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Research Article

The time-dependent diffusion equation: An inverse diffusivity problem

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Abstract

We find a solution of an unknown time-dependent diffusivity $a(t)$ in a linear inverse parabolic problem by a modified genetic algorithm. At first, it is shown that under certain conditions of data, there exists at least one solution for unknown $a(t)$ in $(a(t), T(x,t))$, which is a solution to the corresponding problem. Then, an optimal estimation for unknown $a(t)$ is found by applying the least-squares method and a modified genetic algorithm. Results show that an excellent estimation can be obtained by the implementation of a modified real-valued genetic algorithm within an Intel Pentium (R) dual-core CPU with a clock speed of 2.4 GHz.

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1 Introduction

This article considers helium diffusion kinetics, which is important for materials in which helium measurements are made, particularly for thermochronology (see [2, 4, 3]). Solutions of an inverse problem for the helium production-diffusion equation are explored in this research article. This solution is a time-temperature path derived from apatite fission-track length distribution to characterize the response of apatite helium ages to thermal histories involving partial retention. Mathematically, the helium production-diffusion equation without any source of radiogenic production in a rectangle domain $\Omega = \{(x, y) : 0 < x < 1,\ 0 < y < 1\}$ can be written as (see [2, 4, 3])

$$\frac{\partial T}{\partial t} = a(t)(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}), \qquad (x, y) \in \Omega,\ t > 0, \tag{1a}$$

$$T(x, y, 0) = f(x, y), \qquad (x, y) \in \overline{\Omega}, \tag{1b}$$

$$T(0, y, t) = g_0(y, t), \qquad 0 \le y \le 1,\ t \ge 0, \tag{1c}$$

$$T(1, y, t) = g_1(y, t), \qquad 0 \le y \le 1,\ t \ge 0, \tag{1d}$$

$$\frac{\partial T}{\partial y}(x, 0, t) = h_0(x, t), \qquad 0 \le x \le 1,\ t \ge 0, \tag{1e}$$

$$\frac{\partial T}{\partial y}(x, 1, t) = h_1(x, t), \qquad 0 \le x \le 1,\ t \ge 0, \tag{1f}$$

where $f(x, y)$, $g_0(y, t)$, $g_1(y, t)$, $h_0(x, t)$, and $h_1(x, t)$ are known functions and $a(t)$ is the time-dependent diffusivity and should be determined in this inverse problem.

Here the special case $h_0 = h_1 = 0$ is considered, that is,

$$\frac{\partial T}{\partial t} = a(t)(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}), \qquad (x, y) \in \Omega,\ t > 0, \tag{2a}$$

$$T(x, y, 0) = f(x, y), \qquad (x, y) \in \Omega, \tag{2b}$$

$$T(0, y, t) = g_0(y, t), \qquad 0 \le y \le 1,\ t \ge 0, \tag{2c}$$

$$T(1, y, t) = g_1(y, t), \qquad 0 \le y \le 1,\ t \ge 0, \tag{2d}$$

$$\frac{\partial T}{\partial y}(x, 0, t) = \frac{\partial T}{\partial y}(x, 1, t) = 0, \qquad 0 \le x \le 1,\ t \ge 0, \tag{2e}$$

together with an overspecified condition for fixed points $x_0, y_0 \in (0, 1)$, as Cannon considered in [7, p. 191],

$$a(t)\frac{\partial T}{\partial x}(x_0, y_0, t) = p(t) \qquad (t \ge 0). \tag{3}$$

Problem (1) in one-dimensional case arises from the heat conduction in a material that is eroded or damaged. This erosion can be caused, for example, by radioactive. Duo to the erosion, thermal characteristics of the material including its heat capacity, diffusivity, and conductivity change based on the level of the erosion. These characteristics can be related to time, where $a(t)$ is the thermal diffusivity, which is the ratio between the thermal conductivity and the heat capacity. Lesnic [14] investigated an inverse problem for determining the temperature and the time-dependent thermal diffusivity from various additional nonlocal information (see also [7, Chapter 13]). It should be remarked that the inverse problem (1) with $a(t) = 1$ has been investigated by several authors; see, for example, [5, 20, 8, 22, 23, 9] and for a two-dimensional case [21]. Since solving inverse diffusivity problems using evolutionary algorithms has not been investigated yet, the solution to these problems using a modified genetic algorithm is considered in this article in the hope that this effort triggers other authors' motivation to examine new artificial intelligence techniques.

2 General solution in term of Green's functions

For $\Omega = (0,1) \times (0,1)$ and $\overline{\Omega} = [0,1] \times [0,1]$, take

$$S_1 = \{\mathbf{x} = (x,y) : x \in \{0,1\},\, 0 \le y \le 1\},$$
$$S_2 = \{\mathbf{x} = (x,y) : y \in \{0,1\},\, 0 \le x \le 1\}.$$

Then $\partial\Omega = S_1 \cup S_2$. Consider the two-dimensional heat conduction problem with nonhomogeneous boundary conditions

$$\frac{\partial T}{\partial t}(\mathbf{x},t) = \nabla^2 T(\mathbf{x},t), \qquad \mathbf{x} \in \Omega,\ t > 0, \tag{4a}$$
$$T(\mathbf{x},0) = f(\mathbf{x}), \qquad \mathbf{x} \in \overline{\Omega}, \tag{4b}$$
$$T(\mathbf{x},t) = g(\mathbf{x},t), \qquad \mathbf{x} \in S_1,\ t \ge 0, \tag{4c}$$
$$\frac{\partial T}{\partial y}(\mathbf{x},t) = h(\mathbf{x},t), \qquad \mathbf{x} \in S_2,\ t \ge 0, \tag{4d}$$

where $\mathbf{x} = (x,y)$ and

$$g(\mathbf{x},t) = \begin{cases} g_0(y,t), & \mathbf{x} \in S_1, x = 0, \\ g_1(y,t), & \mathbf{x} \in S_1, x = 1, \end{cases} \qquad h(\mathbf{x},t) = \begin{cases} h_0(x,t), & \mathbf{x} \in S_2, y = 0, \\ h_1(x,t), & \mathbf{x} \in S_2, y = 1. \end{cases}$$

Here g_0, g_1, h_0, h_1, and f are known functions. It is known that (4) possesses a unique solution provided the given initial and boundary data are continuous functions; see, for example, [7, 10, 19]. The aim of this section is to find a representation of the solution of (4). To this end, as in [7], define, for $x \in \mathbb{R}$, $t > 0$,

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\Big(\frac{-x^2}{4t}\Big), \quad \theta(x,t) = \sum_{m=-\infty}^{\infty} K(x+2m,t),$$

and define the function $\Phi(\mathbf{x},t) = K(x,t)K(y,t)$ for $\mathbf{x} = (x,y) \in \mathbb{R}^2$, $t > 0$ by

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi t} \exp\Big(-\frac{||\mathbf{x}||^2}{4t}\Big).$$

It is well known from [12, Chapter 7], that

$$T(\mathbf{x},t) = \Phi *_{\mathbf{x}} f = \int_{\mathbb{R}^2} \Phi(\mathbf{x}-\boldsymbol{\xi},t) f(\boldsymbol{\xi}) d\boldsymbol{\xi} \qquad (\mathbf{x} \in \mathbb{R}^2,\ t > 0), \tag{5}$$

where $T(\mathbf{x},t)$ is a solution for a pure initial value problem in the equation

$$\begin{aligned} \frac{\partial T}{\partial t}(\mathbf{x},t) &= \nabla^2 T(\mathbf{x},t), \\ T(\mathbf{x},0) &= f(\mathbf{x}), \end{aligned}$$

for $\mathbf{x} \in \mathbb{R}^2$ and $t > 0$, provided that the function $f(\mathbf{x})$ is bounded and also (piecewise) continuous in $\mathbb{R}^2$. Now, we extend $T(\mathbf{x},t)$ continuously to $\overline{\Omega}$, and consider the following heat conduction problem in Ω with homogeneous boundary conditions:

$$\frac{\partial T}{\partial t}(\mathbf{x},t) = \nabla^2 T(\mathbf{x},t), \qquad \mathbf{x} \in \Omega,\ t > 0, \tag{6a}$$

$$T(\mathbf{x},0) = f(\mathbf{x}), \qquad \mathbf{x} \in \overline{\Omega}, \tag{6b}$$

$$T(\mathbf{x},t) = 0, \qquad \mathbf{x} \in S_1,\ t \geq 0, \tag{6c}$$

$$\frac{\partial T}{\partial y}(\mathbf{x},t) = 0, \qquad \mathbf{x} \in S_2,\ t \geq 0. \tag{6d}$$

To solve problem (6), one can first extend $f(\mathbf{x})$ to $[-1,1] \times [-1,1]$ such that f is odd in x and even in y. Then we extend f periodically to $\mathbb{R}^2$ by

$$f(x+2m, y+2n) = f(x,y), \qquad (n,m \in \mathbb{Z}).$$

If we define, for $x, \xi \in \mathbb{R}$, $t > 0$,

$$\Theta^{\pm}(x,\xi,t) = \theta(x-\xi,t) \mp \theta(x+\xi,t),$$

then we can obtain from (5) for $T(\mathbf{x},t)$ that

$$T(\mathbf{x},t) = \int_{\Omega} \Theta^{-}(x,\xi,t)\Theta^{+}(y,\eta,t) f(\xi,\eta) d\xi d\eta. \tag{7}$$

Now, define $\mathcal{G}$ by

$$\mathcal{G}(x,y,t;\xi,\eta) = \Theta^{+}(y,\eta,t)\Theta^{-}(x,\xi,t). \tag{8}$$

Since $\theta(x,t)$ is an even function with respect to the variable x for all values of $t > 0$, it is easy to observe for every ξ, η, t that

$$\mathcal{G}(x,y,t;\xi,\eta) = 0, \qquad (x,y) \in S_1,$$
$$\frac{\partial}{\partial y}\mathcal{G}(x,y,t;\xi,\eta) = 0, \qquad (x,y) \in S_2.$$

So $T(\mathbf{x},t)$, defined by (7), is, in fact, the solution of (6). This means that the function $\mathcal{G}$, defined by (8), is the Green function for problem (4), so that the solution of (4) can be written (see [19, Chapter 6] or [13, Chapter 7]) by

$$T(\mathbf{x},t) = \int_\Omega \mathcal{G}(\mathbf{x},t;\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}$$
$$+ \int_0^t d\tau \int_{S_1} \frac{\partial \mathcal{G}}{\partial \mathbf{n}_1}(\mathbf{x},t-\tau;\boldsymbol{\xi})g(\boldsymbol{\xi},\tau)d\eta \tag{9}$$
$$- \int_0^t d\tau \int_{S_2} (\mathbf{x},t-\tau;\boldsymbol{\xi})h(\boldsymbol{\xi},\tau)d\xi, \tag{10}$$

where $\mathbf{x} = (x,y)$, $\boldsymbol{\xi} = (\xi,\eta)$, $\boldsymbol{\xi} \in S_1$ in (9), and $\boldsymbol{\xi} \in S_2$ in (10); and $(\partial/\partial \mathbf{n}_1)$, in (9), denotes the directional derivative on S_1. Therefore, one can write the solution $T(x,y,t)$ of (4) in the following form:

$$\begin{aligned}
T(x,y,t) = &\int_\Omega \mathcal{G}(x,y,t;\xi,\eta)f(\xi,\eta)d\xi d\eta \\
&- 2\int_\Omega g_0(\eta,\tau)\frac{\partial\theta}{\partial x}(x,t-\tau)\Theta^{+}(y,\eta,t-\tau)d\eta d\tau \\
&+ 2\int_\Omega g_1(\eta,\tau)\frac{\partial\theta}{\partial x}(x-1,t-\tau)\Theta^{+}(y,\eta,t-\tau)d\eta d\tau \\
&- 2\int_\Omega h_0(\eta,\tau)\theta(y,t-\tau)\Theta^{-}(x,\xi,t-\tau)d\xi d\tau \\
&- 2\int_\Omega h_1(\eta,\tau)\theta(y-1,t-\tau)\Theta^{-}(x,\xi,t-\tau)d\xi d\tau.
\end{aligned}$$

Assuming that $a(t)$ is given, one should note that under some suitable hypotheses on the initial and boundary conditions, the existence of a unique solution to (6) can be proved.

Theorem 1. Let $\tau > 0$, $\Omega = (0,1)$ and $f \in L^2(\Omega)$. Suppose also that $a \in L^\infty((0,\tau))$ and $a(t) \geq 0$, for almost every $t \in (0,\tau)$. Then there exists a unique function u of (6) satisfying

$$T \in L^2\left((0,\tau); H_0^1(\Omega)\right) \cap C\left([0,\tau]; L^2(\Omega)\right)$$

and

$$\frac{\partial T}{\partial t} \in L^2\left((0,\tau); H_0^{-1}(\Omega)\right),$$

where $H_0^1(\Omega)$ and $H_0^{-1}(\Omega)$ are the usual Sobolev spaces.

Proof. The proof of Theorem 1 follows from [16, Theorem 7.1] (see also [1, 15, 17, 18]). □

Remark 1. If a is given and the assumptions of Theorem 1 hold, then one can obtain a unique solution $T = T(x, y, t; a)$ of (6). Hence some consistency conditions, in boundary, should hold between a and u in (3). Consequently, the coefficient a is uniquely determined.

Remark 2. One should note that Theorem 1 improves [3, Lemma 2.1].

3 Utilizing an appropriate transformation

Turn back to the main problem (2), in which $u(x, y, t)$ and $a(t)$ are unknown. Denote by $\mathcal{F}$ the family of all continuous functions $a : \mathbb{R} \to \mathbb{R}$ such that $a(t) > 0$, for all $t \in \mathbb{R}$. For $a \in \mathcal{F}$, let (see [7])

$$s = \alpha(t) = \int_0^t a(\tau)d\tau \qquad (t \geq 0).$$

Then $\alpha(0) = 0$ and $\alpha'(t) = a(t) > 0$, and thus $s = \alpha(t)$ is a nonnegative invertible function with inverse $t = \beta(s)$. By plugging

$$\begin{gathered} v(x, y, s) = u(x, y, \beta(s)), \\ \phi_0(y, s) = g_0(y, \beta(s)), \quad \phi_1(y, s) = g_1(y, \beta(s)), \\ b(s) = a(\beta(s)), \quad q(s) = p(\beta(s)), \end{gathered}$$

problem (2) is rephrased in the following form:

$$v_s = v_{xx} + v_{yy}, \qquad (x, y) \in \Omega,\ s > 0, \tag{11a}$$
$$v(x, y, 0) = f(x, y), \qquad (x, y) \in \overline{\Omega}, \tag{11b}$$
$$v(0, y, s) = \phi_0(y, s), \qquad 0 \leq y \leq 1,\ s \geq 0, \tag{11c}$$
$$v(1, y, s) = \phi_1(y, s), \qquad 0 \leq y \leq 1,\ s \geq 0, \tag{11d}$$
$$v_y(x, 0, s) = v_y(x, 1, s) = 0, \qquad 0 \leq x \leq 1,\ s \geq 0. \tag{11e}$$

In this setting, the over-specified condition (3) implies that

$$b(s)v_x(0, y_0, s) = q(s) \qquad (s \geq 0).$$

By [7, Chapter 13], the solution of (11) is represented as follows:

$$
\begin{aligned}
v(x,y,s) = &\int_{\Omega} f(\xi,\eta)\mathcal{G}(x,y,s;\xi,\eta)d\xi d\eta \\
&- 2\int_0^s\int_0^1 \frac{\partial\theta}{\partial x}(x,s-\zeta)\Theta^+(y,\eta,s-\zeta)\phi_0(\eta,\zeta)d\eta d\zeta \\
&+ 2\int_0^s\int_0^1 \frac{\partial\theta}{\partial x}(x-1,s-\zeta)\Theta^+(y,\eta,s-\zeta)\phi_1(\eta,\zeta)d\eta d\zeta.
\end{aligned} \tag{12}
$$

Applying the change of the variable $\zeta = \alpha(\tau)$ in (12), one obtains

$$
\begin{aligned}
u(x,y,t) = &\int_{\Omega} f(\xi,\eta)\mathcal{G}(x,y,\alpha(t);\xi,\eta)d\xi d\eta \\
&- 2\int_0^t\int_0^1 \frac{\partial\theta}{\partial x}(x,\alpha(t)-\alpha(\tau))\Theta^+(y,\eta,\alpha(t)-\alpha(\tau))g_0(\eta,\tau)d\eta a(\tau)d\tau \\
&+ 2\int_0^t\int_0^1 g_1(\eta,\tau)\frac{\partial\theta}{\partial x}(x-1,\alpha(t)-\alpha(\tau))\Theta^+(y,\eta,\alpha(t)-\alpha(\tau))d\eta a(\tau)d\tau.
\end{aligned} \tag{13}
$$

For the sake of simplicity, from now on, let

$$
\int_\tau^t a = \int_\tau^t a(\zeta)d\zeta = \alpha(t) - \alpha(\tau).
$$

Using (13) and calculating $u_x(x,y,t)$, one writes $u_x(x,y,t) = J_1 + J_2 + J_3$, where

$$
\begin{aligned}
J_1 &= \int_{\Omega} f(\xi,\eta)\frac{\partial}{\partial x}\mathcal{G}(x,y,\alpha(t);\xi,\eta)d\xi d\eta, \\
J_2 &= -2\int_0^t\int_0^1 \frac{\partial^2\theta}{\partial x^2}(x,\int_\tau^t a)\Theta^+(y,\eta,\int_\tau^t a)g_0(\eta,\tau)d\eta a(\tau)d\tau, \\
J_3 &= 2\int_0^t\int_0^1 \frac{\partial^2\theta}{\partial x^2}(x-1,\int_\tau^t a)\Theta^+(y,\eta,\int_\tau^t a)g_1(\eta,\tau)d\eta a(\tau)d\tau.
\end{aligned}
$$

It is known that $\theta_t(x,t) = \theta_{xx}(x,t)$ (see [7]), so

$$
\frac{\partial^2\theta}{\partial x^2}(x,\int_\tau^t a) = \frac{\partial\theta}{\partial t}(x,\int_\tau^t a) = -\frac{1}{a(\tau)}\frac{\partial}{\partial\tau}\theta(x,\int_\tau^t a).
$$

Since $g_0(y,0) = g_1(y,0) = 0$ and $\theta(x,0) = 0$ for $x \neq 0$ (see [7]), integration by parts yields that

$$
\begin{aligned}
J_2 &= -2\int_0^t\int_0^1 -\frac{\partial}{\partial\tau}\theta(x,\int_\tau^t a)\Theta^+(y,\eta,\int_\tau^t a)g_0(\eta,\tau)d\eta d\tau\\
&= -2\int_0^t\int_0^1 \theta(x,\int_\tau^t a)\frac{\partial}{\partial\tau}\Big[\Theta^+(y,\eta,\int_\tau^t a)g_0(\eta,\tau)\Big]d\eta d\tau,\\
J_3 &= 2\int_0^t\int_0^1 \theta(x-1,\int_\tau^t a)\frac{\partial}{\partial\tau}\Big[\Theta^+(y,\eta,\int_\tau^t a)g_1(\eta,\tau)\Big]d\eta d\tau.
\end{aligned}
$$

Now by applying the over-specified condition (3), we have

$$
\begin{aligned}
\frac{p(t)}{a(t)} &= \int_\Omega f(\xi,\eta)\frac{\partial}{\partial x}\mathcal{G}(x_0,y_0,\alpha(t);\xi,\eta)d\xi d\eta,\\
&\quad -2\int_0^t\int_0^1 \theta(x_0,\int_\tau^t a)\frac{\partial}{\partial\tau}\Big[\Theta^+(y_0,\eta,\int_\tau^t a)g_0(\eta,\tau)\Big]d\eta d\tau\\
&\quad +2\int_0^t\int_0^1 \theta(x_0-1,\int_\tau^t a)\frac{\partial}{\partial\tau}\Big[\Theta^+(y_0,\eta,\int_\tau^t a)g_1(\eta,\tau)\Big]d\eta d\tau.
\end{aligned}
\tag{14}
$$

Thus one obtains the following result.

Theorem 2. Suppose that the function $f(x,y)$ is piecewise continuous on $\overline{\Omega}$, that $g_0(y,t)$, $g_1(y,t)$, $(\partial/\partial t)g_0(y,t)$, and $(\partial/\partial t)g_1(y,t)$ are continuous functions for $0 \le y \le 1$, $t \ge 0$, and that $g_0(y,0) = g_1(y,0) = 0$ for all y [or just $y = y_0$]. Then problem (2) possesses a unique solution (u,a) if and only if the integral equation (14) has a unique solution $a \in \mathcal{F}$.

4 Uniqueness and existence of solution for a certain problem

Now, consider the problem

$$
\begin{aligned}
\frac{\partial T}{\partial t} &= a(t)(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}), && (x,y)\in\Omega,\ t>0, && \text{(15a)}\\
T(x,y,0) &= 0, && (x,y)\in\overline{\Omega}, && \text{(15b)}\\
T(0,y,t) &= 0, && 0\le y\le 1,\ t\ge 0, && \text{(15c)}\\
T(1,y,t) &= g(y,t), && 0\le y\le 1,\ t\ge 0, && \text{(15d)}\\
\frac{\partial T}{\partial y}(x,0,t) &= \frac{\partial T}{\partial y}(x,1,t) = 0, && 0\le x\le 1,\ t\ge 0, && \text{(15e)}
\end{aligned}
$$

and the overspecified condition

$$
a(t)\frac{\partial T}{\partial x}(0,0,t) = p(t), \qquad (t\ge 0).
$$

The solution of (15) is as follows:

$$T(x,y,t) = +2\int_0^t\int_0^1 g(\eta,\tau)\frac{\partial\theta}{\partial x}(x-1,\int_\tau^t a)\Theta^+(y,\eta,\int_\tau^t a)d\eta a(\tau)d\tau.$$

The integral equation (14) becomes

$$\begin{aligned}\frac{p(t)}{a(t)} =&2\int_0^t\int_0^1 \theta(-1,\int_\tau^t a)\frac{\partial}{\partial\tau}\Big[\Theta^+(0,\eta,\int_\tau^t a)g(\eta,\tau)\Big]d\eta d\tau\\ =&2\int_0^t\int_0^1 \theta(-1,\int_\tau^t a)\Big[-a(\tau)\Theta_t^+ g + \Theta^+ g_\tau\Big]d\eta d\tau.\end{aligned} \tag{16}$$

Moreover,

$$\begin{aligned}\frac{p'(t)a(t)-p(t)a'(t)}{a(t)^2} =&2a(t)\int_0^t d\tau\\ &\times\int_0^1 \theta_t(-1,\int_\tau^t a)\Big[-a(\tau)\Theta_t^+ g(\eta,\tau) + \Theta^+\frac{\partial g}{\partial\tau}(\eta,\tau)\Big]\\ &+\theta(-1,\int_\tau^t a)\Big[-a(\tau)\Theta_{tt}^+ g(\eta,\tau) + \Theta_t^+\frac{\partial g}{\partial\tau}(\eta,\tau)\Big]d\eta.\end{aligned}$$

Remark 3. Since $\Theta^+(0,\eta,\cdot) = 2\theta(\eta,\cdot)$ and $\Theta_t^+(0,\eta,\cdot) = 2\theta_t(\eta,\cdot)$, equation (16) can be rephrased in the following forms:

$$\begin{aligned}a(t) =& \frac{p(t)}{2}\Big\{\int_0^t \theta(-1,\int_\tau^t a)\int_0^1[\theta g_\tau - a(\tau)\theta_t g]d\eta d\tau\Big\}^{-1}\\ =& \frac{p(t)}{2}\Big\{\int_0^t \theta(-1,\int_\tau^t a)\int_0^1[\theta g_\tau + a(\tau)\theta_{xx} g]d\eta d\tau\Big\}^{-1}.\end{aligned}$$

Lemma 1. If $g = g(t)$, then one can write (16) in the following form:

$$a(t) = \frac{p(t)}{2}\Big\{\int_0^t \theta(1,\int_\tau^t a)g'(\tau)d\tau\Big\}^{-1}. \tag{17}$$

Proof. If $g = g(t)$, then

$$\begin{aligned}\frac{\partial T}{\partial x}(0,0,t) =& 4\int_0^t \theta(-1,\int_\tau^t a)\Big\{\frac{g'(\tau)}{2} + a(\tau)g(\tau)\Big[\theta_x(1,\int_\tau^t a) - \theta_x(0,\int_\tau^t a)\Big]\Big\}d\tau\\ =& 2\int_0^t \theta(-1,\int_\tau^t a)g'(\tau)d\tau.\end{aligned}$$

□

Corollary 1. Under the assumption that $g = g(t)$ is continuously differentiable and that $g(0) = 0$, problem (15) possesses a unique solution (u,a) if and only if the nonlinear integral equation (17) possesses a unique positive solution $a(t)$ that is continuous for $0 \le t < T$.

Definition 1. Suppose that $g = g(t)$ is continuously differentiable and that $g(0) = 0$. For a positive continuous function $a(t)$, that is, $a \in \mathcal{F}$, define

$$(\mathcal{L}a)(t) = \frac{p(t)}{2}\left\{\int_0^t \theta(1, \int_\tau^t a) g'(\tau) d\tau\right\}^{-1}.$$

Therefore, a is a solution of (17) if and only if a is a fixed point of the operator $\mathcal{L}$, that is, $\mathcal{L}a = a$. In the rest of this section, it is proved that under certain conditions, $\mathcal{L}$ possesses a fixed point.

For a function h defined on $[0, T]$, let

$$\|h\|_t = \sup\{|h(s)| : 0 \leq s \leq t\}.$$

For a positive constant δ, if $\delta \leq a(t)$ and $\delta \leq b(t)$, then, using the mean value theorem, one obtains

$$\begin{aligned}
|(\mathcal{L}a)(t) - (\mathcal{L}b)(t)| &\leq \frac{|p(t)|}{2}\Big|\int_0^t \theta_t(1, \delta(t-\tau)) g'(\tau) d\tau\Big|^{-2} \\
&\quad \times \Big|\int_0^t \theta_t(1, \gamma)\Big[\int_\tau^t a(s) - b(s) ds\Big] g'(\tau) d\tau\Big| \\
&\leq \frac{|p(t)|}{2}\Big|\int_0^t \theta_t(1, \delta(t-\tau)) g'(\tau) d\tau\Big|^{-2} \|\theta_t\|_t \|a - b\|_t t |g(t)|,
\end{aligned}$$

where $\int_\tau^t a \leq \gamma \leq \int_\tau^t b$.

Corollary 2. Suppose for some $r \in (0, 1)$ that

$$\frac{|tp(t)g(t)|}{2}\Big|\int_0^t \theta_t(1, \delta(t-\tau)) g'(\tau) d\tau\Big|^{-2} \|\theta_t\|_t \leq r \qquad (t \in [0, T]). \tag{18}$$

Then the mapping $a \mapsto \mathcal{L}a$ is a contraction.

The main result of this section will be obtained by using (17) and Corollaries 1 and 2.

Theorem 3. Suppose that $p(t)$ and $g = g(t)$ are continuously differentiable, that $g(0) = 0$, and that, for some $r \in (0, 1)$, (18) holds. Then (15) possesses a unique solution (T, a).

5 Discretization of the problem

In this section, the following linear inverse parabolic problem is considered:

$$\frac{\partial T}{\partial t}(x,t) = a(t)(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}), \quad 0 < x, y < 1, 0 < t < t_M, \tag{19a}$$

$$T(x,y,0) = f(x,y), \qquad 0 \le x, y \le 1, \tag{19b}$$

$$T(0,y,t) = g_0(y,t), \qquad 0 \le y \le 1, 0 \le t \le t_M, \tag{19c}$$

$$T(1,y,t) = g_1(y,t), \qquad 0 \le y \le 1, 0 \le t \le t_M, \tag{19d}$$

$$T(x,0,t) = h_0(x,t), \qquad 0 \le x \le 1, 0 \le t \le t_M, \tag{19e}$$

$$T(x,1,t) = h_1(x,t), \qquad 0 \le x \le 1, 0 \le t \le t_M, \tag{19f}$$

with the overspecified conditions for the fixed points x_0,$y_0 \in (0,1)$,

$$T(x_0, y_0, t) = s(t), \qquad 0 \le t \le t_M, \tag{19g}$$

where $f(x,y)$, $g_0(y,t)$, $g_1(y,t)$, $h_0(x,t)$, $h_1(x,t)$, and $\phi(t)$ are known functions while $a(t)$ and $T(x,y,t)$ are unknown in this inverse problem.

Now we can solve problem (19) in the least-square method. Also, we can consider the amount of difference between calculated values of the problem at the location of sensor and measured temperatures, which come from sensor as the cost function. When the cost function approaches to zero, the estimated value for $a(t)$ approaches to exact $a(t)$. Let

$$f(G) = \sum_{j=1}^{m} (T_j - s_j)^2, \tag{20}$$

where T_j, $j = 1, 2, 3, \ldots, m$, are calculated by solving the direct heat problem. To do this, consider a prior guess for $a(t)$. Also $s_j = s(t_j)$, $j = 1, 2, 3, \ldots, m$, are measured temperatures. Here, when (20) reaches its minimum, the estimated value for $a(t)$ is optimal.

Remark 4. In this article, we use an explicit finite difference approximation to discretize problem (19) as follows:

$$\begin{aligned} T_{i,j,s+1} = ka_s(&r_1 T_{i-1,j,s} - 2r_1 Ti,j,s + r_1 Ti+1,j,s + r_2 T_{i,j-1,s} - 2r_2 Ti,j,s \\ &+ r_2 Ti,j+1,s) + T_{i,j,s}, \\ &\quad i = 1, \ldots, N-1, \; j = 1, \ldots, N-1, \; S = 0, \ldots, N-1, \\ T_{i,j,0} =& f(ih, jk), \quad i = 1, \ldots, N-1, \; j = 1, \ldots, N-1, \\ T_{0,j,s} =& g_0(jk, sw), \quad j = 1, \ldots, N-1, \; s = 0, \ldots, N-1, \\ T_{1,j,s} =& g_1(jk, sw), \quad j = 1, \ldots, N-1, \; s = 0, \ldots, N-1, \\ T_{i,0,s} =& h_0(ih, sw), \quad i = 1, \ldots, N-1, \; s = 0, \ldots, N-1, \\ T_{1,j,s} =& h_1(ih, sw), \quad i = 1, \ldots, N-1, \; s = 0, \ldots, N-1, \end{aligned}$$

where $x = ih$, $y = jk$, and $t = sw$.

6 Genetic algorithm

The genetic algorithm was introduced by Holland [11] and has been used to solve a wide range of problems. This algorithm is a searching method that has been inspired by Darwinian principles of biological evolution. This algorithm utilizes an initial stochastic set of candidate solutions as an initial population. Each individual of this set is a numeric vector that estimates a possible solution to the problem. Then, genetic operators such as *Mutation* and *Recombination* are successively applied to initial populations in order to simulate a biological evolution. These operators push all populations toward optimal solutions to the problems. After a predefined number of iterations or termination conditions, the best individual in the population is considered as an optimal solution to the problem.

In the classic genetic algorithms, the vectors are a binary string and present a possible solution to the problem. Indeed there are many problems in which a real-valued vector is needed as an optimal solution. In this article, a real-valued genetic algorithm (RVGA) is used to estimate unknown $a(t)$ in its corresponding time. The steps of our RVGA in this article are as follows:

Step 1. A random set of real-valued vectors are generated as an initial population.

Step 2. Each individual in the population is evaluated by the fitness function.

Step 3. Some individuals are selected based on their fitness by *Tournament Selection*.

Step 4. A recombination operator is applied to the selected individuals to generate offsprings.

Step 5. The Mutation operation is applied to offspring.

Step 6. The offsprings are evaluated accordingly.

Step 7. The population is updated.

Step 8. Steps 3–7 are repeated, until the termination condition satisfies.

7 A modified real-valued genetic algorithm to estimate unknown $a(t)$

In this research article, a modified genetic algorithm is presented to determine unknown $a(t)$. This algorithm employs a real-valued vector as a candidate solution to the problem. In this algorithm, a new step is added to the procedure of the original genetic algorithm after the *Mutation operator* to increase the capability of the algorithm for solving the problems of this article. In the presented algorithm, the jth element of each vector is a real number that estimates $a(j * .001)$ for $j = 0, 1, 2, \ldots, N-1$. Each vector is considered as a chromosome in the population, and accordingly each element of chromosomes is considered as a gene. Therefore, $g_{p,j}$ is the jth gene of chromosome of p. To reach an acceptable estimation for $a(t)$, equation (20) should reach its minimum. To meet this purpose, equation (20) is considered as the fitness function of the algorithm. When the algorithm stops, a vector with the lowest value of fitness is considered as the best solution of $a(t)$. Then this vector is interpolated to estimate $a(t)$ as a polynomial function.

The steps of our presented real-valued genetic algorithm (RVGA) are as follows:

Step 1. A random set of real-valued vectors is generated as the initial population.

Step 2. Each individual in the population is evaluated by the fitness function.

Step 3. Some individuals are selected based on their fitness by *Tournament Selection*.

Step 4. To apply the recombination operator on the pair of parents, we act as follows:

$$g_{ch1,j} = \alpha \times g_{p1,j} + (1-\alpha) \times g_{p2,j}, \qquad j = 1, 2, 3, \ldots, N-1,$$

$$g_{ch2,j} = \beta \times g_{p1,j} + (1-\beta) \times g_{p2,j}, \qquad j = 1, 2, 3, \ldots, N-1,$$

Step 5. To apply the *Mutation* operator on the offsprings, a gene of each offspring is selected randomly and a small random value is added to it.

Step 6. Fitness of offsprings is evaluated.

Step 7. All other offsprings are moved toward the best offspring. Then a small random value is added to a gene of each offspring selected randomly to ensure the variety of the population.

Step 8. The population is updated.

Step 9. Steps 3–7 are repeated, until the termination condition satisfies.

Table 1: Parameters of the real-valued genetic algorithm

Representation	Real-valued vector
Recombination	Arithmetic crossover
Recombination probability	100%
Mutation	Add a random value to one gene
Mutation probability	$1/n$ for each gene
Parents selection	Best 4 out of random 10
Survivor selection	All of the old generations replaced with new offsprings
Number of offsprings	4
Population size	20
Initialization	Random
Termination condition	Number of generations

Table 1 shows the parameters of the RVGA and their roles.

8 Numerical results and discussion

The main purpose of this section is to show the capability of our presented algorithm in the previous section for solving IPP. The capability of the algorithm is investigated by an example of IPP as follows.

Example 1. In this example, for $(x, y, t) \in (0,1) \times (0,1) \times (0,1)$, consider the IPP

$$\frac{\partial T}{\partial t}(x,y,t) = a(t)(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}), \tag{22a}$$

$$T(x,y,0) = (\cos(\pi y) + \sin(\pi x)), \tag{22b}$$

$$T(0,y,t) = e^{-\pi^2 t^2}\cos(\pi y), \tag{22c}$$

$$T(1,y,t) = e^{-\pi^2 t^2}(\cos(\pi y) + \sin(\pi)), \tag{22d}$$

$$T(x,0,t) = e^{-\pi^2 t^2}(\sin(\pi x) + 1), \tag{22e}$$

$$T(x,1,t) = e^{-\pi^2 t^2}(\cos(\pi) + \sin(\pi x)), \tag{22f}$$

and with the following overspecified condition

$$s(t_j) = T(0.1, 0.1, t_j) + \sigma R, \quad t_j = (0.001)j, \quad j = 0,1,2,\ldots,999. \tag{22g}$$

In this problem, the values of $s(t_j)$'s are gained from the exact solution of the problem. It is evident that in the real-world application of this algorithm,

Table 2: Result of the execution of the modified RVGA, with 10 to 100 generations when the noise of the measurement is on the s (overspecified condition).

Gen.	Fitness	Time(s)	S
10	1.2936	371.1635	0.1821
20	1.1459	715.8062	0.2162
30	0.6429	1063.0171	0.0823
40	0.7662	1446.6704	0.0966
50	0.4227	2012.2701	0.0515
60	0.4109	2171.3517	0.0529
70	0.3383	2543.9981	0.0435
80	0.3379	2890.7852	0.0168
90	0.2945	3254.4581	0.0080
100	0.2407	3663.9104	0.0064

these values come from sensors. It is obvious that all sensors have a small error of measurement. Therefore, to simulate these errors, a small random value is added to $s(t_j)$'s to have noisy data.

In this example, $a(t) = 2t$ and

$$T(x, y, t) = e^{-\pi^2 t^2}(\cos(\pi y) + \sin(\pi x)).$$

Remark 5. In this article, we use the following equation to calculate total error S, in our numerical computations (see [6]):

$$S = \Big[\frac{1}{(N-1)} \sum_{i=1}^{N} (\widehat{a_i} - a_i)^2\Big]^{\frac{1}{2}},$$

where $(N-1)$ is the length of estimated vector (chromosome), $\widehat{a_i}, i = 1, 2, \cdots, N$, come from interpolated function, and $a_i, i = 1, 2, \cdots, N$, are the exact values of $a(t)$.

In this section, the population size of the algorithm is 20. Each chromosome in the population has 1000 genes ($t = 0.001, 0.002, 0.003, \ldots, 0.999$). Table 2 shows the result of the execution of the modified RVGA, with 10 to 100 generations when the noise (noisy data = input data + (0.00001)rand(1)) is on s. To study the ability of the algorithm to solve an IPP, we have severally executed it while the noise of the measurement is on the different functions. Table 3 illustrates the result of those executions. Also, Figures 1–6 show the executions.

Remark 6. One should note that by tending to the real value of $a(t)$, the fitness value (the difference of the calculated and measured values in the sensor position) will be reduced (see (20)). Thus the total error S will decrease.

Table 3: Result of the different executions when the noise of the measurement is on the different functions.

Noisy data	Gen.	Fitness	Time(s)	S
$s(t_j)$ *and* f	100	0.1381	3551.8039	0.0216
$s(t_j)$ *and* g_0	100	0.4914	3602.6282	0.0525
$s(t_j)$ *and* g_1	100	0.3511	3649.0969	0.0385
$s(t_j)$ *and* h_0	100	0.4028	3704.7115	0.0627
$s(t_j)$ *and* h_1	100	0.3718	3679.4812	0.0373

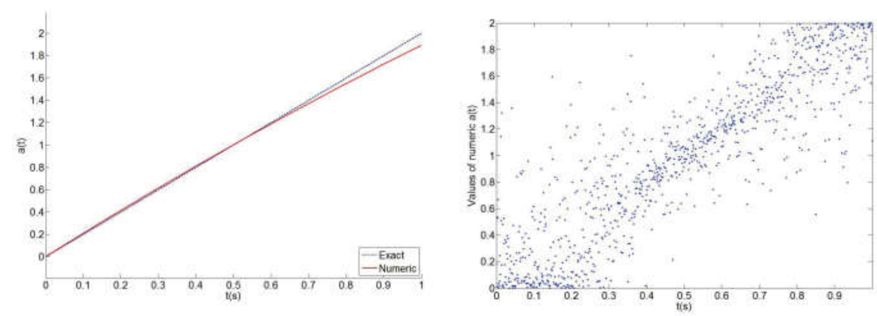

Figure 1: The exact and approximated $a(t)$ (Left) and the best chromosome (Right) with the noisy data on condition (22g).

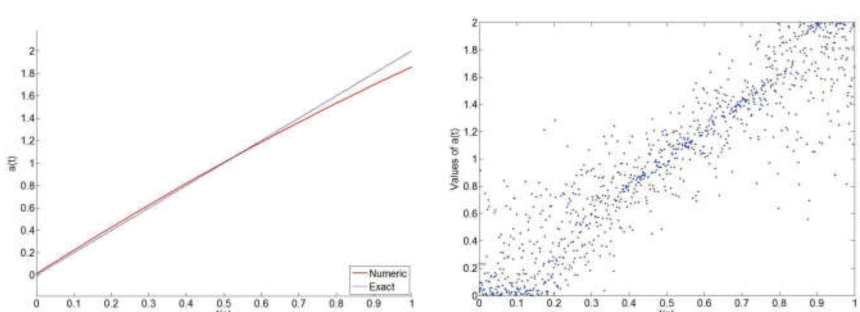

Figure 2: The exact and approximated $a(t)$ (Left) and the best chromosome (Right) with the noisy data on conditions (22g) and (22b).

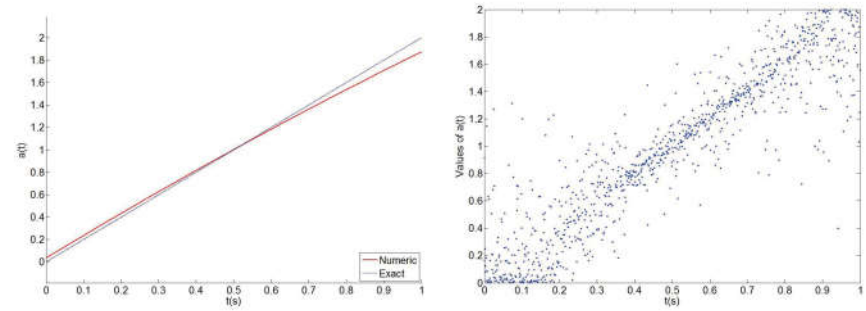

Figure 3: The exact and approximated $a(t)$ (Left) and the best chromosome (Right) with the noisy data on conditions (22g) and (22c).

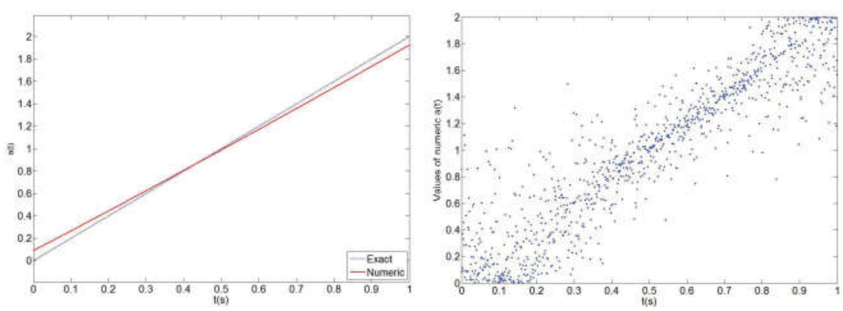

Figure 4: The exact and approximated $a(t)$ (Left) and the best chromosome (Right) with the noisy data on conditions (22g) and (22d).

8.1 Computational time and accuracy comparison

To study computational time and accuracy efficiency, the example of this section is solved by a general real-valued genetic algorithm. Table 4 shows the execution of a general real-valued genetic algorithm for the number of generations from 10 to 100. Also, Figures 7, 8, and 9 illustrate a comparative study of execution time, fitness, and total error (S) between the modified and general real-valued genetic algorithm, respectively.

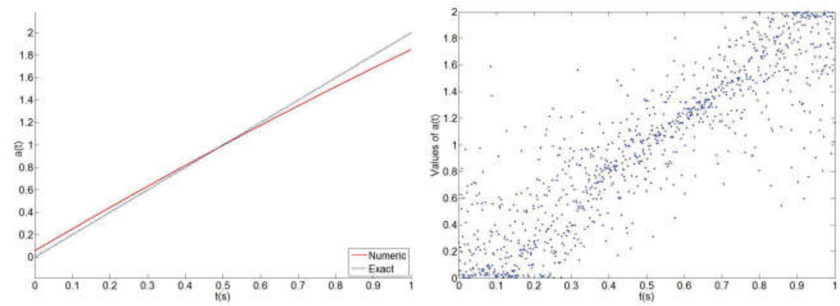

Figure 5: The exact and approximated $a(t)$ (Left) and the best chromosome (Right) with the noisy data on conditions (22g) and (22e).

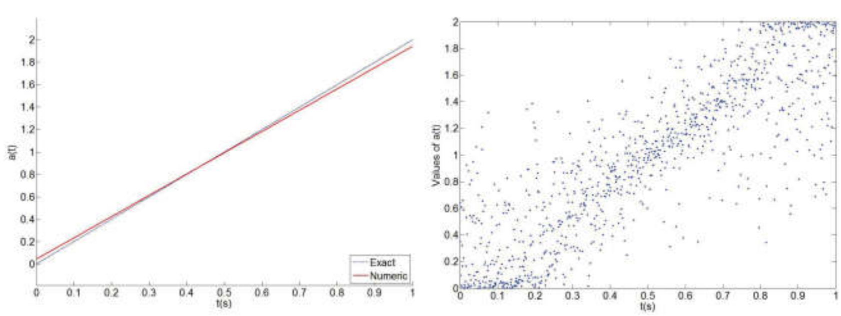

Figure 6: The exact and approximated $a(t)$ (Left) and the best chromosome (Right) with the noisy data on conditions (22g) and (22f).

Table 4: The execution of a general RVGA, with 10 to 100 generations when the noise of the measurement is on the s (overspecified condition).

Gen.	Fitness	Time(s)	S
10	3.0323	318.4011	0.3824
20	2.9706	672.6387	0.3792
30	1.4854	992.8147	0.2140
40	1.4103	1156.1270	0.2041
50	1.1419	1434.9134	0.2043
60	1.1557	1617.6324	0.1285
70	1.0357	1701.0975	0.1253
80	1.1049	1927.8542	0.1077
90	0.9340	2154.6946	0.1174
100	0.9267	2315.6700	0.0939

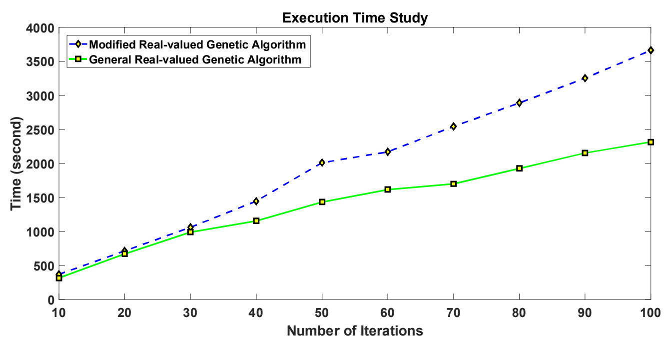

Figure 7: Comparative execution time study between the modified and a general RVGA.

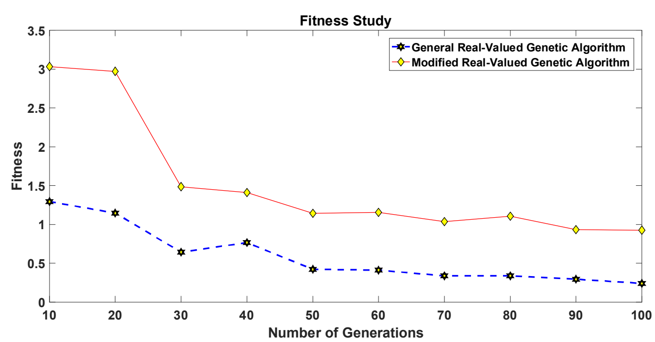

Figure 8: Comparative fitness study between the modified and a general RVGA.

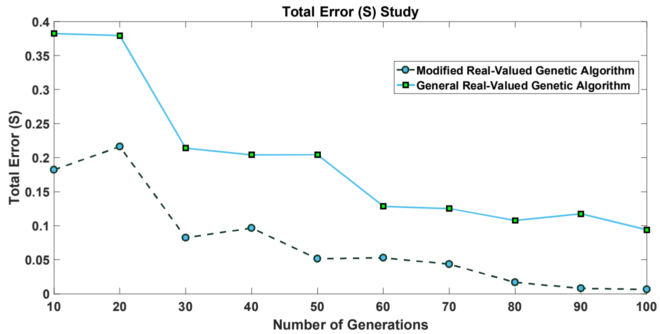

Figure 9: Comparative total error (S) study between the modified and a general RVGA.

It is evident from Figure 7 that the execution time of the modified and the general RVGA increase steadily as the number of generations grows from 10 to 100. Compared to the general RVGA, the modified RVGA generally has a higher run time due to an additional step. In contrast to the execution time, there are significant improvements in the fitness values and accordingly, the total errors (S) in the modified RVGA. This study suggests that the modified algorithm is more suitable for the applications that need a higher amount of accuracy.

9 Discussion

In Table 2, the amount of S (total error) significantly decreases while the number of generations (iterations) is increasing from 10 to 100. Our experiments showed that the amount of S fluctuates around 0.0064 after 100 generations. Therefore, an optimal solution can be obtained in problem (1) with 100 generations while the noise of measurement is only on s (overspecified condition). Table 3 illustrates the results of the algorithm when the noise of measurement is on one of the initial or boundary conditions in addition to s. It is evident that, the amount of total error increases when the number of noisy resources increases. Therefore, the amount of total error varies between 0.0216 and 0.0627.

10 Conclusion

1. The presented modified genetic algorithm is a capable method to find unknown $a(t)$ in an IPP even with noisy data.

2. Results showed that an acceptable estimation can be obtained by the presented algorithm with 100 generations within a CPU with a clock speed of 2.4 GHz.

3. The presented algorithm is stable with respect to a small perturbation in the input data.

4. The execution time of the presented algorithm is acceptable to be used in real-world applications.

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