Mashhad R. J. Math. Sci., Vol. 2(1)(2009) 1-12

Quasi-permutation representations of Borel and parabolic subgroups of Steinberg's triality groups*

M. Ghorbany†
Department of Mathematics, Iran University of Science and Technology, Mazandaran, Iran.

Abstract

If G is a finite linear group of degree n, that is, a finite group of automorphisms of an n-dimensional complex vector space, or equivalently, a finite group of non-singular matrices of order n with complex coefficients, we shall say that G is a quasi-permutation group if the trace of every element of G is a non-negative rational integer. By a quasi-permutation matrix we mean a square matrix over the complex field C with non-negative integral trace. Thus every permutation matrix over C is a quasi-permutation matrix. For a given finite group G, let $c(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the complex numbers and let $r(G)$ denote the minimal degree of a faithful rational valued complex character of G. The purpose of this paper is to calculate $c(G)$ and $r(G)$ for the Borel and parabolic subgroups of Steinberg's triality groups.

Keywords and phrases: Borel subgroup, character table, parabolic subgroup, quasi-permutation, Steinberg's triality group.
AMS Subject Classification 2000: Primary 20E10; Secondary 20C15.

1 Introduction

If F is a subfield of the complex numbers C, then a square matrix over F with non-negative integral trace is called a quasi-permutation matrix over F. Thus

*Received: 18 October 2008; Revised: 15 April 2009
†e-mail: m-ghorbani@iust.ac.ir

every permutation matrix over C is a quasi-permutation matrix. For a given finite group G, let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices.

By a rational valued character we mean a character χ corresponding to a complex representation of G such that $\chi(g) \in Q$, for all $g \in G$. As the values of the character of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of G is then simply a complex representation of G whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from G into $GL(n, Q)$ a rational representation of G and its corresponding character will be called a rational character of G. Let $r(G)$ denote the minimal degree of a faithful rational valued character of G.

Finding the above quantities have been carried out in some papers, for example in [3], [4], [5] and [6] we found these for the groups $GL(2, q)$, $SU(3, q^2)$, $PSU(3, q^2)$, $SP(4, q)$ and $G_2(2^n)$, respectively. In [2] we found the rational character table and the values of $r(G)$ and $c(G)$ for the group $PGL(2, q)$.

In this paper we will apply the algorithms in [1] to the Borel and parabolic subgroups of Steinberg's triality groups.

2 Notation and preliminary results

Let ${}^3D_4(q)$ be the Steinberg's simple triality group defined over a finite field with $q = p^n$ elements, where p is a prime number and n a positive integer.

Let $\mathbf{B}$ be the F-stable Borel subgroup TU of G, where U is the product of all root subgroups of G to positive roots, and let $B = \mathbf{B}^{\mathbf{F}}$ be the corresponding Borel subgroup of G^F.

The group B is the semidirect product of $T = \mathbf{T}^{\mathbf{F}}$ by the unipotent normal subgroup

$$U = \mathbf{U}^{\mathbf{F}} = X_\alpha X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}.$$

The elements of T form a set of representatives for the semisimple conjugacy classes of B and we parametrize these classes according to Table $A.3$ in Appendix A of [9].

The character table of the Borel subgroup B is given by Tables $A.5$ and $A.6$ in Appendix A of [8] and [9].

Let $\mathbf{P} = \langle \mathbf{B}, n_{r_1}, n_{r_3}, n_{r_4} \rangle$ be the F-stable maximal parabolic subgroup of G corresponding to the subset $\{r_1, r_3, r_4\} \subseteq \Delta$ and $P := \mathbf{P}^{\mathbf{F}}$ be the corresponding maximal parabolic subgroup of $G^F =^3 D_4(q)$. Then P is generated by B and n_α and $|P| = q^{12}(q^6-1)(q-1)$.

P is the semidirect product of the Levi complement $L_P = \langle \mathbf{T}^{\mathbf{F}}, X_\alpha, X_{-\alpha} \rangle$ and the unipotent radical $U_P := X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$. The character table of the parabolic subgroup P is given by Tables $A.9$ and $A.10$ in Appendix A of [8] and [9].

Let $Q = \langle \mathbf{B}, \mathbf{n_{r_2}} \rangle$ be the F-stable maximal parabolic subgroup of G corresponding to the subset $\{r_2\} \subseteq \Delta$ and $Q := \mathbf{Q}^{\mathbf{F}}$ be the corresponding maximal parabolic subgroup of $G^F =^3 D_4(q)$. Then Q is generated by B, n_β, and $|Q| = q^{12}(q^3-1)(q^2-1)$.

Q is the semidirect product of the Levi complement $L_Q = \langle \mathbf{T}^{\mathbf{F}}, X_\beta, X_{-\beta} \rangle$ by the unipotent radical $U_Q := X_\alpha X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$.

The character table of the parabolic subgroup Q is given by Tables $A.13$ and $A.14$ in Appendix A of [8] and [9], respectively.

Assume E is a splitting field for G and that F is a subfield of E. If $\chi, \psi \in Irr_E(G)$ we say that χ and ψ are Galois conjugate over F if $F(\chi) = F(\psi)$ and there exists $\sigma \in Gal(F(\chi)/F)$ such that $\chi^\sigma = \psi$, where $F(\chi)$ denotes the field obtained by adding the values $\chi(g)$, for all $g \in G$, to F. It is clear that this defines an equivalence relation on $Irr_E(G)$.

Let η_i for $0 \leq i \leq r$ be the Galois conjugacy classes of irreducible complex characters of G. For $0 \leq i \leq r$, let φ_i be a representative of the class η_i, with $\varphi_o = 1_G$. Write $\Psi_i = \sum_{\chi_i \in \eta_i} \chi_i$ and $K_i = ker \varphi_i$. We know that $K_i = ker \Psi_i$. For $I \subseteq \{0, 1, 2, \cdots, r\}$, put $K_I = \bigcap_{i \in I} K_i$. By definitions of $r(G)$, $c(G)$ and using the above notations we have

$r(G) = \min\{\xi(1) : \xi = \sum_{i=1}^{r} n_i\Psi_i, n_i \geq 0, K_I = 1, \ for\ I = \{i, i \neq 0, n_i > 0\}\}$,

$c(G) = \min\{\xi(1) : \xi = \sum_{i=0}^{r} n_i\Psi_i, n_i \geq 0, K_I = 1, \ for\ I = \{i, i \neq 0, n_i > 0\}\}$,

where $n_0 = -\min\{\xi(g)|g \in G\}$.

$d(\chi), m(\chi)$ and $c(\chi)$ have been defined in [1] [see Definition 3.4]. Here we may redefine them as follows.

Definition 2.1. Let χ be a complex character of G, such that $\ker\chi = 1$ and $\chi = \chi_1 + \cdots + \chi_n$, for some $\chi_i \in Irr(G)$. Then

(1) $d(\chi) = \sum_{i=1}^{n} |\Gamma_i(\chi_i)|\chi_i(1)$

(2) $m(\chi) = \begin{cases} 0, & \text{if } \chi = 1_G, \\ |\min\{\sum_{i=1}^{n}\sum_{\alpha\in\Gamma_i(\chi_i)}\chi_i^\alpha(g) : g \in G\}|, & \text{otherwise,} \end{cases}$

(3) $c(\chi) = \sum_{i=1}^{n}\sum_{\alpha\in\Gamma_i(\chi_i)} \chi_i^\alpha + m(\chi)1_G$.

So

$$r(G) = \min\{d(\chi) : \ker\chi = 1\},$$

and

$$c(G) = \min\{c(\chi)(1) : \ker\chi = 1\}.$$

The proofs of the following statements may be found in [1].

proposition 2.2. Let $\chi \in Irr(G)$, then $\sum_{\alpha\in\Gamma(\chi)} \chi^\alpha$ is a rational valued character of G. Moreover, $c(\chi)$ is a non-negative rational valued character of G and $c(\chi) = d(\chi) + m(\chi)$.

Lemma 2.3. Let $\chi \in Irr(G), \chi \neq 1_G$. Then $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$.

Lemma 2.4. Let $\chi \in Irr(G)$. Then

(1) $c(\chi)(1) \geq d(\chi) \geq \chi(1)$;

(2) $c(\chi)(1) \leq 2d(\chi)$. Equality holds if and only if $Z(\chi)/ker\chi$ is of even order.

3 Quasi-permutation representations

In this section, we calculate $r(G)$ and $c(G)$ for Borel and parabolic subgroups of Steinbergs triality groups ${}^3D_4(q)$. First we shall determine these quantities for odd q.

Theorem 3.1. Let q be a power of an odd prime number. Then

A) If G is a Borel subgroup B of ${}^3D_4(q)$, then

1) $r(G) = \begin{cases} mq^4(q-1) & \text{if } m \leq \frac{q^3-1}{2}, \\ \frac{1}{2}q^4(q^3-1)(q-1) & \text{otherwise}, \end{cases}$

2) $c(G) = \begin{cases} mq^5 & \text{if } m \leq \frac{q^3-1}{2}, \\ \frac{1}{2}q^5(q^3-1) & \text{otherwise}, \end{cases}$

where $m = |\Gamma(B\chi_{17}(k))|$.

B) If G is the maximal parabolic subgroup P of ${}^3D_4(q)$, then

1) $r(G) = q^4(q-1)$;

2) $c(G) = q^5$.

C) If G is the maximal parabolic subgroup Q of ${}^3D_4(q)$, then

1) $r(G) = \begin{cases} mq^3(q^2-1) & if \ \frac{m}{n} \leq q-1 \\ nq^3(q-1)^2(q+1), & otherwise, \end{cases}$

2) $c(G) = \begin{cases} mq^5 & if \ \frac{m}{n} \leq q-1, \\ nq^5(q-1), & otherwise; \end{cases}$

where $m = |\Gamma(Q\chi_{16}(k))|$ and $n = |\Gamma(Q\chi_{17}(k))|$.

Proof. In order to calculate $r(G)$ and $c(G)$, we need to determine $d(\chi), m(\chi)$, and $c(\chi)(1)$, for all characters which are faithful or $\bigcap_\chi Ker\chi = 1$. Since the degrees of faithful characters are minimal, we only need to consider the faithful characters and by Lemmas 2.3 and 2.4 we have

A) Using the character table $A.6$ of [9] for the Borel subgroup B, we have $d(B\chi_{17}(k)) = |\Gamma(B\chi_{17}(k))|B\chi_{17}(k)(1) \geq q^4(q-1)$ and so $c(B\chi_{17})(k)(1) \geq q^5$, $d(B\chi_{18}) = d(B\chi_{19}) = d(B\chi_{20}) = d(B\chi_{21}) = |\Gamma(B\chi_{18})|B\chi_{18}(1) = \frac{1}{2}q^4(q^3-1)(q-1)$ and so $c(B\chi_{18})(1) = c(B\chi_{19})(1) = c(B\chi_{20})(1) = c(B\chi_{21})(1) = \frac{1}{2}q^5(q^3-1)$. The values are set out in Table (I):

Table (I)

χ	$d(\chi)$	$c(\chi)(1)$
$B\chi_{17}(k)$	$\geq q^4(q-1)$	$\geq q^5$
$B\chi_{18}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$\frac{1}{2}q^5(q^3-1)$
$B\chi_{19}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$\frac{1}{2}q^5(q^3-1)$
$B\chi_{20}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$\frac{1}{2}q^5(q^3-1)$
$B\chi_{21}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$\frac{1}{2}q^5(q^3-1)$

For the character $B\chi_{17}(k)$, as $|\Gamma(B\chi_{17}(k))| \leq q^3-1$, where $\Gamma(B\chi_{17}(k)) = \Gamma(Q(B\chi_{17}(k))Q)$, we have

$$q^4(q-1) \leq d(B\chi_{17}(k)) \leq q^4(q-1)(q^3-1).$$

Now by Table (I) and the above inequality we have
$\min\ \{d(\chi):\ Ker\chi = 1\} = d(B\chi_{17}(k)) = mq^4(q-1)$ if $m \leq \frac{q^3-1}{2}$, otherwise, $\min\{d(\chi):\ Ker\chi = 1\} = \frac{1}{2}q^4(q^3-1)(q-1)$. Also
$\min\ \{c(\chi)(1):\ Ker\chi = 1\} = c(B\chi_{17}(k))(1) = mq^5$, if $m \leq \frac{q^3-1}{2}$, otherwise, $\min\{c(\chi):\ Ker\chi = 1\} = \frac{1}{2}q^5(q^3-1)$, where $m = |\Gamma(B\chi_{17}(k))|$.

B) By the character table $A.10$ of [9], we have
$d(P\chi_{15}) = |\Gamma(P\chi_{15})|P\chi_{15}(1) = q^4(q-1)$ and so $c(P\chi_{15})(1) = q^5$,
$d(P\chi_{16}) = |\Gamma(P\chi_{16})|P\chi_{16}(1) = q^7(q-1)$ and so $c(P\chi_{16})(1) = q^8$,
$d(P\chi_{17}) = |\Gamma(P\chi_{17})|P\chi_{17}(1) = \frac{1}{2}q^4(q-1)(q^3+1)$ and so $c(P\chi_{17})(1) = \frac{1}{2}q^5(q^3+1)$,
$d(P\chi_{18}) = |\Gamma(P\chi_{18})|P\chi_{18}(1) = \frac{1}{2}q^4(q-1)(q^3+1)$ and so $c(P\chi_{18})(1) = \frac{1}{2}q^5(q^3+1)$,
$d(P\chi_{19}) = |\Gamma(P\chi_{19})|P\chi_{19}(1) = \frac{1}{2}q^4(q-1)(q^3-1)$ and so $c(P\chi_{19})(1) = \frac{1}{2}q^5(q^3-1)$,
$d(P\chi_{20}) = |\Gamma(P\chi_{20})|P\chi_{20}(1) = \frac{1}{2}q^4(q-1)(q^3-1)$ and so $c(P\chi_{20})(1) = \frac{1}{2}q^5(q^3-1)$,
$d(P\chi_{21}(k)) = |\Gamma(P\chi_{21}(k))|P\chi_{21}(k)(1) \geq q^4(q-1)(q^3+1)$ and so $c(P\chi_{21})(k)(1) \geq q^5(q^3+1)$,
$d(P\chi_{22}(k)) = |\Gamma(P\chi_{22}(k))|P\chi_{22}(k)(1) \geq q^4(q-1)(q^3-1)$ and so $c(P\chi_{22})(k)(1) \geq q^5(q^3-1)$.
The values are set out in Table (II):

Table (II)

χ	$d(\chi)$	$c(\chi)(1)$
$P\chi_{15}$	$q^4(q-1)$	q^5
$P\chi_{16}$	$q^7(q-1)$	q^8
$P\chi_{17}$	$\frac{1}{2}q^4(q-1)(q^3+1)$	$\frac{1}{2}q^5(q^3+1)$
$P\chi_{18}$	$\frac{1}{2}q^4(q-1)(q^3+1)$	$\frac{1}{2}q^5(q^3+1)$
$P\chi_{19}$	$\frac{1}{2}(q^7-q^4)$	$\frac{1}{2}q^5(q^3-1)$
$P\chi_{20}$	$\frac{1}{2}q^4(q-1)(q^3-1)$	$\frac{1}{2}q^5(q^3-1)$
$P\chi_{21}(k)$	$\geq q^4(q-1)(q^3+1)$	$\geq q^5(q^3+1)$
$P\chi_{22}(k)$	$\geq q^4(q-1)(q^3-1)$	$\geq q^5(q^3-1)$

Now by Table (II) and the above inequality we have
$\min\{d(\chi) :\ Ker\chi = 1\} = d(P\chi_{15}) = q^4(q-1))$ and
$\min\{c(\chi)(1) :\ Ker\chi = 1\} = c(P\chi_{15})(1) = q^5$.

C) By the character table $A.14$ of [9], we have
$d(Q\chi_{16}(k)) = |\Gamma(Q\chi_{16}(k))|Q\chi_{16}(k)(1) \geq q^3(q^2-1)$ and so $c(Q\chi_{16}(k))(1) \geq q^5$,
$d(Q\chi_{17}(k)) = |\Gamma(Q\chi_{17}(k))|Q\chi_{17}(k)(1) \geq q^3(q^2-1)(q-1)$ and so $c(Q\chi_{17}(k))(1) \geq q^5(q-1)$,
$d(\sum_{k=0}^{1} Q\chi_{18}(k)) = |\Gamma(\sum_{k=0}^{1} Q\chi_{18}(k))|(\sum_{k=0}^{1} Q\chi_{18}(k))(1) = 2q^3(q^2-1)(q^3-1)$ and so $c(\sum_{k=0}^{1} Q\chi_{18}(k))(1) = 2q^5(q^3-1)$,
$d(\sum_{k=0}^{1} Q\chi_{19}(k)) = |\Gamma(\sum_{k=0}^{1} Q\chi_{19}(k))|(\sum_{k=0}^{1} Q\chi_{19}(k))(1) = 2q^3(q^2-1)(q^3-1)$ and so $c(\sum_{k=0}^{1} Q\chi_{19}(k))(1) = 2q^5(q^3-1)$,
$d(\sum_{k=1}^{q-1} Q\chi_{20}(k)) = |\Gamma(\sum_{k=1}^{q-1} Q\chi_{20}(k))|(\sum_{k=1}^{q-1} Q\chi_{20}(k))(1) \geq q^3(q-1)(q^2-1)(q^3-1)$ and so $c(\sum_{k=1}^{q-1} Q\chi_{20}(k))(1) \geq q^5(q-1)(q^3-1)$.

The values are set out in Table (III):

Table (III)

χ	$d(\chi)$	$c(\chi)(1)$
$Q\chi_{16}(k)$	$\geq q^3(q^2-1)$	$\geq q^5$
$Q\chi_{17}(k)$	$\geq q^3(q^2-1)(q-1)$	$\geq q^5(q-1)$
$\sum_{k=0}^{1} Q\chi_{18}(k)$	$2q^3(q^2-1)(q^3-1)$	$2q^5(q^3-1)$
$\sum_{k=0}^{1} Q\chi_{19}(k)$	$2q^3(q^2-1)(q^3-1)$	$2q^5(q^3-1)$
$\sum_{k=1}^{q-1} Q\chi_{20}(k)$	$\geq q^3(q-1)(q^2-1)(q^3-1)$	$\geq q^5(q-1)(q^3-1)$

For the character $Q\chi_{16}(k)$, as $|\Gamma(Q\chi_{16}(k))| \leq q^3-1$, where $\Gamma(Q\chi_{16}(k)) = \Gamma(Q(Q\chi_{16}(k))Q)$, we have

$$q^3(q^2-1) \leq d(Q\chi_{16}(k)) \leq q^3(q^2-1)(q^3-1).$$

Thus, for the character $Q\chi_{17}(k)$, as $|\Gamma(Q\chi_{17}(k))| \leq q^2+q+1$, where $\Gamma(Q\chi_{17}(k)) = \Gamma(Q(Q\chi_{17}(k)) : Q)$, we have

$$q^3(q-1)(q^2-1) \leq d(Q\chi_{17}(k)) \leq q^3(q^2-1)(q^3-1).$$

Now by Table (III) and the above inequality, we have
min $\{d(\chi) :\ Ker\chi = 1\} = d(Q\chi_{16}(k)) = mq^3(q^2-1)$ if $\frac{m}{n} \leq q-1$, otherwise,
$\min\{d(\chi) :\ Ker\chi = 1\} = d(Q\chi_{17}(k)) = nq^3(q^2-1)(q-1)$, and
min $\{c(\chi)(1) :\ Ker\chi = 1\} = c(Q\chi_{16}(k))(1) = mq^5$, if $\frac{m}{n} \leq q-1$, otherwise ,
$\min\{c(\chi) :\ Ker\chi = 1\} = c(Q\chi_{17}(k))(1) = nq^5(q-1)$, where $m = |\Gamma(Q\chi_{16}(k))|$ and $n = |\Gamma(Q\chi_{17}(k))|$.

In the following theorem, we have constructed the values of $r(G)$ and $c(G)$ for the case when q is even.

Theorem 3.2. **A)** Let G be the Borel subgroup B of ${}^3D_4(2^n)$, then

1) $r(G) = |\Gamma(B\chi_{15}(k))|q^4(q-1)$

2) $c(G) = |\Gamma(B\chi_{15}(k))|q^5$.

B) Let G be the maximal parabolic subgroup P of ${}^3D_4(2^n)$, then

1) $r(G) = q^4(q-1)$

2) $c(G) = q^5$.

C) Let G be the maximal parabolic subgroup Q of ${}^3D_4(2^n)$, then

1) $r(G) = \begin{cases} mq^3(q^2-1), & \text{if } \frac{m}{n} \leq q-1, \\ nq^3(q-1)^2(q+1), & \text{otherwise,} \end{cases}$

2) $c(G) = \begin{cases} mq^5, & \text{if } \frac{m}{n} \leq q-1, \\ nq^5(q-1) & \text{otherwise,} \end{cases}$

where $m = |\Gamma(Q\chi_{14}(k))|$ and $n = |\Gamma(Q\chi_{15}(k))|$.

Proof. The quasi-permutation representations of Borel subgroup B and maximal parabolic subgroups P and Q of ${}^3D_4(2^n)$ are constructed by the same method as in Theorem 3.1. So in order to calculate $r(G)$ and $c(G)$, we need to determine $d(\chi), m(\chi)$, and $c(\chi)(1)$, for all characters which are faithful or $\bigcap_\chi Ker\chi = 1$. Since the degrees of faithful characters are minimal, we only need to consider the faithful characters. By Lemmas 2.3, 2.4, and the character table $A.6$ of [8], we have

A) $d(B\chi_{15}(k)) = |\Gamma(B\chi_{15}(k))|B\chi_{15}(k)(1) \geq q^4(q-1)$ and so $c(B\chi_{15})(k)(1) \geq q^5$, $d(B\chi_{16}) = |\Gamma(B\chi_{16})|B\chi_{16}(1) = q^4(q^3-1)(q-1)$ and so $c(B\chi_{16})(1) = q^5(q^3-1)$. For the character $B\chi_{15}(k)$, as $|\Gamma(B\chi_{15}(k))| \leq q^3-1$, where $\Gamma(B\chi_{15}(k)) = \Gamma(Q(B\chi_{15}(k)) : Q)$, we have

$$q^4(q-1) \leq d(B\chi_{15}(k)) \leq q^4(q-1)(q^3-1).$$

Now, we have

$\min\{d(\chi) :\ Ker\chi = 1\} = d(B\chi_{15}(k)) = |\Gamma(B\chi_{15}(k))|q^4(q-1)$ and $\min\{c(\chi)(1) : Ker\chi = 1\} = c(B\chi_{15}(k))(1) = |\Gamma(B\chi_{15}(k))|q^5$.

B) By the character table $A.10$ of [8] we obtain

$d(P\chi_{15}) = |\Gamma(P\chi_{15})|P\chi_{15}(1) = q^4(q-1)$ and so $c(P\chi_{15})(1) = q^5$,

$d(P\chi_{16}) = |\Gamma(P\chi_{16})|P\chi_{16}(1) = q^7(q-1)$ and so $c(P\chi_{16})(1) = q^8$,

$d(P\chi_{17}) = |\Gamma(P\chi_{17})|P\chi_{17}(1) \geq q^4(q-1)(q^3+1)$ and so $c(P\chi_{17})(1) \geq q^5(q^3+1)$,

$d(P\chi_{18}) = |\Gamma(P\chi_{18})|P\chi_{18}(1) \geq q^4(q-1)(q^3-1)$ and so $c(P\chi_{18})(1) \geq q^5(q^3-1)$

The values are set out in Table (IV):

Table (IV)

χ	$d(\chi)$	$c(\chi)(1)$
$P\chi_{15}$	$q^4(q-1)$	q^5
$P\chi_{16}$	$q^7(q-1)$	q^8
$P\chi_{17}$	$\geq q^4(q-1)(q^3+1)$	$\geq q^5(q^3+1)$
$P\chi_{18}$	$\geq q^4(q-1)(q^3-1)$	$\geq q^5(q^3-1)$

Now by Table (IV) we have
$\min\{d(\chi):\ Ker\chi=1\}=d(P\chi_{15})=q^4(q-1))$ and
$\min\{c(\chi)(1):\ Ker\chi=1\}=c(P\chi_{15})(1)=q^5$.
C) By the character table $A.14$ of [8] we may calculate the following
$d(Q\chi_{14}(k))=|\Gamma(Q\chi_{14}(k))|Q\chi_{14}(k)(1)\geq q^3(q^2-1)$ and so $c(Q\chi_{14}(k))(1)\geq q^5$,
$d(Q\chi_{15}(k))=|\Gamma(Q\chi_{15}(k))|Q\chi_{15}(k)(1)\geq q^3(q-1)^2(q+1)$ and so $c(Q\chi_{15}(k))(1)\geq q^5(q-1)$ and
$d(\sum_{k=1}^{q}Q\chi_{16}(k))=|\Gamma(\sum_{k=1}^{q}Q\chi_{16}(k))|(\sum_{k=1}^{q}Q\chi_{16}(k))(1)=q^4(q-1)(q+1)(q^3-1)$ so $c(\sum_{k=1}^{q}Q\chi_{16}(k))(1)=q^6(q-1)(q^2+q+1)$.
For the character $Q\chi_{14}(k)$, as $|\Gamma(Q\chi_{14}(k))|\leq q^3-1$, where $\Gamma(Q\chi_{14}(k))=\Gamma(Q(Q\chi_{14}(k)):Q)$, we have

$$q^3(q^2-1)\leq d(Q\chi_{14}(k))\leq q^3(q^2-1)(q^3-1).$$

So for the character $Q\chi_{15}(k)$, as $|\Gamma(Q\chi_{15}(k))|\leq q^2+q+1$, where $\Gamma(Q\chi_{15}(k))=\Gamma(Q(Q\chi_{15}(k)):Q)$, we have

$$q^3(q-1)(q^2-1)\leq d(Q\chi_{15}(k))\leq q^3(q^2-1)(q^3-1).$$

Now by the above inequality we have
$\min\{d(\chi):\ Ker\chi=1\}=d(Q\chi_{14}(k))=mq^3(q^2-1)$ if $\frac{m}{n}\leq q-1$, otherwise, $\min\{d(\chi):\ Ker\chi=1\}=d(Q\chi_{15}(k))=nq^3(q^2-1)(q-1)$. and
$\min\{c(\chi)(1):\ Ker\chi=1\}=c(Q\chi_{14}(k))(1)=mq^5$, if $\frac{m}{n}\leq q-1$, otherwise , $\min\{c(\chi):\ Ker\chi=1\}=c(Q\chi_{15}(k))(1)=nq^5(q-1)$, where $m=|\Gamma(Q\chi_{14}(k))|$ and $n=|\Gamma(Q\chi_{15}(k))|$.

References

[1] Behravesh, H., Quasi-Permutation representations of p-groups of class 2, *Journal of London Math. Soc.* **55**(2)(1997), 251–260.

[2] Behravesh, H., Daneshkhah, A., Darafsheh, M.R. and Ghorbany, M., The rational character table and quasi-permutation representations of the group $PGL(2, q)$, *Italian Journal of Pure and Applied Mathematics* **11**(2001), 9–18.

[3] Darafsheh, M.R., Ghorbany, M., Daneshkhah, A. and Behravesh, H., Quasi-permutation representation of the group $GL(2, q)$, *Journal of Algebra* **243**(2001), 142–167.

[4] Darafsheh, M.R. and Ghorbany, M., Quasi-permutation representations of the groups $SU(3, q^2)$ and $PSU(3, q^2)$, *Southeast Asian Bulletin of Mathematics* **26**(2002), 395–406.

[5] Darafsheh, M.R. and Ghorbany, M., Special representations of the group $SP(4, q)$ with minimal degrees, *Acta Math. Hungar.* **102**(4)(2004), 287–296.

[6] Ghorbany, M., Special representations of the group $G_2(2^n)$ with minimal degrees, *Southeast Asian Bulletin of Mathematics* **30**(2006), 663–670.

[7] Gow, R., Schur indices of some groups of Lie type, *Journal of Algebra* **42**(1976), 102–120.

[8] Himstedt,F., Character tables of parabolic subgroups of Steinberg's triality groups ${}^3D_4(2^n)$, *Journal of Algebra* **281**(1)(2007), 254-283.

[9] Himstedt, F., Character tables of parabolic subgroups of Steinberg's triality groups, *Journal of Algebra* **281**(2004), 774–822.

[10] Isaacs, I.M., Character Theory of Finite Groups, Academic Press, New York, 1976.

[11] Wong, W.J., Linear groups analogous to permutation groups, *Journal of Austral. Math. Soc (Sec. A)* **3**(1963), 180–184.