Iranian Journal of Numerical Analysis and Optimization
Vol. 10, No. 2, (2020), pp 63–85
DOI:10.22067/ijnao.v10i2.68360

Research Article

Comparison of homotopy perturbation transform method and fractional Adams–Bashforth method for the Caputo–Prabhakar nonlinear fractional differential equations

M.H. Derakhshan* and A. Aminataei

Abstract

We study two numerical techniques based on the homotopy perturbation transform method (HPTM) and the fractional Adams–Bashforth method (FABM) for solving a class of nonlinear time-fractional differential equations involving the Caputo–Prabhakar fractional derivatives. In this manuscript, the convergence for numerical solutions obtained using HPTM and the convergence and stability for numerical solutions obtained using FABM are investigated. We compare the solutions obtained by the HPTM and the FABM for some nonlinear time-fractional differential equations. Moreover, some numerical examples are demonstrated in order to show the validity and reliability of the suggested methods.

AMS(2010): 65D15; 65D25.

Keywords: Nonlinear time-fractional differential equations; Fractional Homotopy perturbation transform method; Fractional Adams–Bashforth method; Caputo–Prabhakar fractional derivative.

---

*Corresponding author
Received 2 March 2020; revised 15 May 2020; accepted 31 May 2020

Mohammadhossein Derakhshan
Faculty of Mathematics, K. N. Toosi University of Technology, P. O. Box: 16765-3381, Tehran, Iran. e-mail: M.h.derakhshan.20@gmail.com

Azim Aminataei
Faculty of Mathematics, K. N. Toosi University of Technology, P. O. Box: 16765-3381, Tehran, Iran. e-mail: ataei@kntu.ac.ir

1 Introduction

In recent years, fractional calculus, which is a generalization of the integer-order calculus or integration of integer order to arbitrary order (real or complex order), plays an important, fundamental and significant role in mathematical modeling and solving the models obtained in different fields of mathematics, sciences, physics, finance and artificial intelligence, and engineering; see [11, 39, 3, 10, 31, 38, 2]. Nowadays, fractional differential equations have been applied extensively as an important and crucial tool to describe many different types of complex mechanical, physical behaviors, sciences, engineering, anomalous diffusion, vibration, control, viscoelasticity, electrochemistry, and others, since most of the models in different research areas and engineering applications are nonlinear and obtaining their solutions has an important role.

In this manuscript, we consider a numerical method based on the homotopy perturbation transform method (HPTM), which is a combination of homotopy analysis method and Laplace transform scheme [29, 40] for solving the following nonlinear time-fractional partial differential equation (FPDE) with $m-1<\mu\leq m$:

$$\begin{aligned}&{}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(t)+\mathbf{A}u(t)+\mathbf{N}u(t)=h(t),\\&u^{(k)}(0)=u_0^k,\quad k=0,1,2,\dots,m-1,\, m\in\mathbb{N},\end{aligned}\tag{1}$$

where $\mathbf{A}u(t)$ and $\mathbf{N}u(t)$ are linear and nonlinear operators applied to $u(t)$, respectively, and $h(t)$ is a continuous function. Also ${}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}$ denotes the Caputo–Prabhakar fractional derivative defined by Garra et al. (see [11])

$$\begin{aligned}{}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(t)&=\int_0^t(t-\tau)^{-\mu}E^{-\gamma}_{\rho,1-\mu}(\omega(t-\tau)^{\rho})\dot{u}(\tau)d\tau\\&=\mathbf{E}^{-\gamma}_{\rho,1-\mu,\omega,0^+}\frac{d}{dt}u(t),\quad 0<\mu\leq 1,\end{aligned}\tag{2}$$

where $\mathbf{E}$ is the Prabhakar. It is obtained by modifying the Riemann–Liouville integral operator [31] by extending its kernel with a nonsingular three-parameter Mittag–Leffler function (ML), which is known as the Prabhakar function and its integral is (see [11])

$$(\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+}u)(t)=\int_0^t(t-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-\tau)^{\rho})u(\tau)d\tau,$$

where $E^{\gamma}_{\rho,\mu}$ is the Prabhakar function and its function is (see [11])

$$E^{\gamma}_{\rho,\mu}(t)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!\Gamma(\rho n+\mu)}t^n,\quad \Re(\rho),\Re(\mu)>0,\,\gamma>0,\quad t\in\mathbb{C},\tag{3}$$

and $(\gamma)_n$ is the Pochhammer symbol, which is formulated as [15]

$$(\gamma)_0 = 1, \quad (\gamma)_n = \gamma(\gamma+1)\dots(\gamma+n-1), \quad n \in \mathbb{N}.$$

Most of our interest in studying ML functions is related to their importance in fractional calculus which arise in modeling a complex susceptibility in the response of disordered materials and heterogeneous systems [26], the response in anomalous dielectrics of Havriliak–Negami type [12, 37], in fractional viscoelasticity [14], in the discussion of stochastic processes [9], in probability theory [16], in the description of dynamical models of spherical stellar systems [1], and fractional or integral differential equations [25, 6, 5, 7, 8, 4, 18]. In this manuscript, we consider the following nonlinear time-fractional order differential equations with initial conditions:

$$\begin{aligned} &{}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+} u(t) = f(t), \\ &u^{(k)}(0) = u_0^k, \quad k = 0, 1, 2, \dots, m-1, \, m \in \mathbb{N}, \end{aligned} \tag{4}$$

and in order to solve the nonlinear FPDE (4) with $m - 1 < \mu \leq m$, we apply a numerical technique based on the fractional Adams–Bashforth method (FABM) to obtain the numerical solution of (4). The present work aim is to compare the numerical solutions that are obtained by applying the FABM and HPTM to solve nonlinear time-fractional order differential equations. Several numerical-analytical methods have been applied to solve linear and nonlinear differential equations of fractional derivatives in order to obtain exact solutions and numerical solutions; for instance, the Hermite collocation method [30], optimal homotopy asymptotic method [17], q-homotopy analysis transform technique [36], homotopy analysis Laplace transform method [23, 24], the homotopy analysis Sumudu transform method for solving the linear and nonlinear Fokker-Planck equations [33], and other methods [19, 20, 28, 27, 34, 21] were suggested.

This paper is organized as follows. Some necessary definitions and mathematical preliminaries of the fractional calculus are introduced in Section 2. In Section 3, we introduce a numerical method based on the HPTM. In Section 4, the FABM for time-fractional linear and nonlinear differential equations is introduced to obtain the numerical solution. In Section 5, the proposed methods are applied to some nonlinear time-fractional differential equations to verify the validity and applicability of the proposed methods. Finally, a brief conclusion is added at the end of the manuscript.

2 Preliminaries

In this section, we study some important and basic properties of fractional calculus theory such as Laplace transform, definitions, and lemmas, which are applied in the next sections.

Definition 1. [see [22, 31]] Let $0 < \alpha \leq 1$, let $f \in L^1[a,b]$, and let $0 < t < b \leq \infty$. Then the left-sided and the right-sided Riemann–Liouvillel fractional integrals and derivatives of order α are defined as follows,

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau,$$
$$I^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau)(\tau-t)^{\alpha-1} d\tau,$$
$$D^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_a^t f(\tau)(t-\tau)^{-\alpha} d\tau,$$
$$D^{\alpha}_{b^-} f(t) = -\frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_t^b f(\tau)(\tau-t)^{-\alpha} d\tau,$$

where $\Gamma(\cdot)$ is the Gamma function. Also, the left-sided and the right-sided Caputo fractional derivatives of order α are given by

$${}^C D^{\alpha}_{a^+} f(t) = I^{1-\alpha}_{a^+} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} \frac{d}{d\tau} f(\tau) d\tau,$$
$${}^C D^{\alpha}_{b^-} f(t) = -I^{1-\alpha}_{b^-} \frac{d}{dt} f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\tau-t)^{-\alpha} \frac{d}{d\tau} f(\tau) d\tau.$$

Lemma 1. [4] Let $\rho, \mu, \omega, \gamma \in \mathbb{C}$, let $\Re(\mu) > 0$, and let $\Re(\rho) > 0$. Then the Laplace transform of (2) for $m-1 < \mu \leq m$ is given,

$$\mathcal{L}\Big({}^C\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+} u(t); s\Big) = s^{\mu}(1-\omega s^{-\rho})^{\gamma} U(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}(1-\omega s^{-\rho})^{\gamma} u^k(0^+), \tag{5}$$

where $U(s) = \int_0^{\infty} e^{-st} u(t) dt$. Also, the Laplace transformation of the Prabhakar function (3) is denoted by (see [11, 15])

$$\mathcal{L}\Big(t^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega t^{\rho}); s\Big) = s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}.$$

Lemma 2. [22] Let $\rho, \mu, \omega, \gamma \in \mathbb{C}$ and let $\Re(\mu) > 0, \Re(\rho) > 0$. Then,

$$\int_0^t (t-y)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-y)^{\rho}) y^{\nu-1} dy = \Gamma(\nu) t^{\mu+\nu-1} E^{\gamma}_{\rho,\mu+\nu}(\omega t^{\rho}). \tag{6}$$

Lemma 3. [22] Let $\rho, \mu, \nu, \omega, \sigma, \gamma \in \mathbb{C}$ and let $\Re(\mu) > 0, \Re(\rho) > 0, \Re(\nu) > 0$. Then the following relation is hold for any summable function $u \in L(a,b)$,

$$\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+}\mathbf{E}^{\sigma}_{\rho,\nu,\omega,0^+}u = \mathbf{E}^{\gamma+\sigma}_{\rho,\mu+\nu,\omega,0^+}u. \tag{7}$$

Also, by substituting $\sigma = -\gamma$ in (7), the following relation is obtained,

$$\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+}\mathbf{E}^{-\gamma}_{\rho,\nu,\omega,0^+}u = I^{\mu+\nu}_{0^+}u. \tag{8}$$

3 HPTM to solve the nonlinear fractional order differential equations

In this section, we obtain a combined numerical method based on the homotopy analysis method and the Laplace transform method for solving the nonlinear FPDE in Caputo–Prabhakar fractional derivatives sense. Using formula (5) for $m = 1$ and applying the Laplace transform on both sides of equation (1), we get

$$\begin{aligned}
&\mathcal{L}\Big({}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(t) + \mathbf{A}u(t) + \mathbf{N}u(t) - h(t); s\Big) = 0,\\
&s^{\mu}(1-\omega s^{-\rho})^{\gamma}U(s) - s^{\mu-1}(1-\omega s^{-\rho})^{\gamma}u(0^+) + \mathcal{L}\Big(\mathbf{A}u(t) + \mathbf{N}u(t); s\Big) = H(s),\\
&U(s) = s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}H(s) + \frac{1}{s}u(0) - (s^{-\mu}(1-\omega s^{-\rho})^{-\gamma})\\
&\qquad \times \Big[\mathcal{L}\Big(\mathbf{A}u(t) + \mathbf{N}u(t); s\Big)\Big].
\end{aligned} \tag{9}$$

By applying the Laplace inverse transform on (9), we obtain

$$\begin{aligned}
u(t) =& \mathcal{L}^{-1}\Big[s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}H(s) + \frac{1}{s}u(0); t\Big] - \mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma})\\
&\times \Big[\mathcal{L}\Big(\mathbf{A}u(t) + \mathbf{N}u(t); s\Big)\Big]; t\Big].
\end{aligned} \tag{10}$$

To obtain the solution of (10), we use the HPTM. Suppose that the solution of (1) is $u(t)$, which can be expressed as the following infinite series:

$$u(t) = \sum_{n=0}^{\infty} p^n u_n(t), \tag{11}$$

where $u_n(t)$ for $n = 0, 1, 2, \ldots$ are known functions. Also, the nonlinear term of (1) is $\mathbf{N}u(t)$, which can be presented as the following infinite series:

$$\mathbf{N}u(t) = \sum_{n=0}^{\infty} p^n \mathbf{H}_n(u(t)), \tag{12}$$

where $\mathbf{H}_n(u(t)) = \left\{ \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\Big(\sum_{i=0}^{\infty} p^i u_i \Big) \right]_{p=0} \right\}$ are polynomials of the Adomian polynomials type; see [13]. Now, substituting (11) and (12) into (10), we conclude that

$$\begin{aligned}\sum_{n=0}^{\infty} p^n u_n(t) =& H(t) - \mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}) \\ &\times \Big[\mathcal{L}\Big(\mathbf{A}\sum_{n=0}^{\infty} p^n u_n(t) + \mathbf{N}\sum_{n=0}^{\infty} p^n u_n(t); s\Big)\Big]; t\Big], \qquad (13)\end{aligned}$$

where $H(t) = \mathcal{L}^{-1}\Big[s^{-\mu}(1-\omega s^{-\rho})^{-\gamma} H(s) + \frac{1}{s}u(0); t\Big]$. By equaling the coefficients on powers of p on the both sides of relation (13), we obtain the series solution as follows:

$$\begin{aligned}
p^0 &: u_0(t) = H(t),\\
p^1 &: u_1(t) = -\mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}) \times \Big[\mathcal{L}\Big(\mathbf{A}u_0(t) + \mathbf{H}_0(u(t)); s\Big)\Big]; t\Big],\\
p^2 &: u_2(t) = -\mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}) \times \Big[\mathcal{L}\Big(\mathbf{A}u_1(t) + \mathbf{H}_1(u(t)); s\Big)\Big]; t\Big],\\
&\vdots\\
p^{n+1} &: u_{n+1}(t) = -\mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}) \times \Big[\mathcal{L}\Big(\mathbf{A}u_n(t) + \mathbf{H}_n(u(t)); s\Big)\Big]; t\Big].
\end{aligned}$$

So, the solution of (1) by using the HPTM is obtained as follows:

$$u(t) = \lim_{p\to 1}\lim_{n\to\infty}\sum_{k=0}^{n} p^k u_k(t) = \lim_{n\to\infty}\sum_{k=0}^{n} u_k(t).$$

Theorem 1. Let $S_n = \sum_{k=0}^{n} u_k(t)$ and let $u(t) = \sum_{k=0}^{\infty} u_k(t)$ such that $u(t), u_k(t) \in C\Big([0,1]\Big)$ and for any $n = 1, 2, 3, \ldots$, $\parallel u_{n+1}(t) \parallel \leq \lambda \parallel u_n(t) \parallel$, $0 < \lambda < 1$. Then the solution of (1) using the HPTM is converges.

Proof. To prove convergence, we need to show that the series S_n is a Cauchy sequence. We have

$$\begin{aligned}\parallel S_{n+1} - S_n \parallel =& \parallel \sum_{k=0}^{n+1} u_k(t) - \sum_{k=0}^{n} u_k(t) \parallel \\ =& \parallel u_{n+1}(t) \parallel \leq \lambda \parallel u_n(t) \parallel \leq \lambda^2 \parallel u_{n-1}(t) \parallel \leq \cdots \\ \leq& \lambda^{n+1} \parallel u_0(t) \parallel .\end{aligned}$$

Then for any $m, n \in \mathbb{N}, n > m$, we obtain

$$\begin{aligned}&\parallel S_n - S_m \parallel \\ &\quad = \parallel (S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + (S_{n-2} - S_{n-3}) + \cdots + (S_{m+1} - S_m) \parallel\end{aligned}$$

$$\leq \| S_n - S_{n-1} \| + \| S_{n-1} - S_{n-2} \| + \| S_{n-2} - S_{n-3} \| + \| S_{m+1} - S_m \|$$
$$\leq \lambda^n \| u_0(t) \| + \lambda^{n-1} \| u_0(t) \| + \cdots + \lambda^{m+1} \| u_0(t) \|$$
$$\leq \frac{1-\lambda^{n-m}}{1-\lambda} \lambda^{m+1} \| u_0(t) \| \leq \frac{\lambda^{m+1}}{1-\lambda} \| u_0(t) \|, \quad 1-\lambda^{n-m} < 1.$$

As $m, n \to \infty$ then $\| S_n - S_m \| \to 0$, since $u_0(t)$ is bounded and $0 < \lambda < 1$. So the series $S_n = \sum_{k=0}^{n} u_k(t)$ is a Cauchy sequence in the Banach space and therefore converges. □

4 The Adams–Bashforth scheme to numerically solve nonlinear fractional order differential equations

In this section, we consider the fractional differential equation (4) with initial conditions and apply the Adams–Bashforth scheme in order to numerically solve this equation. Using relation (8) in Lemma 3 and applying the integral transform operator $\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+}$ to both sides of the fractional differential equation (4), the solutions of this equation are obtained as follows:

$$\begin{aligned}
&\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+}\left({}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+} u(t)\right) = \mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+} f(t),\\
&\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+}\left(\mathbf{E}^{-\gamma}_{\rho,1-\mu,\omega,0^+} \frac{d}{dt} u(t)\right) = \mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+} f(t),\\
&u(t) - u(0) = \int_0^t (t-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-\tau)^{\rho}) f(\tau) d\tau,\\
&u(t) = u(0) + \int_0^t (t-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-\tau)^{\rho}) f(\tau) d\tau. \qquad (14)
\end{aligned}$$

Now, we first put $t = t_{n+1}$, $n = 0, 1, 2, 3, \dots$ and then $t = t_n$, $n = 0, 1, 2, 3, \dots$ in relation (14). Now we obtain

$$u(t_{n+1}) = u(0) + \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho}) f(\tau) d\tau, \qquad (15)$$
$$u(t_n) = u(0) + \int_0^{t_n} (t_n-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_n-\tau)^{\rho}) f(\tau) d\tau. \qquad (16)$$

By subtracting two relations (15) and (16), we obtain

$$\begin{aligned}
u(t_{n+1}) =& u(t_n) + \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho}) f(\tau) d\tau\\
&- \int_0^{t_n} (t_n-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_n-\tau)^{\rho}) f(\tau) d\tau. \qquad (17)
\end{aligned}$$

To solve (17), we approximate the function $f(\tau)$ by the Lagrange interpolation as follows:

$$\begin{aligned} P(\tau) &\simeq \frac{\tau - t_{n-1}}{t_n - t_{n-1}} f(t_n) + \frac{\tau - t_n}{t_{n-1} - t_n} f(t_{n-1}), \\ P(\tau) &\simeq \frac{\tau - t_{n-1}}{h} f(t_n) - \frac{\tau - t_n}{h} f(t_{n-1}), \\ & \qquad t_j = jh,\ j = 0,1,2,3,\ldots,\ h = t_n - t_{n-1}. \end{aligned} \tag{18}$$

Thus, by substituting (18) in (17), we obtain

$$\begin{aligned} u(t_{n+1}) =& u(t_n) \qquad (19) \\ &+ \Big[\frac{f(t_n)}{h} \times \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1} - \tau)^{\rho})(\tau - t_{n-1}) d\tau \\ &\quad - \frac{f(t_{n-1})}{h} \times \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1} - \tau)^{\rho})(\tau - t_n) d\tau\Big] \\ &- \Big[\frac{f(t_n)}{h} \times \int_0^{t_n} (t_n - \tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_n - \tau)^{\rho})(\tau - t_{n-1}) d\tau \\ &\quad - \frac{f(t_{n-1})}{h} \times \int_0^{t_n} (t_n - \tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_n - \tau)^{\rho})(\tau - t_n) d\tau\Big], \\ =& u(t_n) + \Big[\frac{f(t_n)}{h} A_{\mu,1} - \frac{f(t_{n-1})}{h} A_{\mu,2}\Big] - \Big[\frac{f(t_n)}{h} A_{\mu,3} - \frac{f(t_{n-1})}{h} A_{\mu,4}\Big], \qquad (20) \end{aligned}$$

where

$$A_{\mu,1} = \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1} - \tau)^{\rho})(\tau - t_{n-1}) d\tau, \tag{21}$$

$$A_{\mu,2} = \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1} - \tau)^{\rho})(\tau - t_n) d\tau, \tag{22}$$

$$A_{\mu,3} = \int_0^{t_n} (t_n - \tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_n - \tau)^{\rho})(\tau - t_{n-1}) d\tau, \tag{23}$$

$$A_{\mu,4} = \int_0^{t_n} (t_n - \tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_n - \tau)^{\rho})(\tau - t_n) d\tau. \tag{24}$$

To calculate relations (21)–(24), first we calculate the integral (21) as follows:

$$
\begin{aligned}
A_{\mu,1} &= \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho})(\tau - t_{n-1}) d\tau \\
&= \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho}) \tau d\tau \\
&\quad - t_{n-1} \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho}) d\tau. \qquad (25)
\end{aligned}
$$

By applying (6) in relation (25), we obtain

$$
\begin{aligned}
A_{\mu,1} =& \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho})(\tau - t_{n-1}) d\tau \\
=& \Gamma(2)(t_{n+1})^{\mu+1} E^{\gamma}_{\rho,\mu+2}(\omega(t_{n+1})^{\rho}) - t_{n-1} \times \Gamma(1)(t_{n+1})^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega(t_{n+1})^{\rho}) \\
=& (n+1)^{\mu+1} h^{\mu+1} E^{\gamma}_{\rho,\mu+2}(\omega((n+1)h)^{\rho}) \\
& - (n-1)h \times \Big[(n+1)^{\mu} h^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega((n+1)h)^{\rho})\Big] \\
=& (n+1)^{\mu} h^{\mu+1} \times \Big[(n+1) E^{\gamma}_{\rho,\mu+2}(\omega((n+1)h)^{\rho}) \\
& - (n-1) E^{\gamma}_{\rho,\mu+1}(\omega((n+1)h)^{\rho})\Big]. \qquad (26)
\end{aligned}
$$

Similarly, for the residual terms, we have

$$
\begin{aligned}
A_{\mu,2} =& \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho})(\tau - t_{n}) d\tau \\
=& (n+1)^{\mu} h^{\mu+1} \\
& \times \Big[(n+1) E^{\gamma}_{\rho,\mu+2}(\omega((n+1)h)^{\rho}) - n E^{\gamma}_{\rho,\mu+1}(\omega((n+1)h)^{\rho})\Big], \qquad (27) \\
A_{\mu,3} =& \int_0^{t_{n}} (t_{n}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n}-\tau)^{\rho})(\tau - t_{n-1}) d\tau \\
=& n^{\mu} h^{\mu+1} \times \Big[n E^{\gamma}_{\rho,\mu+2}(\omega(nh)^{\rho}) - (n-1) E^{\gamma}_{\rho,\mu+1}(\omega(nh)^{\rho})\Big], \qquad (28) \\
A_{\mu,4} =& \int_0^{t_{n}} (t_{n}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n}-\tau)^{\rho})(\tau - t_{n}) d\tau \\
=& n^{\mu+1} h^{\mu+1} \times \Big[E^{\gamma}_{\rho,\mu+2}(\omega(nh)^{\rho}) - E^{\gamma}_{\rho,\mu+1}(\omega(nh)^{\rho})\Big]. \qquad (29)
\end{aligned}
$$

Substituting relations (26)–(29) in (19), we obtain the approximation scheme to solve numerically (4).

Theorem 2.[Convergence] Let $u(t)$ be a solution of (4) and let $f(t)$ be continuous and bounded that is introduced in (4). Then the numerical solution of $u(t)$ is given by

$$
\begin{aligned}
u(t_{n+1}) =& u(t_n) + \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho}) \\
&\Big[\frac{\tau - t_{n-1}}{t_n - t_{n-1}} f(t_n) + \frac{\tau - t_n}{t_{n-1}-t_n} f(t_{n-1})\Big] d\tau \\
&- \int_0^{t_n} (t_n-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_n-\tau)^{\rho}) \\
&\Big[\frac{\tau - t_{n-1}}{t_n - t_{n-1}} f(t_n) + \frac{\tau - t_n}{t_{n-1}-t_n} f(t_{n-1})\Big] d\tau \\
&+ \mathbf{R}(t,\mu,n),
\end{aligned}
$$

where $\mathbf{R}(t,\mu,n)$ is the error function and $\parallel \mathbf{R}(t,\mu,n) \parallel_{\infty} < M$.

Proof. Using (17) and (18), we have

$$
\begin{aligned}
u(t_{n+1}) =& u(t_n) + \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho}) f(\tau) d\tau \\
&- \int_0^{t_n} (t_n-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_n-\tau)^{\rho}) f(\tau) d\tau \\
=& u(t_n) \\
&+ \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho}) \\
&\Big[\frac{\tau - t_{n-1}}{t_n - t_{n-1}} f(t_n) + \frac{\tau - t_n}{t_{n-1}-t_n} f(t_{n-1}) + \frac{f^{(n+1)}(\tau)}{(n+1)!} \prod_{i=0}^{n} (\tau - t_i)\Big] d\tau \\
&- \int_0^{t_n} (t_n-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_n-\tau)^{\rho}) \\
&\Big[\frac{\tau - t_{n-1}}{t_n - t_{n-1}} f(t_n) + \frac{\tau - t_n}{t_{n-1}-t_n} f(t_{n-1}) + \frac{f^{(n)}(\tau)}{(n)!} \prod_{i=0}^{n-1} (\tau - t_i)\Big] d\tau \\
=& u(t_n) + \mathbf{L}(t,\mu,n) + \mathbf{R}(t,\mu,n),
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbf{L}(t,\mu,n) \\
&= \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho}) \\
&\quad \Big[\frac{\tau - t_{n-1}}{t_n - t_{n-1}} f(t_n) + \frac{\tau - t_n}{t_{n-1}-t_n} f(t_{n-1})\Big] d\tau \\
&\quad - \int_0^{t_n} (t_n-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t_n-\tau)^{\rho}) \\
&\quad \Big[\frac{\tau - t_{n-1}}{t_n - t_{n-1}} f(t_n) + \frac{\tau - t_n}{t_{n-1}-t_n} f(t_{n-1})\Big] d\tau,
\end{aligned}
$$

$$\begin{aligned}
&\mathbf{R}(t,\mu,n)\\
&\quad=\int_0^{t_{n+1}}(t_{n+1}-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho})\Big(\frac{f^{(n+1)}(\tau)}{(n+1)!}\prod_{i=0}^{n}(\tau-t_i)\Big)d\tau\\
&\qquad-\int_0^{t_n}(t_n-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t_n-\tau)^{\rho})\Big(\frac{f^{(n)}(\tau)}{(n)!}\prod_{i=0}^{n-1}(\tau-t_i)\Big)d\tau.
\end{aligned}$$

To establish convergence, we need to show that $\parallel \mathbf{R}(t,\mu,n)\parallel_\infty < M,\ M\in\mathbb{N}$. For this, we have

$$\begin{aligned}
&\parallel \mathbf{R}(t,\mu,n)\parallel_\infty\\
&=\parallel\int_0^{t_{n+1}}(t_{n+1}-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho})\Big(\frac{f^{(n+1)}(\tau)}{(n+1)!}\prod_{i=0}^{n}(\tau-t_i)\Big)d\tau\\
&\quad-\int_0^{t_n}(t_n-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t_n-\tau)^{\rho})\Big(\frac{f^{(n)}(\tau)}{(n)!}\prod_{i=0}^{n-1}(\tau-t_i)\Big)d\tau\parallel_\infty\\
&<\parallel\int_0^{t_{n+1}}(t_{n+1}-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho})\Big(\frac{f^{(n+1)}(\tau)}{(n+1)!}\prod_{i=0}^{n}(\tau-t_i)\Big)d\tau\parallel_\infty\\
&\quad+\parallel\int_0^{t_n}(t_n-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t_n-\tau)^{\rho})\Big(\frac{f^{(n)}(\tau)}{(n)!}\prod_{i=0}^{n-1}(\tau-t_i)\Big)d\tau\parallel_\infty\\
&<\max_{t\in[0,t_{n+1}]}\frac{\left|f^{(n+1)}(t)\right|}{(n+1)!}\parallel\prod_{i=0}^{n}(t-t_i)\parallel_\infty\\
&\qquad\int_0^{t_{n+1}}(t_{n+1}-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t_{n+1}-\tau)^{\rho})d\tau\\
&\quad+\max_{t\in[0,t_{n+1}]}\frac{\left|f^{(n)}(t)\right|}{(n)!}\parallel\prod_{i=0}^{n-1}(t-t_i)\parallel_\infty\int_0^{t_n}(t_n-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t_n-\tau)^{\rho})d\tau.
\end{aligned}$$

Using Lemma 2, we obtain

$$\begin{aligned}
\parallel \mathbf{R}(t,\mu,n)\parallel_\infty &< \max_{t\in[0,t_{n+1}]}\frac{\left|f^{(n+1)}(t)\right|}{(n+1)!}\parallel\prod_{i=0}^{n}(t-t_i)\parallel_\infty t^{\mu}_{n+1}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}_{n+1})\\
&\quad+\max_{t\in[0,t_{n+1}]}\frac{\left|f^{(n)}(t)\right|}{(n)!}\parallel\prod_{i=0}^{n-1}(t-t_i)\parallel_\infty t^{\mu}_{n}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}_{n})
\end{aligned}$$

$$
\begin{aligned}
&< \sup_{t\in[0,t_{n+1}]}\Big\{\max_{t\in[0,t_{n+1}]}\frac{\left|f^{(n+1)}(t)\right|}{(n+1)!},\max_{t\in[0,t_{n+1}]}\frac{\left|f^{(n)}(t)\right|}{n!}\Big\}\\
&\quad\times\Big[\frac{n!h^{n+1}}{4}t_{n+1}^{\mu}E_{\rho,\mu+1}^{\gamma}(\omega t_{n+1}^{\rho})+\frac{(n-1)!h^{n}}{4}t_{n}^{\mu}E_{\rho,\mu+1}^{\gamma}(\omega t_{n}^{\rho})\Big]\\
&= M.
\end{aligned}
$$

□

Theorem 3.[Stability condition] If

$$
\frac{M_1\ n!h^n}{4}\Big[\frac{(n+1)t_{n+1}^{\mu}E_{\rho,\mu+1}^{\gamma}(\omega t_{n+1}^{\rho})}{h}+\frac{t_{n}^{\mu}E_{\rho,\mu+1}^{\gamma}(\omega t_{n}^{\rho})}{h^2}\Big]\to 0
$$

as $n\to\infty$ and $h\to 0$, then, the FABM is stable.

Proof. Using (17), we have

$$
\begin{aligned}
\parallel u(t_{n+1})-u(t_n)\parallel_\infty &= \parallel \int_0^{t_{n+1}}(t_{n+1}-\tau)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t_{n+1}-\tau)^{\rho})f(\tau)d\tau\\
&\quad -\int_0^{t_n}(t_n-\tau)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t_n-\tau)^{\rho})f(\tau)d\tau\parallel_\infty\\
&< \parallel \int_0^{t_{n+1}}(t_{n+1}-\tau)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t_{n+1}-\tau)^{\rho})f(\tau)d\tau\parallel_\infty\\
&\quad +\parallel \int_0^{t_n}(t_n-\tau)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t_n-\tau)^{\rho})f(\tau)d\tau\parallel_\infty .
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|u(t_{n+1})-u(t_n)\parallel_\infty\\
&\quad<\parallel \int_0^{t_{n+1}}(t_{n+1}-\tau)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t_{n+1}-\tau)^{\rho})\Big[\sum_{k=0}^{n}\prod_{k=0}^{n}\frac{(\tau-t_k)f(t_k)}{(-1)^k h}\Big]d\tau\parallel_\infty\\
&\qquad+\parallel \int_0^{t_n}(t_n-\tau)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t_n-\tau)^{\rho})\Big[\sum_{k=0}^{n-1}\prod_{k=0}^{n-1}\frac{(\tau-t_k)f(t_k)}{(-1)^k h}\Big]d\tau\parallel_\infty\\
&\quad<\parallel \mathbf{P}(t,\mu,n)\parallel_\infty+\parallel \mathbf{R}_n^{\mu}(t)\parallel_\infty,
\end{aligned}
$$

where $\parallel \mathbf{P}(t,\mu,n)\parallel_\infty$ and $\parallel \mathbf{R}_n^{\mu}(t)\parallel_\infty$ are calculated as follows:

$$
\begin{aligned}
&\|\mathbf{P}(t,\mu,n)\parallel_\infty\\
&\quad=\parallel \int_0^{t_{n+1}}(t_{n+1}-\tau)^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega(t_{n+1}-\tau)^{\rho})\Big[\sum_{k=0}^{n}\prod_{k=0}^{n}\frac{(\tau-t_k)f(t_k)}{(-1)^k h}\Big]d\tau\parallel_\infty
\end{aligned}
$$

$$\leq \sum_{k=0}^{n} \frac{\| f(t_k) \|_\infty \, t_{n+1}^{\mu} E_{\rho,\mu+1}^{\gamma}(\omega t_{n+1}^{\rho})}{h} \prod_{k=0}^{n} \Big| t - t_k \Big|$$
$$\leq \sum_{k=0}^{n} \frac{\| f(t_k) \|_\infty \, t_{n+1}^{\mu} E_{\rho,\mu+1}^{\gamma}(\omega t_{n+1}^{\rho}) n! h^n}{4h}$$

and

$$\begin{aligned}
&\|\mathbf{R}_n^{\mu}(t) \|_\infty \\
&= \| \int_0^{t_n} (t_n - \tau)^{\mu-1} E_{\rho,\mu}^{\gamma}(\omega(t_n - \tau)^{\rho}) \Big[\sum_{k=0}^{n-1} \prod_{k=0}^{n-1} \frac{(\tau - t_k) f(t_k)}{(-1)^k h} \Big] d\tau \, \|_\infty \\
&\leq \sum_{k=0}^{n-1} \frac{\| f(t_k) \|_\infty \, t_n^{\mu} E_{\rho,\mu+1}^{\gamma}(\omega t_n^{\rho})}{h} \prod_{k=0}^{n-1} \Big| t - t_k \Big| \\
&\leq \sum_{k=0}^{n-1} \frac{\| f(t_k) \|_\infty \, t_n^{\mu} E_{\rho,\mu+1}^{\gamma}(\omega t_n^{\rho}) (n-1)! h^{n-1}}{4h}.
\end{aligned}$$

Then

$$\begin{aligned}
&\|u(t_{n+1}) - u(t_n) \|_\infty \\
&\quad < \sum_{k=0}^{n} \frac{\| f(t_k) \|_\infty \, t_{n+1}^{\mu} E_{\rho,\mu+1}^{\gamma}(\omega t_{n+1}^{\rho}) n! h^n}{4h} \\
&\qquad + \sum_{k=0}^{n-1} \frac{\| f(t_k) \|_\infty \, t_n^{\mu} E_{\rho,\mu+1}^{\gamma}(\omega t_n^{\rho}) (n-1)! h^{n-1}}{4h} \\
&\quad < \frac{M_1 \; n! h^n}{4} \Big[\frac{(n+1) t_{n+1}^{\mu} E_{\rho,\mu+1}^{\gamma}(\omega t_{n+1}^{\rho})}{h} + \frac{t_n^{\mu} E_{\rho,\mu+1}^{\gamma}(\omega t_n^{\rho})}{h^2} \Big],
\end{aligned}$$

where $M_1 = \max_{t \in [0, t_{n+1}]} \Big| f(t) \Big|$. Therefore,

$$\frac{M_1 \; n! h^n}{4} \Big[\frac{(n+1) t_{n+1}^{\mu} E_{\rho,\mu+1}^{\gamma}(\omega t_{n+1}^{\rho})}{h} + \frac{t_n^{\mu} E_{\rho,\mu+1}^{\gamma}(\omega t_n^{\rho})}{h^2} \Big] \to 0$$

as $n \to \infty,\ h \to 0$ in which $h = \frac{1}{n+1}$ is considered. □

5 Illustrative examples

In this section, we compare the obtained solution with the numerical simulations of the applied methods on five test problems in order to show the accuracy and efficiency of the proposed techniques.

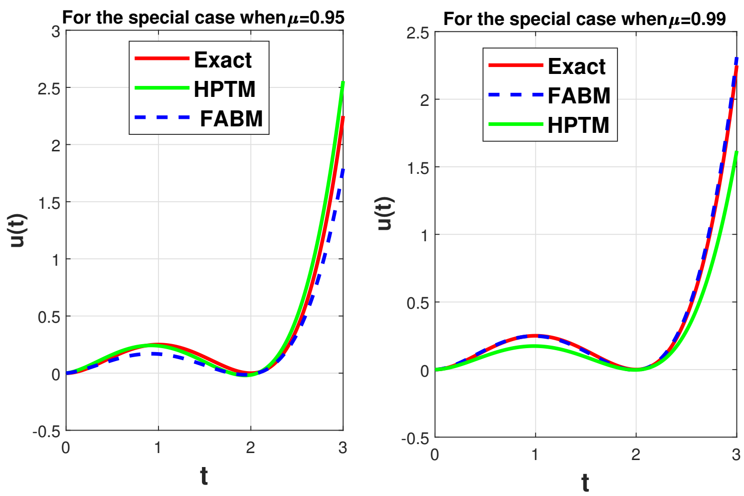

Figure 1: Comparison between numerical solution and the exact solution of Example 1 when $\mu = 0.95$, 0.99, $\rho = \omega = 1$.

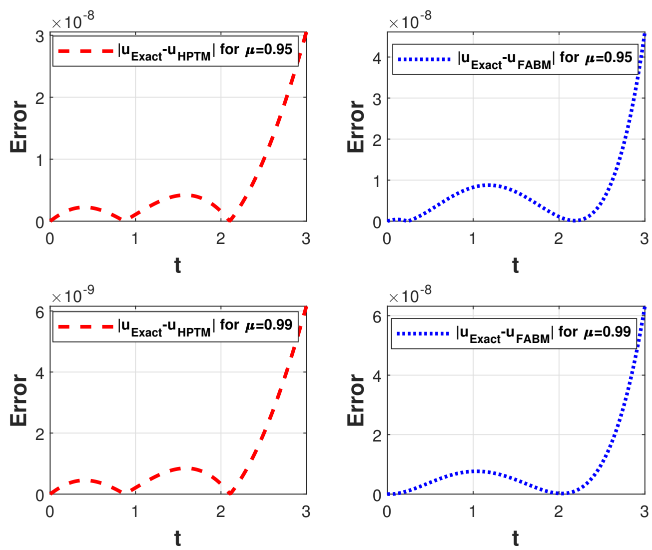

Figure 2: The value of absolute error of Example 1 when $\mu = 0.95$, 0.99, $\rho = \omega = 1$.

Example 1. We consider the following nonlinear fractional order differential equation:

Table 1: The exact solution, the numerical solutions, and the absolute errors of Example 1 for different values of t when $\mu = 0.95$, $\rho = \omega = 1$.

t	$u_{Exact}(t)$	$u_{HPTM}(t)$	$u_{FABM}(t)$	$\lvert u_{Exact}(t) - u_{HPTM}(t)\rvert$	$\lvert u_{Exact}(t) - u_{FABM}(t)\rvert$
0	0	0	0	0	0
0.01	0.0001	0.0011	0.0008	0.001009960804482	0.000677271813138
0.02	0.0004	0.0024	0.0017	0.002022248070965	0.001297961649675
0.03	0.0009	0.0039	0.0027	0.003027289212479	0.001857141698735
0.04	0.0015	0.0056	0.0039	0.004020110685205	0.002353085479644
0.05	0.0024	0.0074	0.0052	0.004997320737974	0.002785155766582
0.06	0.0034	0.0093	0.0065	0.005956376904834	0.003153291833384
0.07	0.0046	0.0115	0.0080	0.006895275398590	0.003457792029013
0.08	0.0059	0.0137	0.0096	0.007812388169833	0.003699199718883
0.09	0.0074	0.0161	0.0113	0.008706366104405	0.003878235523084

Table 2: The exact solution, the numerical solutions, and the absolute errors of Example 1 for different values of t when $\mu = 0.99$, $\rho = \omega = 1$.

t	$u_{Exact}(t)$	$u_{HPTM}(t)$	$u_{FABM}(t)$	$\lvert u_{Exact}(t) - u_{HPTM}(t)\rvert$	$\lvert u_{Exact}(t) - u_{FABM}(t)\rvert$
0	0	0	0	0	0
0.01	0.0001	0.0003	0.0002	0.000201585080800	0.000111408806560
0.02	0.0004	0.0008	0.0006	0.000403349846034	0.000164732892224
0.03	0.0009	0.0015	0.0010	0.000603578912785	0.000160544488950
0.04	0.0015	0.0023	0.0016	0.000801354185865	0.000099955930106
0.05	0.0024	0.0034	0.0024	0.000996039906343	0.000015740815560
0.06	0.0034	0.0046	0.0032	0.001187154171698	0.000185164079811
0.07	0.0046	0.0059	0.0042	0.001374313737802	0.000406881133539
0.08	0.0059	0.0075	0.0052	0.001557204785139	0.000679428650402
0.09	0.0074	0.0091	0.0064	0.001735565417799	0.001001324957541

$$ {}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+} u(t) = f(t) - u^2(t), $$

subject to the initial condition

$$ u(0) = 0 $$

and

$$ \begin{aligned} f(t) =& (2t - 3t^2 - t^3) + \Big[2t\Big(1 - \mu + \frac{\mu t^{\mu}}{\Gamma(\mu+2)}\Big) \\ & - 3t^2\Big(2t\Big(1 - \mu + \frac{2\mu t^{\mu}}{\Gamma(\mu+3)}\Big) + t^3\Big(1 - \mu + \frac{6\mu t^{\mu}}{\Gamma(\mu+4)}\Big)\Big)\Big]^2. \end{aligned} $$

Under these conditions, the analytical solution is given by (see [41])

$$ \begin{aligned} u(t) =& \Big[2t\Big(1 - \mu + \frac{\mu t^{\mu}}{\Gamma(\mu+2)}\Big) \\ & - 3t^2\Big(2t\Big(1 - \mu + \frac{2\mu t^{\mu}}{\Gamma(\mu+3)}\Big) + t^3\Big(1 - \mu + \frac{6\mu t^{\mu}}{\Gamma(\mu+4)}\Big)\Big)\Big]^2. \end{aligned} $$

We apply the proposed schemes in this article and solve this problem. Figure 1 shows the analytical and numerical results of Example 1 obtained by applying the HPTM and the FABM together with the exact solution. Also the absolute error for $\mu = 0.95,\, 0.99$ with several values of t is shown in Tables 1 and 2, and for $\mu = 0.95,\, 0.99$ with several values of t, it is shown in Figure 2. The numerical results demonstrate that the FABM method is an efficient algorithm.

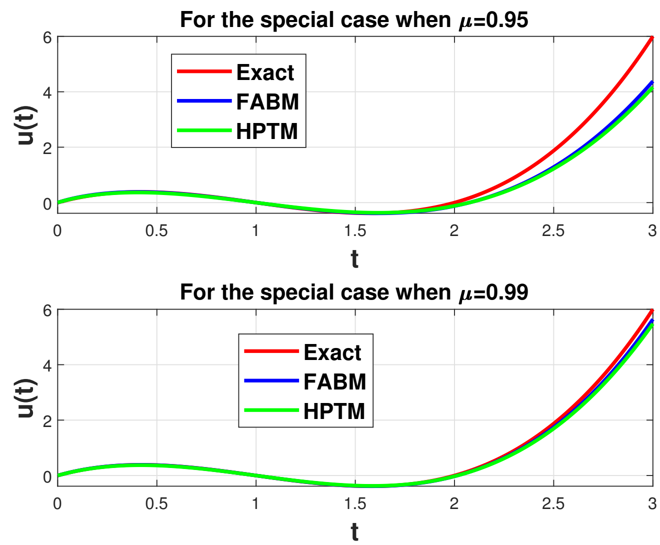

Figure 3: Comparison between numerical solution and the exact solution of Example 2 when $\mu = 0.95,\, 0.99,\, \rho = \omega = 1$.

Example 2. We consider the following nonlinear fractional order differential equation:

$$ {}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+} u(t) = f(t) - u^2(t), $$

subject to the initial condition

$$ u(0) = 0 $$

and

$$ f(t) = (2t - 3t^2 + t^3)^2 + \frac{2t^{1-\mu}}{\Gamma(2-\mu)} - \frac{6t^{2-\mu}}{\Gamma(3-\mu)} + \frac{6t^{3-\mu}}{\Gamma(4-\mu)}. $$

Under these conditions, the analytical solution is given by (see [41])

$$ u(t) = 2t - 3t^2 + t^3. $$

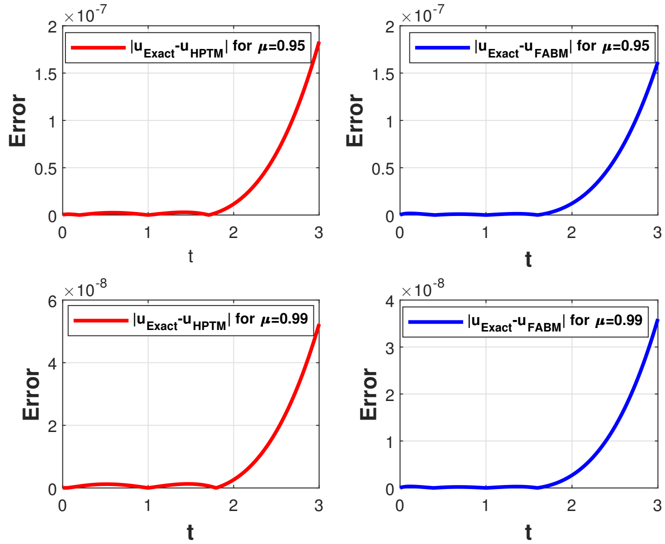

Figure 4: The value of absolute error of Example 2 when $\mu = 0.95, 0.99$, $\rho = \omega = 1$.

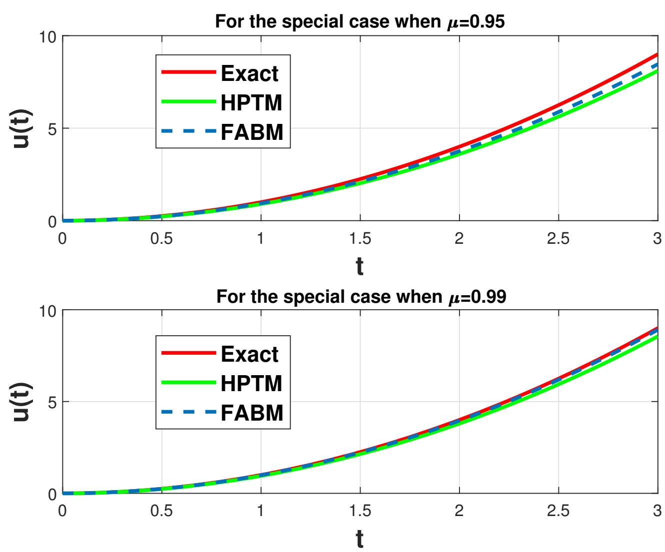

Figure 5: Comparison between numerical solution and the exact solution of Example 3 when $\mu = 0.95, 0.99$, $\rho = \omega = 1$.

We apply the proposed schemes in this article and solve this problem. Figure 3, shows the analytical and numerical results obtained by applying the HPTM

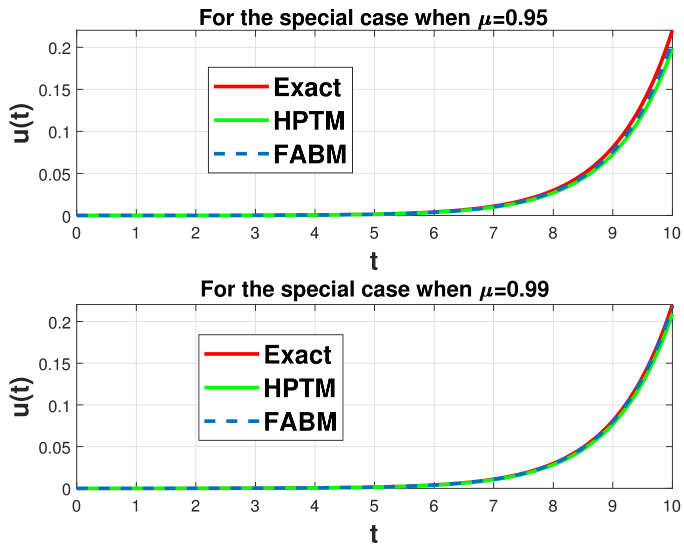

Figure 6: Comparison between numerical solution and the exact solution of Example 4 when $\mu = 0.95$, 0.99, $\rho = \omega = 1$.

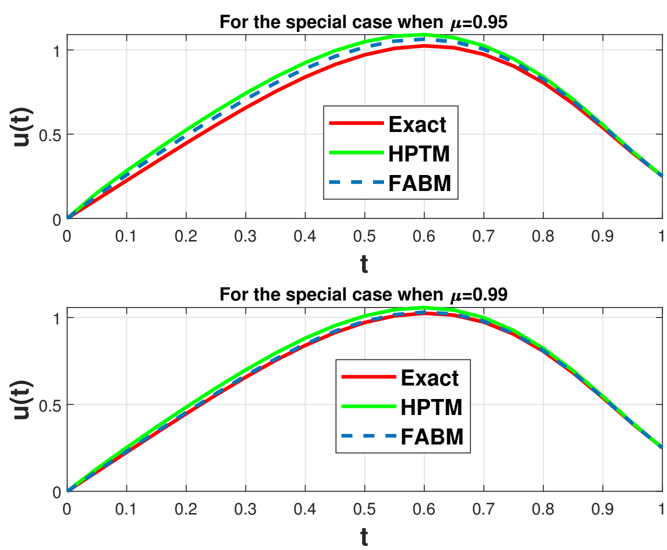

Figure 7: Comparison between numerical solution and the exact solution of Example 5 when $\mu = 0.95$, 0.99, $\rho = \omega = 1$.

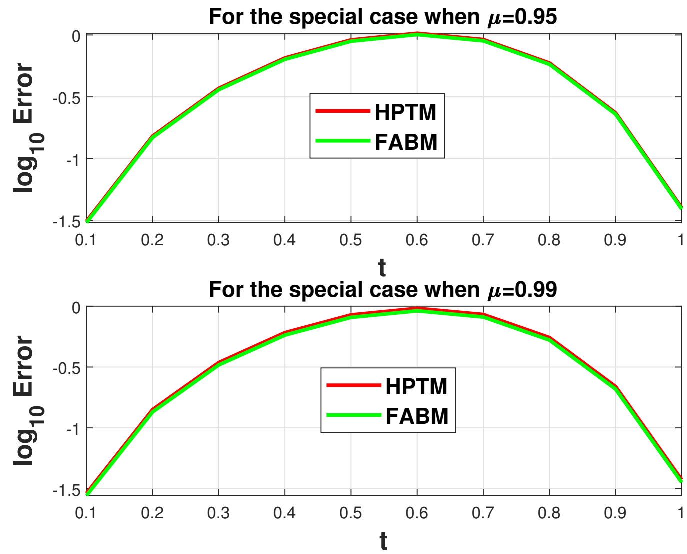

Figure 8: The value of absolute error of Example 5 when $\mu = 0.95, 0.99$, $\rho = \omega = 1$.

and the FABM together with the exact solution of Example 2. Also the absolute error for $\mu = 0.95, 0.99$ with several values of t is shown in Figure 4. The numerical results demonstrate that the FABM method is an efficient algorithm.

Example 3. We consider the following equation:

$$ {}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(t) + u(t) = \frac{2t^{2-\mu}}{\Gamma(3-\mu)} + t^2, \quad u(0) = 0, \ 0 < \mu \le 1. $$

For this example, the exact solution $u(t) = t^2$ was presented in [32]. This example is solved by using numerical methods introduced in this article for different values of μ, and their diagrams are shown in Figure 5.

Example 4. We consider the following equation:

$$ {}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(t) - \frac{1}{2}u(t) = \frac{1}{2}e^{t}u(\frac{t}{2}), \quad u(0) = 1, \ 0 < \mu \le 1, \ 0 \le t \le 10. $$

The exact analytical solution of this example is $u(t) = e^t$, which was presented in [18]. This example is solved by using numerical methods introduced in this article for different values of μ, and their diagrams are shown in Figure 6.

Example 5. We consider the following nonlinear fractional initial value equation:

$$ {}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(t) + u^{\frac{3}{2}}(t) = f(t), \quad u(0) = 1, \ 0 < \mu \le 1, \ 0 < t < 1, $$

where

$$f(t) = \frac{40320t^{8-\mu}}{\Gamma(9-\mu)} - 3\frac{\Gamma(5+\frac{\mu}{2})t^{4-\frac{\mu}{2}}}{\Gamma(5-\frac{\mu}{2})} + \frac{9\Gamma(\mu+1)}{4} + \left[\frac{3t^{\frac{\mu}{2}}}{2} - t^4\right]^3.$$

The exact solution of this example is $u(t) = t^8 - 3t^{4+\frac{\mu}{2}} + \frac{9t^\mu}{4}$, which was given in [35]. Diagrams of the numerical solutions for different values of μ are shown in Figure 7. We calculated the logarithm of the absolute error for different values of μ, and their diagrams are shown in Figure 8.

6 Conclusions

In this paper, the FABM and the HPTM were presented to study the initial value problem of the linear and nonlinear fractional differential equations involving the Caputo–Prabhakar fractional derivatives for numerically solving this type of equations. In addition, the two proposed methods were compared. Numerical examples and graphical representations were presented for testing the validity and accuracy of suggested methods. These examples showed that the FABM is a strong tool for obtaining the numerical solutions of linear and nonlinear fractional differential equations. Furthermore, the stability property of the FABM and the convergence analysis for both methods were considered. The numerical results were stable and converged well for different values of t for all examples.

References

1. An, J., Van Hese, E. and Baes, M. *Phase-space consistency of stellar dynamical models determined by separable augmented densities*, Mon. Not. R. Astron. Soc. 422(1) (2012), 652–664.

2. Atangana, A. and Owolabi, K.M. *New numerical approach for fractional differential equations*, Math. Model. Nat. Phenom, 13(1) (2018), Paper No. 3, 21 pp.

3. Balcl, E., Öztürk, İ. and Kartal, S. *Dynamical behaviour of fractional order tumor model with Caputo and conformable fractional derivative*, Chaos Solitons Fractals, 123 (2019), 43–51.

4. Derakhshan, M.H., Ahmadi Darani, M., Ansari, A. and Khoshsiar Ghaziani, R. *On asymptotic stability of Prabhakar fractional differential systems*, Comput. Methods Differ. Equ. 4(4) (2016), 276–284.

5. Derakhshan, M.H. and Ansari, A. *On Hyers-Ulam stability of fractional differential equations with Prabhakar derivatives*, Anal. 38(1) (2018), 37–46.

6. Derakhshan, M.H. and Ansari, A. *Fractional Sturm-Liouville problems for Weber fractional derivatives*, Int. J. Comput. Math. 96(2) (2019), 217–237.

7. Derakhshan, M.H. and Ansari, A. *Numerical approximation to Prabhakar fractional Sturm-Liouville problem*, Comput. Appl. Math, 38(2) (2019), 71.

8. Derakhshan, M.H., Ansari, A. and Ahmadi Darani, M. *On asymptotic stability of Weber fractional differential systems*, Comput. Methods Differ. Equ. 6(1) (2018), 30–39.

9. D'Ovidio, M. and Polito, F. *Fractional diffusion-telegraph equations and their associated stochastic solutions*, Theory. Probab. Appl. 64(4) (2018), 552–574.

10. Dumitru, B., Kai, D. and Enrico, S. *Fractional calculus: Models and numerical methods* , World Scientific, 2012.

11. Garra, R., Gorenflo, R., Polito, F. and Tomovski, Ž. *Hilfer-Prabhakar derivatives and some applications*, Appl. math. comput. 242 (2014), 576–589.

12. Garrappa, R., Mainardi, F. and Guido, M. *Models of dielectric relaxation based on completely monotone functions*, Fract. Calc. Appl. Anal. 19(5) (2016), 1105–1160.

13. Ghorbani, A. *Beyond Adomian polynomials: He polynomials*, Chaos. Solitons. Fractals, 39(3) (2009), 1486–1492.

14. Giusti, A. and Colombaro, I. *Prabhakar-like fractional viscoelasticity*, Commun. Nonlinear. Sci. Numer. Simul. 56 (2018), 138–143.

15. Gorenflo, R., Kilbas, A.A., Mainardi, F. and Rogosin, S.V. *Mittag-Leffler functions, related topics and applications*, Springer Monographs in Mathematics. Springer, Heidelberg, 2014.

16. Górska, K., Horzela, A., Bratek, L., Penson, K.A. and Dattoli, G. *The probability density function for the Havriliak-Negami relaxation*, J. Phys. A, Math. Theor. 51(13) (2018) Article ID 135202, 15 pp.

17. Hamarsheh, M., Ismail, A.I. and Odibat, Z. *An analytic solution for fractional order Riccati equations by using optimal homotopy asymptotic method*, Appl. Math. Sci. 10(23) (2016), 1131–1150.

18. Heydari, M., Loghmani, G.B. and Hosseini, S.M. *Operational matrices of Chebyshev cardinal functions and their application for solving delay differential equations arising in electrodynamics with error estimation*, Appl. Math. Model, 37(14–15) (2013), 7789–7809.

19. Hou, T. and Leng, H. *Numerical analysis of a stabilized Crank-Nicolson/Adams–Bashforth finite difference scheme for Allen-Cahn equations*, Appl. Math. Lett. 102 (2020), 106–150.

20. Karaagac, B. *Two step Adams Bashforth method for time-fractional Tricomi equation with non-local and non-singular Kernel*, Chaos. Solitons. Fractals, 128 (2019), 234–241.

21. Karunakar, P. and Chakraverty, S. *Solutions of time-fractional third- and fifth-order Korteweg-de-Vries equations using homotopy perturbation transform method*, Eng. Comput. 36(7) (2019), 2309–2326.

22. Kilbas, A.A., Saigo, M. and Saxena, R.K. *Generalized Mittag-Leffler function and generalized fractional calculus operators*, Integral. Transform. Spec. Funct. 15(1) (2004), 31–49.

23. Kumar, S., Kumar, A. and Odibat, Z.M. *A nonlinear fractional model to describe the population dynamics of two interacting species*, Math. Methods. Appl. Sci. 40(11) (2017), 4134–4148.

24. Li, C., Kumar, A., Kumar, S. and Yang, X.J. *On the approximate solution of nonlinear time-fractional KdV equation via modified homotopy analysis Laplace transform method*, J. Nonlinear. Sci. Appl. 9 (2016), 5463–5470.

25. Liemert, A., Sandev, T. and Kantz, H. *Generalized Langevin equation with tempered memory kernel*, Physica. A. 466 (2017), 356–369.

26. Miskinis, P. *The Havriliak-Negami susceptibility as a nonlinear and nonlocal process*, Phys. Scr. 2009.

27. Owolabi, K.M. and Atangana, A. *Analysis and application of new fractional Adams–Bashforth scheme with Caputo-Fabrizio derivative*, Chaos. Solitons. Fractals, 105 (2017), 111–119.

28. Pan, Y., He, Y. and Mikkola, A. *Accurate real-time truck simulation via semirecursive formulation and Adams–Bashforth-Moulton algorithm*, Acta. Mech. Sin. 35 (2019), 641–652.

29. Pandey, R. K. and Mishra, H.K. *Homotopy analysis Sumudu transform method for time-fractional third order dispersive partial differential equation*, Adv. Comput. Math. 43(2) (2017), 365–383.

30. Pirim, N.A. and Ayaz, F. *A new technique for solving fractional order systems: Hermite collocation method* Appl. Math, 7(18)(2016), 2307–2323.

31. Podlubny, I. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.

32. Qureshi, S. and Kumar, P. *Using Shehu integral transform to solve fractional order Caputo type initial value problems*, J. Appl. Math. Comput. Mech. 18(2) (2019), 75–83.

33. Rathore, S., Kumar, D., Singh, J. and Gupta, S. *Homotopy analysis Sumudu transform method for nonlinear equations*, Int. J. Ind. Math. 4(4) (2012), 301–314.g

34. Ratti, I. *Comparative study of nonlinear partial differential equation using homotopy perturbation transform method (HPTM) using He's polynomial and mixture of Elzaki transform and partial differential transform method (PDTM)*, Int. J. Appl. Eng. Res. 13(18) (2018), 14037–14040.

35. Saadatmandi, A. and Dehghan, M. *A new operational matrix for solving fractional-order differential equations*, Comput. Math. Appl. 59(3) (2010), 1326–1336.

36. Singh, J., Kumar, D., Swroop, R. and Kumar, S. *An efficient computational approach for time-fractional Rosenau–Hyman equation*, Neural. Comput. Appl. 30(10) (2018), 3063–3070.

37. Stanislavsky, A. and Weron, K. *A typical case of the dielectric relaxation responses and its fractional kinetic equation*, Fract. Calc. Appl. Anal. 19(1) (2016), 212–228.

38. Subashini, R., Ravichandran, C., Jothimani, K. and Baskonus, H.M. *Existence results of Hilfer integro-differential equations with fractional order*, Discrete. Contin. Dyn. Syst. 13(3) (2020), 911–923.

39. Touchent, K.A., Hammouch, Z., Mekkaoui, T. and Belgacem, F. *Implementation and convergence analysis of homotopy perturbation coupled with sumudu transform to construct solutions of local-fractional pdes*, Fractal. Fract. 2(3) (2018), 22.

40. Vahidi, J. *The combined Laplace-homotopy analysis method for partial differential equations*, J. Math. Comput. Sci. 16(1) (2016), 88–102.

41. Yépez-Martínez, H., and Gómez-Aguilar, J.F. *Numerical and analytical solutions of nonlinear differential equations involving fractional operators with power and Mittag-Leffler kernel*, Math. Model. Nat. Phenom. 13(1) (2018), Paper No. 13, 17 pp.