{10}|f(s_{k+1})| / |f(s_k)|}{\log_{10}|f(s_k)| / |f(s_{k-1})|},$$

and $|f(s_k)|$. All computation programs for the methods were written and executed in the MAPLE 2017 software environment using Intel Celeron(R) computer with a 2GB RAM processor. The error tolerance $|f(s_k)| \leq 200^{-200}$ was used in terminating program execution. The computational outputs are maintained at 2000 digits precision in all computations. The NE $f_i(s) = 0$ used for the test and their respective solutions s^* presented in 15 digits precision are as given in Examples 1–5. For comparison, computation results are presented in Tables 1 and 2.

Consider the following examples:

Example 1. [6] $f_1(s) = s^2 - e^s - 3s + 2 = 0, \qquad s^* = 0.257530285439860\ldots$.

Example 2. [6] $f_2(s) = \sin^2(s) - s^2 + 1 = 0, \qquad s^* = 1.404491648215341\ldots.$

Example 3. [13] $f_3(s) = \arctan(s) = 0, \qquad s^* = 0.$

Example 4. [6] $f_4(s) = (s-1)^3 - 2 = 0, \qquad s^* = 2.259921049894873\ldots.$

Example 5. [13] $f_4(s) = \sqrt{\frac{1}{s}} + 2\log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{R\sqrt{s}}\right) = 0$, where $\frac{\epsilon}{D} = 10^{-4}$ and $R = 10^5$, $s^* = 0.004129319146099\ldots.$

Table 1: Methods results comparison for Examples 1– 2

Methods	s_0	$\lvert f(s_1)\rvert$	$\lvert f(s_2)\rvert$	$\lvert f(s_3)\rvert$	$\lvert f(s_4)\rvert$	$\lvert f(s_5)\rvert$	ρ_{coc}
SBM		$5.3e-4$	$7.1e-18$	$2.3e-73$	$2.7e-295$	$-$	4.0
HKM		$7.3e-3$	$3.4e-16$	$1.6e-66$	$7.1e-268$	-	4.0
SEM		$5.9e-3$	$1.7e-13$	$1.1e-55$	$1.7e-224$	-	4.0
OOM		$1.7e-2$	$2.9e-11$	$2.5e-46$	$1.4e-186$	-	4.0
GM1		$2.8e-4$	$3.4e-19$	$7.5e-79$	$1.7e-317$	-	4.0
GM2	-0.3	$3.0e-3$	$1.1e-14$	$1.6e-60$	$8.4e-244$	-	4.0
M1		$1.6e-4$	$4.8e-20$	$4.8e-82$	$4.3e-330$	-	4.0
M2		$2.3e-3$	$3.9e-15$	$3.1e-62$	$1.2e-250$	-	4.0
M3		$7.3e-5$	$2.3e-21$	$2.5e-87$	$3.1e-351$	-	4.0
M4		$2.1e-3$	$2.6e-15$	$5.1e-63$	$1.5e-253$	-	4.0
M5		$2.4e-3$	$4.0e-15$	$3.3e-62$	$1.6e-250$	-	4.0
SBM		$5.3e-2$	$4.5e-7$	$2.8e-27$	$4.0e-108$	$1.8e-431$	4.0
HKM		$3.7e-2$	$6.2e-8$	$5.0e-31$	$2.1e-123$	$6.3e-493$	4.0
SEM		$1.2e-2$	$1.5e-9$	$4.1e-37$	$2.2e-147$	$1.7s-588$	4.0
OOM		$1.0e-2$	$1.6e-5$	$1.7e-20$	$2.2e-80$	$3.6e-320$	4.0
GM1		$6.1e-2$	$1.0e-6$	$1.1e-25$	$1.1e-101$	$1.4e-405$	4.0
GM2	2	$1.4e-2$	$2.8e-9$	$3.9e-36$	$1.6e-143$	$4.7e-573$	4.0
M1		$5.6e-2$	$7.5e-7$	$3.0e-26$	$6.7e-104$	$1.8e-414$	4.0
M2		$6.1e-2$	$1.2e-6$	$1.9e-25$	$1.3e-100$	$2.9e-401$	4.0
M3		$5.9e-2$	$9.3e-7$	$6.8e-26$	$1.9e-102$	$1.1e-408$	4.0
M4		$2.7e-2$	$2.1e-9$	$9.8e-38$	$5.1e-151$	$3.6e-604$	4.0
M5		$1.5e-2$	$2.4e-10$	$1.9e-41$	$6.8e-166$	$1.1e-663$	4.0

From the numerical results in Tables 1 and 2, we observe that all the particular methods constructed from the developed class of methods (25) solved the test problems very well. Furthermore, the new methods' computational order of convergence displayed in the last column of Tables 1 and 2 agree with the theoretical CO established in Section 3. The methods computation results outputs are also highly competitive with the methods compared.

5 Conclusion

The new three-parameter class of IM put forward in this manuscript was shown to have a capacity of approximating the solution of NE with CO four requiring three evaluations of distinct functions in an iteration cycle.

Table 2: Methods results comparison for Examples 3– 5

Methods	s_0	$\|f(s_1)\|$	$\|f(s_2)\|$	$\|f(s_3)\|$	$\|f(s_4)\|$	$\|f(s_5)\|$	ρ_{coc}
SBM		$6.8e-3$	$2.3e-12$	$1.1e-59$	$2.2e-296$	-	4.0
HKM		$4.7e-3$	$3.7e-13$	$1.1e-63$	$2.3e-316$	-	4.0
SEM		$4.7e-3$	$5.0e-13$	$6.8e-62$	$3.2e-312$	-	4.0
OOM		$6.1e-2$	$1.5e-7$	$1.6e-35$	$2.6e-175$	-	4.0
GM1		$9.3e-3$	$1.6e-11$	$2.2e-55$	$1.1e-274$	-	4.0
GM2	-0.5	$1.3e-3$	$7.3e-16$	$4.6e-77$	$4.8e-383$	-	4.0
M1		$9.2e-3$	$1.1e-11$	$2.4e-56$	$1.2e-279$	4.0	
M2		$2.9e-3$	$3.1e-14$	$4.8e-69$	$4.4e-343$	-	4.0
M3		$3.1e-3$	$4.4e-14$	$2.7e-68$	$2.2e-339$	-	4.0
M4		$1.5e-3$	$1.4e-15$	$9.6e-76$	$1.3e-376$	-	4.0
M5		$3.1e-3$	$4.5e-14$	$3.2e-68$	$5.6e-339$	-	4.0
SBM		$2.5e-1$	$2.5e-5$	$3.8e-21$	$2.0e-84$	$1.3e-337$	4.0
HKM		$1.5e-1$	$1.8e-6$	$4.8e-26$	$2.2e-104$	$1.0e-417$	4.0
SEM		$8.2e-1$	$7.2e-2$	$4.0e-7$	$3.0e-28$	$1.0e-112$	4.0
OOM		$4.6e-1$	$7.5e-4$	$1.0e-14$	$4.0e-58$	$8.5e-232$	4.0
GM1		$2.9e-1$	$5.7e-5$	$1.3e-19$	$3.3e-78$	$1.4e-312$	4.0
GM2	3	$3.2e-2$	$1.1e-8$	$1.5e-34$	$5.9e-138$	$1.3e-551$	4.0
M1		$2.7e-1$	$4.2e-5$	$3.8e-20$	$2.6e-80$	$5.8e-321$	4.0
M2		$3.5e-2$	$2.3e-9$	$4.4e-38$	$5.8e-153$	$1.7e-612$	4.0
M3		$3.4e-2$	$6.6e-10$	$1.0e-40$	$5.2e-164$	$3.8e-657$	4.0
M4		$9.6e-2$	$9.4e-8$	$1.2e-31$	$3.5e-127$	$2.2e-509$	4.0
M5		$1.2e-2$	$3.3e-11$	$1.9e-45$	$2.1e-182$	$3.2e-730$	4.0
SBM		$3.7e-1$	$1.4e-5$	$2.6e-23$	$3.7e-94$	$1.5e-377$	4.0
HKM		$1.6e-3$	$8.5e-16$	$7.6e-65$	$4.3e-261$	-	4.0
SEM				Failed to converge			
OOM		$6.1e-1$	$3.4e-4$	$2.8e-17$	$1.4e-69$	$8.7e-279$	4.0
GM1		$4.5e-1$	$3.7e-5$	$2.0e-21$	$1.6e-86$	$6.8e-347$	4.0
GM2	0.002	$2.9e-1$	$1.5e-5$	$8.0e-23$	$7.3e-92$	$5.2e-368$	4.0
M1		$4.1e-1$	$2.9e-5$	$7.8e-22$	$4.3e-88$	$4.0e-353$	4.0
M2		1.0509	$8.1e-4$	$2.4e-16$	$1.9e-66$	$7.4e-267$	4.0
M3		$5.1e-1$	$3.3e-5$	$4.9e-22$	$2.6e-89$	$2.0e-358$	4.0
M4		$1.2e-1$	$1.4e-7$	$1.9e-31$	$8.1e-127$	$2.4e-508$	4.0
M5		$7.3e-1$	$1.7e-4$	$5.0e-19$	$3.6e-77$	$9.7e-310$	4.0

Several famous existing methods, such as Chun [4], Ahmad [1], Babajee [2], Jarrat [8], and others, are typical members of the class of developed methods. For future research, the implementation of the method developed herein in Banach spaces may be considered.

Conflicts of Interest

There is no existence of any form of conflict between the authors as per the publication of this manuscript.

Acknowledgment

The authors are thankful to the Editor and anonymous reviewers for their constructive comments that led to the improvement of the manuscript.

Funding

The first author is supported by TETFund Nigeria via the Institution Based Research (IBR) Fund.

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How to cite this article

Ogbereyivwe, O. and Izevbizua, O., A three-free-parameter class of power series based iterative method for approximation of nonlinear equations solution. *Iran. j. numer. anal. optim.*, 2023; 13(2): 157-169. https://doi.org/10.22067/ijnao.2022.74901.1095