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Research Article

A fourth-order optimal numerical approximation and its convergence for singularly perturbed time delayed parabolic problems

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Abstract

This paper presents a numerical solution for a time delay parabolic problem (reaction-diffusion) containing a small parameter. The numerical method combines the implicit Crank–Nicolson scheme for the time derivative on the uniform mesh and the central difference scheme for the spatial derivative on the Shishkin-type meshes. It is shown to be second-order uniformly convergent in time and space. Then Richardson extrapolation technique is applied to enhance the accuracy from second-order to fourth-order. The error analysis is carried out, and the method is proved to be uniformly convergent. These two methods are applied to two test examples in support of the theoretical results.

AMS subject classifications (2020): 65M06, 65M12.

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1 Introduction

In singularly perturbed delay partial differential equations (SPPDEs), a small parameter affects the highest derivative of partial differential equations (PDEs) and also contains more than one delay term in the time direction. These SPPDEs have existence in subjects like physical/chemical science, biology, and ecology. Time delay (large) diffusion problems (the present time depends upon the past) appear in mathematical models in population dynamics [13, 26, 17] and also in biological modeling [29].

Consider the following time delay reaction-diffusion initial-boundary-value problems (IBVPs):

$$\begin{cases} \left(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,s}\right) z(y,\ s) = -\mathfrak{b}(y,\ s) z(y,\ s-\gamma) + \mathfrak{f}(y,s), & (y,s) \in \mathfrak{D}, \\ z(y,s) = \Theta_b(y,\ s), \quad (y,s) \in \mathfrak{d}_b, & \\ z(0,s) = \Theta_l(s), \quad \text{on} \quad \mathfrak{d}_l = \{(0,s) : 0 \le s \le \mathcal{S}\}, & \\ z(1,s) = \Theta_r(s), \quad \text{on} \quad \mathfrak{d}_r = \{(1,s) : 0 \le s \le \mathcal{S}\}, & \end{cases} \tag{1}$$

where $L_{\varepsilon,y} z(y,\ s) = -\varepsilon z_{yy}(y,\ s) + \mathfrak{p}(y) z(y,\ s)$. Here $= (0,\ 1)$, $\mathfrak{D} = \Pi \times (0,\ \mathcal{S}]$, $\mathfrak{d} = \mathfrak{d}_l \cup\ \mathfrak{d}_b \cup \mathfrak{d}_r$. Moreover, $\gamma > 0$ is a given constant. This model (1) is singularly perturbed, parabolic in nature with $0 < \varepsilon \ll 1$. Moreover, $\mathfrak{d}_r$ and $\mathfrak{d}_l$ are the right and the left sides of the domain $\mathfrak{D}$, and $\mathfrak{d}_b = [0,\ 1] \times [-\gamma,\ 0]$. The functions $\mathfrak{b}(y,\ s)$ $(\mathfrak{b}(y,\ s) > 0)$, $\mathfrak{p}(y)(\mathfrak{p}(y) \ge \beta > 0)$, the source term $\mathfrak{f}(y,\ s)$ on $\overline{\mathfrak{D}}$ and $\Theta_l(s)$, $\Theta_r(s)$, $\Theta_b(y,\ s)$ on $\mathfrak{d}$, are bounded, sufficiently smooth functions. The assumption on the time $\mathcal{S}$ is $\mathcal{S} = m\gamma$ for $m \in \mathcal{N}$. We also assume that suitable compatibility conditions hold true on the given boundary and initial data to ensure a unique solution for (1) that exhibits boundary layers along with end points of y [1, 19]. These conditions are $\Theta_b(0,0) = \Theta_l(0)$, $\Theta_b(1,0) = \Theta_r(0)$, and

$$\frac{d\Theta_l(0)}{ds} - \varepsilon \frac{\partial^2 \Theta_b(0,\ 0)}{\partial y^2} + \mathfrak{p}(0)\Theta_b(0,\ 0) = -\mathfrak{b}(0,0)\Theta_b(0,\ -\gamma) + \mathfrak{f}(0,\ 0),$$
$$\frac{d\Theta_r(0)}{ds} - \varepsilon \frac{\partial^2 \Theta_b(1,\ 0)}{\partial y^2} + \mathfrak{p}(1)\Theta_b(1,\ 0) = -\mathfrak{b}(1,\ 0)\Theta_b(1,\ -\gamma) + \mathfrak{f}(1,\ 0).$$

Due to the small parameter ε involved in the problem (1), the classical methods on equal step length for solving (1) fail to give accurate results. They are mostly unstable and unacceptable [22, 27]. Hence, we choose the fitted mesh idea through the nonuniform mesh as given in [14, 20, 22, 25, 27] and the references therein. One can refer [1, 7, 10, 9, 11, 18, 16, 23] for some order enhancing methods of IBVPs. Some articles are available, which discuss both the analytical and the numerical techniques for SPPDEs in literature. Ansari, Bakr, and Shishkin [1] numerically solved the problem (1) on the Shishkin mesh. Das and Natesan [6] gave details of numerical results for the time delay parabolic convection diffusion problem. Indeed, the methods discussed above using numerical techniques are of first- or second-order accurate. Hence, order enhancing techniques are needed for the problem (1), which is the main aim of this work.

Richardson extrapolation through averaging the numerical solutions computed on two embedded meshes provides a good approximation to the exact solution. This helps to increase the accuracy and the order of convergence. Mohapatra and Natesan [21] used the extrapolation technique for solving singularly perturbed delay BVPs while Shishkin and Shishkina [28] applied on time dependent IBVPs of reaction-diffusion type. This technique is used in [5] for convection-diffusion singularly perturbed parabolic problems on the adaptive mesh. This article aims to get a fourth-order accurate solution for (1) using the extrapolation technique. Initially, the central difference scheme is used on Shishkin-type meshes and the Crank–Nicolson method on temporal direction on uniform mesh. Here, problem (1) is solved with M number of mesh points in spatial and N number of mesh points in time direction. After that (1) is solved by the above methods with $2M$ and $2N$ number of mesh points. Then the rate of convergence increases from second- to fourth-order globally by taking a proper combination of these two solutions.

The rest portion is arranged as follows. In Section 2, we describe the continuous solution to the problem. Section 3 studies the construction of the numerical schemes. In Section 4, we implement the post-process ideas and the theoretical analysis. Section 5 presents the numerical results through plots and tables. We denote $\mathcal{C}$ and the subscripted $\mathcal{C}$'s as constants, which are positive, independent of the small parameter (ε) and spatial and time mesh sizes. The error is represented in the supremum norm ($\|\cdot\|_\infty$). It is defined as $||\mathfrak{h}||_\infty = \sup_{(y,s)\in\mathfrak{D}} |\mathfrak{h}(y,s)|$ for any function $\mathfrak{h}$ on the domain $\mathfrak{D}$.

2 Analytic solution and its behavior

As the perturbed parameter $\varepsilon \to 0$ in (1), the problem reduces as given below:

$$\begin{cases} \dfrac{\partial z_0(y,\ s)}{\partial s} + \mathfrak{p}(y) z_0(y,\ s) = -\mathfrak{b}(y,\ s) z_0(y, s-\gamma) + \mathfrak{f}(y,\ s), & (y,\ s) \in \mathfrak{D}, \\ z_0(y,\ s) = \theta_b(y,\ s), & (y,\ s) \in \mathfrak{d}_b. \end{cases} \tag{2}$$

It is clear that the solution to (1) has boundary layers on $\mathfrak{d}_l$ and $\mathfrak{d}_r$. The characteristics of (2) are the vertical lines $y = C$, which implies that boundary layers arising in the solution are parabolic type. The operator $\left(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\right)$ in (1) satisfies the following lemma known as the maximum principle.

Lemma 1. Suppose that $\Psi(y,\ s) \in \mathcal{C}^0(\overline{\mathfrak{D}}) \cap \mathcal{C}^2(\mathfrak{D})$ satisfies $\left(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\right)\Psi(y,\ s) \geq 0$ in $\mathfrak{D}$ and that $\Psi(y,\ s) \geq 0$ on $\mathfrak{d}$. Then $\Psi(y,\ s) \geq 0$ for all $(y,\ s) \in \overline{\mathfrak{D}}$.

Proof. The proof of this lemma is available in [1]. □

2.1 Solution decomposition

The continuous solution $z(y,\ s)$ to (1) is decomposed as $z = v_r + v_s$. The regular component v_r expresses as $v_r(y,\ s) = v_{r0}(y,\ s)+\varepsilon v_{r1}(y,\ s),\ (y,\ s) \in \overline{\mathfrak{D}}$, where v_{r0} and v_{r1} are the solutions to the following problem:

$$\begin{cases} (v_{r0})_s(y,\ s) + \mathfrak{p}(y)v_{r0}(y,\ s) = -\mathfrak{b}(y,\ s)v_{r0}(y, s-\gamma) + f(y,\ s),\ (y,\ s) \in \mathfrak{D}, \\ v_{r0}(y,\ s) = \Theta_b(y,\ s),\ (y,\ s) \in \mathfrak{d}_b, \\ \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big)v_{r1}(y,\ s) = -\mathfrak{b}(y,\ s)v_{r1}(y, s-\gamma) + (v_{r0})_{yy},\ (y,\ s) \in \mathfrak{D}, \\ v_{r1}(y,\ s) = 0, \quad (y,\ s) \in \Gamma. \end{cases} \tag{3}$$

The regular component $v_r(y,\ s)$ satisfies the following problem:

$$\begin{cases} \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big)v_r(y,\ s) = -\mathfrak{b}(y,\ s)v(y, s-\tau) + \mathfrak{f}(y,\ s), & (y,\ s) \in \mathfrak{D}, \\ v_r(y,\ s) = \theta_b(y,\ s), & (y,\ s) \in \mathfrak{d}_b, \\ v_r(0,t) = v_{r0}(0,\ s), & \text{on} \quad \mathfrak{d}_l = \{(0,\ s) : 0 \le s \le \mathcal{S}\}, \\ v_r(1,t) = v_{r0}(1,\ s), & \text{on} \quad \mathfrak{d}_r = \{(1,\ s) : 0 \le s \le \mathcal{S}\}, \end{cases} \tag{4}$$

and the singular component $v_s(y,\ s)$ satisfies the PDE

$$\begin{cases} \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big)v_s(y,\ s) = -\mathfrak{b}(y,\ s)v_s(y, s-\gamma), & (y,\ s) \in \mathfrak{D}, \\ v_s(y,\ s) = 0, & (y,\ s) \in \Gamma_b, \\ v_s(0,\ s) = \Theta_l - v_{r0}(0,\ s), & \text{on} \quad \mathfrak{d}_l = \{(0,\ s) : 0 \le s \le \mathcal{S}\}, \\ v_s(1,\ s) = \Theta_r - v_{r0}(1,\ s), & \text{on} \quad \mathfrak{d}_r = \{(1,\ s) : 0 \le s \le \mathcal{S}\}. \end{cases} \tag{5}$$

Now, we can write $v_s = v_{sl} + v_{sr}$, where v_{sl} is the boundary layer part on $\mathfrak{d}_l$ and v_{sr} is the boundary layer part on $\mathfrak{d}_r$. Here v_{sl} and v_{sr} are defined by

$$\begin{cases} \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big)v_{sl}(y,\ s) = -\mathfrak{b}(y,\ s)v_{sl}(y,\ s-\gamma),\ \ (y,\ s) \in \mathfrak{D}, \\ v_{sl}(0,\ s) = \theta_l - v_{r0}(0,\ s)\ (y,\ s) \in \mathfrak{d}_l\ v_{sl}(y,\ s) = 0,\ (y,\ s) \in \mathfrak{d}_b \bigcup\ \mathfrak{d}_r, \\ \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big)v_{sr}(y,\ s) = -\mathfrak{b}(y,\ s)v_{sr}(y,\ s-\gamma),\ \ (y,\ s) \in \mathfrak{D}, \\ v_{sr}(1,\ s) = \Theta_r - v_{r0}(1,\ s),\ (y,s) \in \mathfrak{d}_r\ v_{sr}(y,s) = 0,\ (y,s) \in \mathfrak{d}_l \bigcup \mathfrak{d}_b. \end{cases} \tag{6}$$

Theorem 1. For all integers $l > 0$ and $m > 0$ with $0 \le l + 2m \le 8$, v_r and v_s, defined in (3) and (5), respectively, satisfy

$$\left\|\frac{\partial^{l+m} v_s}{\partial y^l \partial s^m}\right\|_\infty \le \mathcal{C}(1+\varepsilon^{2-l/2}), \quad (y,\ s) \in \mathfrak{D},$$

and

$$\left\|\frac{\partial^{l+m} v_r}{\partial y^l \partial s^m}\right\|_\infty \le \mathcal{C}\varepsilon^{-l/2}\Big(\exp(-y\sqrt{\beta/\varepsilon}) + \exp(-(1-y)\sqrt{\beta/\varepsilon})\Big), \quad (y,\ s) \in \mathfrak{D},$$

$$\left\|\frac{\partial^{l+m} v_{sl}}{\partial y^l \partial s^m}\right\|_\infty \le \mathcal{C}\varepsilon^{-l/2}\Big(\exp(-y\sqrt{\beta/\varepsilon}))\Big), \quad (y,\ s) \in \mathfrak{D},$$

$$\left\|\frac{\partial^{l+m} v_{sr}}{\partial y^l \partial s^m}\right\|_\infty \le \mathcal{C}\varepsilon^{-l/2}\Big(\exp(-(1-y)\sqrt{\beta/\varepsilon})\Big), \quad (y,\ s) \in \mathfrak{D}.$$

Proof. One may refer [1] for the details. □

3 Discretization

On $[0,\ \mathcal{S}]$, the uniform time step Δs is used in the time direction such that $\Pi_s^N = \{s_n = n\Delta s, \quad n = 0, \dots, N, \quad s_N = \mathcal{S}, \quad \Delta s = \mathcal{S}/N\}$, $\Pi_s^p = \{s_j = j\Delta s, \quad j = 0, \dots, p, \quad s_p = \gamma, \quad \Delta s = \gamma/p\}$. Here, p represents the number of mesh points in $[-\gamma, 0]$ and N denotes the number of mesh points in temporal direction on the interval $[0,\ \mathcal{S}]$. The step size (Δs) satisfies $p\Delta s = \gamma$, where $p > 0$ is an integer $s_n = n\Delta s$, $n \ge -p$.

3.1 Discretization of the spatial domain

Let $\xi = \min\left\{\frac{1}{4},\ \rho_0\sqrt{\varepsilon}\ln M\right\}$, where $\rho_0 \ge 2/\beta$ is a mesh transition parameter. We divide $\overline{\Pi} = [0, 1]$ into three subdomains as $\overline{\Pi} = \overline{\Pi_l} \cup \overline{\Pi_c} \cup \overline{\Pi_r}$, where $\Pi_l = [0, \xi], \Pi_c = (\xi, 1-\xi]$, and $\Pi_r = (1-\xi, 1]$. Without loss of generality, we assume that M is even and that $M \ge 8$.

Shishkin mesh(S-mesh)

One can find the construction of the S-mesh in [22, 27] briefly described as follows: let us define S-mesh as $\Pi_y^M = \{y_i \in [0, 1], 0 \le i \le M\}$, where

$$y_i = \begin{cases} \dfrac{4i\xi}{M} & \text{for} \quad 0 \le i \le M/4, \\ \dfrac{2i(1-2\xi)}{M} & \text{for} \quad \dfrac{M}{4}+1 \le i \le \dfrac{3M}{4}, \\ \dfrac{4i\xi}{M} & \text{for} \quad \dfrac{3M}{4}+1 \le i \le M. \end{cases}$$

If $\xi = \dfrac{1}{4}$, then the mesh has equal step length and otherwise when $\xi = \rho_0\sqrt{\varepsilon}\ln M$, the mesh is changing at near the end of $\mathfrak{d}_l$ and $\mathfrak{d}_r$, where $y_i - y_{i-1} = 4\xi M^{-1}$. Therefore, the mesh is piecewise uniform.

Bakhvalov-Shishkin mesh (B-S-mesh)

The idea of constructing the B-S mesh is available in [4, 27]. The mesh is constructed as follows:

$$y_i = \begin{cases} \dfrac{2}{\beta}\Bigg(-\ln\Big(1 - 2(1-\tfrac{1}{M})\tfrac{i}{M}\Big)\Bigg), & \text{for} \quad 0 \le i \le M/4-1, \\ y_{M/4-1} + \Big(\dfrac{y_{M/4+1} - y_{M/4-1}}{M/2+2}\Big)\big(i - M/4+1\big) & \text{for} \quad \dfrac{M}{4} \le i \le \dfrac{3M}{4}, \\ 1 - \dfrac{2}{\beta}\Bigg(-\ln\Big(1 - 2(1-\tfrac{1}{M})\tfrac{M-i}{M}\Big)\Bigg) & \text{for} \quad \dfrac{3M}{4}+1 \le i \le M. \end{cases}$$

We define the numerical domain $\mathfrak{D}^M = \Pi_y^M \times \Pi_s^N$ on $\mathfrak{D}$ and $\mathfrak{d}^M = \Pi_y^M \times \Pi_s^p$ on $\mathfrak{d}$.

3.2 Semidiscretization

The Crank–Nicolson scheme for the time variable of (1) is given by

$$\begin{cases} z^{-j} = \Theta_b(y, -s_j) \qquad \text{for} \quad j = 0, \dots p, \quad y \in \overline{\mathfrak{D}}, \\ \Big(I + \frac{\Delta s}{2}\mathfrak{L}_{\varepsilon,y}\Big) z^{n+1} = \frac{\Delta s}{2}\Big(- \mathfrak{b}^{n+1} z^{n-p+1} - \mathfrak{b}^n z^{n-p} + \mathfrak{f}^{n+1} + \mathfrak{f}^n\Big) + \Big(I - \frac{\Delta s}{2}\mathfrak{L}_{\varepsilon,y}\Big) z^n, \\ z^{n+1}(0) = \Theta_l(s_{n+1}), \quad z^{n+1}(1) = \Theta_r(s_{n+1}), \end{cases} \tag{7}$$

where $\mathfrak{f}^n = \mathfrak{f}(y, t_n)$, $\mathfrak{b}^n = \mathfrak{b}(y,\ s_n)$, $z^n = z(y,\ s_n)$ is the semidiscrete approximation to $z(y,\ s)$ of (1) at $s_n = n\Delta s$. Let $e^{n+1} = z^{n+1} - \widetilde{z}^{n+1}$ be the local truncation error of (7) and let $\widetilde{z}^{n+1}$ be the solution to

$$\begin{cases} \widetilde{z}^{-j} = \Theta_b(y, -t_j) \qquad \text{for} \quad j = 0, \dots p, \quad x \in \overline{\mathfrak{D}}, \\ \Big(I + \frac{\Delta s}{2}\mathfrak{L}_{\varepsilon,y}\Big) \widetilde{z}^{n+1} = \frac{\Delta s}{2}\Big(- \mathfrak{b}^{n+1} \widetilde{z}^{n-p+1} - \mathfrak{b}^n \widetilde{z}^{n-p} + \mathfrak{f}^{n+1} + \mathfrak{f}^n\Big) + \Big(I - \frac{\Delta s}{2}\mathfrak{L}_{\varepsilon,y}\Big) (\widetilde{z}^2)^n, \\ \widetilde{z}^{n+1}(0) = \Theta_l(s_{n+1}), \quad \widetilde{z}^{n+1}(1) = \Theta_r(s_{n+1}). \end{cases}$$

Hence, the global error at s^n is given by $E^n = z(y,s^n) - z^n(y)$.

3.3 Bounds on the solution and its derivatives

Consider the global error $E^{n+1} = e^{n+1} + RE^n$, associated with the scheme (7). The transition operator

$$R = \left(I + \frac{\Delta s}{2}\mathfrak{L}_{\varepsilon,y}\right)^{-1}\left(I - \frac{\Delta s}{2}\mathfrak{L}_{\varepsilon,y}\right),$$

is defined as follows: set $z^n = E_n$ as the initial data with null boundary condition and zero source term $\mathfrak{f}$. After one time step of (7), let RE^n be the solution obtained. Using this, we have $E^{n+1} = \sum_{k=0}^{M} R^{n-k}e_{k+1}$. Thus, we claim

$$||R^j||_\infty \leq \mathcal{C} \quad \text{for all} \quad j = 0,1,\ldots,n. \tag{8}$$

Then it follows that $\sup_{n\Delta s\leq \mathcal{S}} ||E^{n+1}||_\infty \leq \mathcal{C}(\Delta s)^2$. Hence, the scheme (7) is second-order accurate. It may be noted here that (8) is a stability condition. The details of this argument were given in [3, 8].

Lemma 2. If $\left|\frac{\partial^i}{\partial s^i}z(y,\ s)\right| \leq \mathcal{C}$ for all $(y,\ s) \in \overline{\mathfrak{D}}$, for $0 \leq i \leq 3$, then the local error with the method (7) satisfies

$$\|e^{n+1}\| \leq \mathcal{C}(\Delta s)^3. \tag{9}$$

Proof. It can be shown using the same argument discussed in [3]. □

Theorem 2. The global error estimate E_n associated with (7) is given by $\|E^n\|_\infty \leq \mathcal{C}(\Delta s)^2$ for all $n \leq \mathcal{S}/\Delta s$.

Proof. See [8]. □

Theorem 3. The derivatives of $z(y,\ s)$ satisfy the bounds

$$\left|\frac{\partial^{l+m}z}{\partial y^l\partial s^m}\right| \leq \mathcal{C}\varepsilon^{-l/2}, \quad (y,\ s) \in \mathfrak{D} \quad \text{for all } l > 0, m > 0 \text{ with } 0 \leq l + 2m \leq 8.$$

Proof. Refer to [1] for the details. □

3.4 Totally finite difference scheme

The second-order approximation for the operator is given by

$$D_y^+ D_y^- \omega_j^n = \frac{2}{h_j + h_{j+1}} \left(\frac{\omega_{j+1}^n - \omega_j^n}{h_{j+1}} - \frac{\omega_j^n - \omega_{j-1}^n}{h_j} \right).$$

In time, the backward difference scheme is $D_s^- \omega_j^n = \dfrac{\omega_j^n - \omega_j^{n-1}}{\Delta s}$, where $\omega_j^n = \omega(y_j, s_n)$. To solve (1), the Crank–Nicolson scheme for the time scale and the central difference scheme for the space are combined as

$$2D_t^- Z_i^{n+1} + \mathfrak{L}_\varepsilon Z_i^{n+1} = -\mathfrak{b}_i^{n+1} Z_i^{n-p+1} - \mathfrak{b}_i^n Z_i^{n-p} + \mathfrak{f}_i^n + \mathfrak{f}_i^{n+1} - \mathfrak{L}_\varepsilon Z_i^n \quad \text{for} \quad 1 \le i < M. \tag{10}$$

Here,

$$\mathfrak{L}_\varepsilon Z_i^n = -\varepsilon D_y^+ D_y^- Z_i^n + \mathfrak{p}_i Z_i^n, \quad \mathfrak{f}_i^n = \mathfrak{f}(y_i, s_n), \quad \mathfrak{b}_i^n = \mathfrak{b}(y_i, s_n), \quad \mathfrak{p}_i = \mathfrak{p}(y_i).$$

After rearranging the terms in (10) and combining (7), the fully discrete scheme is

$$\begin{cases} \dfrac{1}{2}\Big(r_i^- Z_{i-1}^{n+1} + r_i^o Z_i^{n+1} + r_i^+ Z_{i+1}^{n+1} \Big) = g_i^M, \quad 1 \le i < M, \\ Z_0^{n+1} = \Theta_l(s_{n+1}), \quad Z_N^{n+1} = \Theta_r(s_{n+1}), \\ Z_i^{-j} = \Theta_b(y_i, -s_j) \quad \text{for} \quad j = 0, \ldots, p \quad \text{and} \quad i = 1 \le i < M, \end{cases} \tag{11}$$

$$r_i^- = \Delta s\Big(-\frac{2\varepsilon}{\widehat{h}_i h_i} \Big),\ r_i^o = \Delta s\Big(\frac{2\varepsilon}{h_{i+1} h_i} + \mathfrak{b}_i^{n+1} \Big) + 1,\ r_i^+ = \Delta s\Big(-\frac{2\varepsilon}{\widehat{h}_i h_{i+1}} \Big),$$

$$g_i^M = \frac{\Delta s}{2}\Big(-\mathfrak{b}_i^{n+1} Z_i^{n-p+1} - \mathfrak{b}_i^n Z_i^{n-p} + \mathfrak{f}_i^{n+1} + \mathfrak{f}_i^n \Big) + \Big(1 - \frac{\Delta s}{2} \mathfrak{L}_\varepsilon \Big) Z_i^n \quad \text{for} \quad 0 < i \le M-1.$$

The difference equation (11) at $n+1$ time level forms a tridiagonal system of $M-1$ equations with the same number of unknowns. This system has properties like $r_i^- < 0$, $r_i^o > 0$, $r_i^+ < 0$ for $1 \le i \le M-1$. The Thomas algorithm [24] is used to solve the tridiagonal system.

3.5 Error analysis

Theorem 4. Let z be the analytical solution to (1) and let Z be the numerical solution (11). Then on S-mesh, the error of the scheme (11) satisfies the following estimate:

$$\max_{i,n}\left|(z-Z)(y_i,\ s_n)\right| \leq \mathcal{C}\Big((M^{-1}\ln M)^2+\Delta s^2\Big) \quad \text{for} \quad i=1,\dots,M-1.$$

On B-S-mesh, it satisfies

$$\max_{i,n}\left|(z-Z)(y_i,\ s_n)\right| \leq \mathcal{C}\Big(M^{-2}+\Delta s^2\Big) \quad \text{for} \quad i=1,\dots,M-1,$$

where $Z(y_i,s_n)=Z_i^n$ for $(y_i,s_n)\in\mathfrak{D}^M$.

Proof. The proof is divided into various steps for each time level. On the first interval $s\in[0,\gamma]$ (the time discretization n varies from 0 to p), the term $\mathfrak{f}(y,\ s)-\mathfrak{c}(y,\ s)\Theta_b(s,s-\gamma)$ in the right side of (1) is independent of ε. Now since $(y_i,\ s_n)\in\mathfrak{D}_1^M=\Pi_y^M\times[0,\gamma)$, using the convergence results given in [3] and Theorem 2, we obtain on S-mesh

$$\max_{i,n}\left|(z-Z)(y_i,\ s_n)\right| \leq \mathcal{C}\Big(M^{-2}\ln^2 M+(\Delta s)^2\Big) \quad \text{for} \quad 0<i<M.$$

Following a similar procedure done in [31, 30] for the error bounds on B-S mesh, we consider the mesh generating function for B-S mesh $\varsigma(\xi)=(2/\beta)\xi\chi(s)$, where $\chi(s)=-\ln\Big(1-2(1-\frac{1}{M})(s)\Big)$. The mesh generating function satisfies $\varsigma(\xi)'\leq\mathcal{C}M$, and assume that $\varepsilon\leq M^{-1}$. The spatial mesh size h_i on the layer region satisfies $h_i\leq\mathcal{C}M^{-1}$ and $h_i\leq\mathcal{C}\varepsilon$, for $i=0,1,\dots,(M/4)-1$ and for $i=(3M/4)+1,\dots,M$. Now using the bounds on the derivative of z given Theorem (3), we get on B-S-mesh,

$$\max_{i,n}\left|(z-Z)(y_i,\ s_n)\right| \leq \mathcal{C}\Big(M^{-2}+\Delta s^2\Big) \quad \text{for} \quad i=1,\dots,M-1,$$

Now, the term $z(y,\ s)$ depends on $z(y,\ s-\gamma)$, which is unknown for $s\geq\gamma$. So, the above process is not applicable for $s\geq\gamma$. To get the error over the interval $[\gamma,\ 2\gamma]$, using the convergence results in [1] and Theorem 2, we obtain the desired bound. □

4 Post processing technique

To enhance the order for the difference scheme (10), we use the Richardson extrapolation technique. First, we solve the discrete problems (11) on the fine mesh $\mathfrak{D}^{2M}=\overline{\Pi}_y^{2M}\times\overline{\Pi}_s^{2N}$ with $2M$ and $2N$ mesh intervals in the spatial and time direction respectively, where $\overline{\Pi}_y^{2M}$ is the Shishkin-type mesh and is obtained by halving each mesh interval of $\overline{\Pi}_y^{M}$ with a fixed transition parameter. Clearly, $\mathfrak{D}^M=\{(y_i,\ s_n)\}\subset\mathfrak{D}^{2M}=\{(\overline{y}_i,\overline{s}_n)\}$. Therefore, the corresponding S-mesh is $\overline{\Pi}_y^{2M}=\{\overline{y}_i\in(0,1),0\leq i\leq 2M\}$ by

$$\overline{y}_i = \begin{cases} \dfrac{2i\xi}{M} & \text{for } 0 \le i \le M/2, \\ \dfrac{i(1-2\xi)}{M} & \text{for } \dfrac{M}{2}+1 \le i \le \dfrac{3M}{2}, \\ \dfrac{2i\xi}{M} & \text{for } \dfrac{3M}{2}+1 \le i \le 2M, \end{cases}$$

and the B-S-mesh is

$$\overline{y}_i = \begin{cases} \dfrac{2}{\beta}\Bigg(-\ln\Big(1-2(1-\frac{1}{2M})\frac{i}{2M}\Big)\Bigg) & \text{for} \quad 0 \le i \le M/2-1, \\ \overline{y}_{M/2-1} + \Big(\dfrac{\overline{y}_{M/2+1}-\overline{y}_{M/2-1}}{M+2}\Big)\big(i-M/2+1\big) & \text{for} \quad \dfrac{M}{2} \le i \le \dfrac{3M}{2}, \\ 1-\dfrac{2}{\beta}\Bigg(-\ln\Big(1-2(1-\frac{1}{2M})\frac{2M-i}{2M}\Big)\Bigg) & \text{for} \quad i \le \dfrac{3M}{2}+1 \le 2M, \end{cases}$$

and $\overline{s}_n - \overline{s}_{n-1} = \Delta s/2$ for $\overline{s_n} \in \overline{\Pi}_s^{2N}$. Now, from Theorem 4, the error is

$$\begin{aligned}(Z-z)(y_i,\ s_n) &= \mathcal{C}\Big((M^{-1}\ln M)^2 + \Delta s^2\Big) + o\Big((M^{-1}\ln M)^2 + \Delta s^2\Big), \\ &= \mathcal{C}\big(\tfrac{M^{-1}\xi}{\rho_0\sqrt{\varepsilon}}\big)^2 + \mathcal{C}\Delta s^2 + o\Big((M^{-1}\ln M)^2 + \Delta s^2\Big)\end{aligned} \tag{12}$$

for $(y_i,\ s_n) \in \mathfrak{D}^M$. Let $\overline{Z}(\overline{y}_i, \overline{s}_n)$ be the solution to (11) on the domain $\mathfrak{D}^{2M}$. From Theorem 4, we get

$$(\overline{Z}-z)(\overline{y}_i, \overline{s}_n) = \mathcal{C}\Big((2M)^{-2}\big(\tfrac{\xi}{\rho_0\sqrt{\varepsilon}}\big)^2 + \big(\tfrac{\Delta s}{2}\big)^2\Big) + o\Big((M^{-1}\ln M)^2 + \big(\tfrac{\Delta s}{2}\big)^2\Big) \tag{13}$$

for $(\overline{y}_i, \overline{s}_n) \in \mathfrak{D}^{2M}$. Now, the elimination of $o(M^{-2})$ term from (12) and (13) leads to the following approximation:

$$z(y_i, s_n) - \frac{1}{3}\Big(4\overline{Z} - Z\Big)(y_i,\ s_n) = o\Big((M^{-1}\ln M)^2 + \Delta s^2\Big),\ (y_i, s_n) \in \mathfrak{D}^M. \tag{14}$$

Therefore, we use the extrapolation formula as

$$Z_{extp}(y_i,\ s_n) = \frac{1}{3}\Big(4\overline{Z} - Z\Big)(y_i,\ s_n), \quad (y_i,\ s_n) \in \mathfrak{D}^M. \tag{15}$$

Theorem 5. Let Z_{extp} be the solution by extrapolation technique (15) and let z be the solution to problem (1). Also, assume that $\sqrt{\varepsilon} \le M^{-1}$. Then we have the following error bound on S-mesh

$$\Big|z(y_i,\ s_n) - Z_{extp}(y_i,\ s_n)\Big| \le \mathcal{C}\Big(M^{-4}\ln^4 M + \Delta s^4\Big) \quad \text{for} \quad 1 \le i \le M-1.$$

Proof. The term in right side of (1), $\mathfrak{f}(y,\ s) - \mathfrak{b}(y,\ s)\Theta_b(y,\ s-\gamma)$ is independent of ε in the interval $[0, \gamma]$, where the time discretization parameter n

varies from 0 to p. One can find the complete analysis of the extrapolation technique in [28].

Since $z(y,\ s)$ depends on $z(y,\ s-\gamma)$, which is unknown for $s \geq \gamma$, the approach in [28] is not applicable in the interval $[\gamma, 2\gamma]$ and $s \geq \gamma$. Therefore the following delay parabolic equation involving small parameter considered on domain $\mathfrak{D}_2 = \Pi \times (\gamma, 2\gamma)$:

$$\begin{cases} \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big) z(y,\ s) = -\mathfrak{b}(y,\ s) z(y,\ s-\gamma) + \mathfrak{f}(y,\ s), \quad (y,\ s) \in \mathfrak{D}_2, \\ z(y,\ s) = z_b(y,\ s), \quad (y,\ s) \in \mathfrak{D}_1 = \Pi \times (0, \gamma), \\ z(0,\ s) = \Theta_l(s), \quad z(1,\ s) = \Theta_r(s), \quad s \in [\gamma, 2\gamma], \end{cases} \tag{16}$$

where $z_b(y,\ s)$ is a continuous solution in $\mathfrak{D}_1$. Applying the scheme for (16) on $\mathfrak{D}_2$ as given in (10), we have

$$\begin{aligned} 2D_t^- Z_i^{n+1} - \varepsilon \delta_y^2 Z_i^{n+1} + \mathfrak{p}_i Z_i^{n+1} &= -\mathfrak{b}_i^{n+1} Z_i^{n-p+1} - \mathfrak{b}_i^n Z_i^{n-p} \\ &\quad + \mathfrak{f}_i^n + \mathfrak{f}_i^{n+1} - \mathfrak{L}_\varepsilon Z_i^n, \\ Z_0^n = \Theta_l(s_n), \quad Z_N^n &= \Theta_r(s_n), \quad s_n \in [\gamma, 2\gamma], \\ Z_i^{-j} &= Z_b(y_i,\ s_n), \quad (y_i,\ s_n) \in \mathfrak{D}_1^M, \end{aligned} \tag{17}$$

where $\delta_y^2 = D_y^- D_y^+$, $\mathfrak{f}_i^n = \mathfrak{f}(y_i,\ s_n)$, $(y_i,\ s_n) \in \mathfrak{D}_2^M$ and $Z_b^1(\cdot, \cdot)$ is the numerical solution in $\mathfrak{D}_1^M$.

Decompose z in (16) on the domain $\mathfrak{D}_2$ as $z = w_r + w_s$, where w_r is the smooth component and w_s is the singular component. Again we write $w_r = w_{r0} + \varepsilon w_{r1}$, satisfying

$$\begin{cases} \dfrac{\partial w_{r0}(y,\ s)}{\partial s} + \mathfrak{p}(y) w_{r0}(y,\ s) = -\mathfrak{b}(y,\ s) w_{r0}(y,\ s-\gamma) + \mathfrak{f}(y,\ s),\ (y,\ s) \in \mathfrak{D}_2, \\ w_{r0}(y,\ s) = z(y,\ s), \quad (y,\ s) \in \mathfrak{D}_1,\ w_{r0}(0, s) = z(0,\ s), \quad s \in [\gamma, 2\gamma], \end{cases} \tag{18}$$

and

$$\begin{cases} \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big) w_{r1}(y, s) = -\mathfrak{b}(y, s) w_{r1}(y, s-\gamma) + (w_{r0})_{yy},\ (y, s) \in \mathfrak{D}_2, \\ w_{r1}(y,\ s) = 0, (y,\ s) \in \mathfrak{D}_1,\ w_{r1}(0, s) = 0,\ w_{r1}(1, s) = 0,\ s \in [\gamma, 2\gamma]. \end{cases} \tag{19}$$

Therefore

$$\begin{cases} \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big) w_r(y,\ s) = -\mathfrak{b}(y,\ s) w_r(y, s-\gamma) + \mathfrak{f}(y,\ s),\ (y,\ s) \in \mathfrak{D}_2, \\ w_r(y,\ s) = z(y,\ s), \quad (y,\ s) \in \mathfrak{D}_1, \\ w_r(0,\ s) = w_{r0}(0, s), \quad w_r(1,\ s) = w_{r0}(1,\ s), \quad s \in [\gamma, 2\gamma]. \end{cases} \tag{20}$$

Then, the singular component w_s satisfies

$$\begin{cases} \left(\dfrac{\partial}{\partial s}+\mathfrak{L}_{\varepsilon,y}\right)w_s(y,\ s)=-\mathfrak{b}(y,\ s)w_s(y,\ s-\gamma), \quad (y,\ s)\in\mathfrak{D}_2,\\ w_s(y,\ s)=0, \quad (y,\ s)\in\mathfrak{D}_1, w_s(0,s)=\Theta_l(s)-w_{r0}(0,\ s),\\ w_s(1,\ s)=\Theta_r(s)-w_{r0}(1,\ s),\ s\in[\gamma,2\gamma]. \end{cases} \tag{21}$$

Write $w_s=w_{sl}+w_{sr}$, where w_{sl} is the left boundary layer component on $\mathfrak{d}_l$ and w_{sr} is the right boundary layer component on $\mathfrak{d}_r$. Also, w_{sl} and w_{sr} are satisfying the following PDEs:

$$\begin{cases} \left(\dfrac{\partial}{\partial s}+\mathfrak{L}_{\varepsilon,y}\right)w_{sl}(y,\ s)=-\mathfrak{b}(y,\ s)w_l(y,\ s-\gamma), \ \ (y,\ s)\in\mathfrak{D}_2,\\ w_{sl}(0,\ s)=\Theta_l-w_{r0}(0,\ s)\ s\in[\gamma,2\gamma],\ w_{sl}(y,\ s)=0,\ (y,\ s)\in\mathfrak{D}_{1l},\\ \left(\dfrac{\partial}{\partial s}+\mathfrak{L}_{\varepsilon,y}\right)w_{sr}(y,\ s)=-\mathfrak{b}(y,\ s)w_r(y,\ s-\gamma), \quad (y,\ s)\in\mathfrak{D}_2,\\ w_{sr}(1,s)=\Theta_r-w_{r0}(1,s),\ s\in[\gamma,2\gamma],\ w_{sr}(y,s)=0,\ (y,s)\in\mathfrak{D}_{1r}, \end{cases} \tag{22}$$

where $\mathfrak{D}_{1l}=(0,\xi)\times[0,\gamma]$ and $\mathfrak{D}_{1r}=(1-\xi,1)\times[0,\gamma]$. Since, $\mathfrak{D}_2\subset\mathfrak{D}$, the estimates given in Theorem 1, can be used for w_r and w_s. Decompose the numerical solution Z to (17) as $Z=W_r+W_s$, where W_r is the smooth part and W_s is the singular part. Thus

$$\begin{aligned} \left(D_s^+ + D_s^- +\mathfrak{L}_\varepsilon\right){W_r}_i^{n+\frac12} &= -\mathfrak{b}_i^{n+\frac12}{W_r}_i^{n+\frac12-p}+f_i^{n+\frac12},\ (y_i,\ s_{n+\frac12})\in\mathfrak{D}_2^M, \\ {W_r}_i^n &= Z_b(y_i,\ s_n), \quad (y_i,\ s_n)\in\mathfrak{D}_1^M,\\ {W_r}_0^n &= w_r(0,\ s_n), \quad {W_r}_N^n=w_r(1,\ s_n), \quad s_n\in[\gamma,2\gamma], \end{aligned} \tag{23}$$

and therefore, W_s satisfies

$$\begin{aligned} \left(D_s^+ + D_s^- +\mathfrak{L}_\varepsilon\right){W_s}_i^{n+\frac12} &= -b_i^n W_i^{n+\frac12-p}, \quad (y_i,s_{n+\frac12})\in\mathfrak{D}_2^M,\\ {W_s}_i^n &= 0, \quad (y_i,\ s_n)\in\mathfrak{D}_1^M,\\ {W_s}_0^n &= \Theta_l(s_n)-w_r(0,\ s_n),\\ {W_s}_N^n &= \Theta_r(s_n)-w_r(1,\ s_n), \quad s_n\in[\gamma,2\gamma]. \end{aligned} \tag{24}$$

Now, we write $W_s=W_{sl}+W_{sr}$, where W_{sl} is the boundary layer on $\mathfrak{d}_l^M$ and W_{sr} is the boundary layers on $\mathfrak{d}_r^M$. Hence W_{sl} and W_{sr} are defined by

$$\begin{aligned} &\left(D_s^+ + D_s^- +\mathfrak{L}_\varepsilon\right)W_{sl}(y_i,\ s_{n+\frac12})=-\mathfrak{b}_i^n W_{sl}(y_i,\ s_{n+\frac12-p}),\ \ (y_i,\ s_{n+\frac12})\in\mathfrak{D}_2^M,\\ &W_{sl}(0,s_n)=\Theta_l-w_r(0,\ s_n)\ (y_i,\ s_n)\in\mathfrak{d}_l, \quad W_{sl}(y_i,\ s_n)=0,\ (y_i,\ s_n)\in\mathfrak{D}_{1l}^M,\\ &\left(D_s^+ + D_s^- +\mathfrak{L}_\varepsilon\right)W_{sr}(y_i,\ s_{n+\frac12})=-\mathfrak{b}_i^n W_{sr}(y_i,\ s_{n+\frac12-p}),\ \ (y_i,\ t_{n+\frac12})\in\mathfrak{D}_2^M,\\ &W_{sr}(1,\ s_n)=\Theta_r-w_r(1,\ s_n),\ (y_i,\ s_n)\in\mathfrak{d}_r, \quad W_{sr}=0, \quad (y_i,\ s_n)\in\mathfrak{D}_{1r}^M, \end{aligned} \tag{25}$$

where $\mathfrak{D}_{1l}^M$ is the discretized domain of $\mathfrak{D}_{1l}$ and $\mathfrak{D}_{1r}^M$ is the discretized domain of $\mathfrak{D}_{1l}$. Similarly $\mathfrak{d}_l^M$ is the discretized domain of $\mathfrak{d}_l$ and $\mathfrak{d}_r^M$ is the discretized domain of $\mathfrak{d}_r$. □

4.1 Error bound for W_r

Lemma 3. The local truncation error in $\mathfrak{D}_2^M$ associated to the smooth component satisfies

$$\begin{aligned}\Big(2D_s^- + \mathfrak{L}_\varepsilon\Big)&(w_r - W_r)(y_i,\ s_n)\\ &= \mathcal{C}\Big((M^{-1}\ln M)^2 + \Delta s^2\Big) + \frac{\varepsilon}{12}(y_i - y_{i-1})^2\frac{\partial^4 w_r}{\partial y^4}(y_i,\ s_{n-\frac{1}{2}})\\ &\quad + \frac{1}{12}\Delta s^2\frac{\partial^3 w_r}{\partial s^3}(y_i,\ s_n) + O(M^{-4} + \Delta s^4).\end{aligned}$$

Proof. Using (20) and (23), we get

$$\begin{aligned}\Big(2D_s^- + \mathfrak{L}_\varepsilon\Big)&(w_r - W_r)(y_i,\ s_n)\\ &= \mathfrak{b}_i^n(z(y_i,\ s_{n-p-\frac{1}{2}}) - Z_b(y_i,\ s_{n-p-\frac{1}{2}})) + \Big(\frac{\partial}{\partial s} - 2D_s^-\Big)w_r(y_i,\ s_n)\\ &\quad + \frac{\partial}{\partial s}w_r(y_i,\ s_{n-1}) - +(\mathfrak{L}_{\varepsilon,y} - \mathfrak{L}_\varepsilon)w_r(y_i,\ s_{n-\frac{1}{2}}).\end{aligned}$$

Now, by using the estimate in Theorem 4 and Taylor's expansion, the desired result can be achieved. Refer to [8] for more details. □

Let the function $E_d(y,\ s)$, for $d = 1, 2$, satisfy the IBVPs (refer approach of Kellar [12]):

$$\begin{cases}\Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big)E_1(y,\ s) = \dfrac{\varepsilon}{12}\dfrac{\partial^4 w_r(y,\ s)}{\partial y^4} & \text{in } \mathfrak{D},\\ E_1(y,\ s) = 0, & (y,\ s) \in \mathfrak{b}_b,\\ E_1(0,\ s) = 0, \quad E_1(1,\ s) = 0, & s \in [0, \mathcal{S}],\end{cases} \tag{26}$$

$$\begin{cases}\Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big)E_2(y,\ s) = \dfrac{1}{12}\dfrac{\partial^3 w_r(y,\ s)}{\partial s^3} & \text{in } \mathfrak{D},\\ E_2(y,\ s) = 0, & (y,\ s) \in \Gamma_b,\\ E_2(0,\ s) = 0, \quad E_2(1,\ s) = 0, & s \in [0, \mathcal{S}].\end{cases} \tag{27}$$

Now E_d can be decomposed as $E_d = \eta_d + \vartheta_d$, where η_d and ϑ_d are the regular and singular layer parts of E_d. Now from Theorem 1, we have

$$\left\|\frac{\partial^{l+m}\eta_d}{\partial y^l \partial s^m}\right\|_\infty \le \mathcal{C}(1 + \varepsilon^{2-l/2}) \quad for \quad 0 \le l + 2m \le 8.$$

One can get these bounds using a similar procedure as done in [1]. Taking $(y,\ s) \in \mathfrak{D}_2$, (26) and (27) reduce to the following IBVPs:

$$\begin{cases} \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big) E_1(y,\ s) = \dfrac{\varepsilon}{12}\dfrac{\partial^4 w_r(y,\ s)}{\partial y^4} \quad \text{in } \mathfrak{D}_2, \\ E_1(y,\ s) = E_{1\gamma}, \qquad\qquad\qquad (y,\ s) \in \mathfrak{D}_1, \\ E_1(0,\ s) = 0, \quad E_1(1,\ s) = 0, \quad s \in [\gamma, 2\gamma], \end{cases}$$

$$\begin{cases} \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big) E_2(y,\ s) = \dfrac{1}{12}\dfrac{\partial^3 w_r(y,\ s)}{\partial s^3} \quad \text{in } \mathfrak{D}_2, \\ E_2(y,\ s) = E_{2\gamma}, \qquad\qquad\qquad (y,\ s) \in \mathfrak{D}_1, \\ E_2(0,\ s) = 0, \quad E_2(1,\ s) = 0, \quad s \in [\gamma, 2\gamma], \end{cases}$$

where $E_{1\gamma}(\cdot,\cdot)$ and $E_{2\gamma}(\cdot,\cdot)$ are the respective solutions in $\mathfrak{D}_1$. Therefore, the components η_d and ϑ_d, $d = 1, 2$, satisfy

$$\begin{cases} \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big)\eta_1 = \dfrac{\varepsilon}{12}\dfrac{\partial^4 w_r(y,\ s)}{\partial y^4} \quad \text{in } \mathfrak{D}_2, \\ \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big)\eta_2 = \dfrac{1}{12}\dfrac{\partial^3 w_r(y,\ s)}{\partial s^3} \quad \text{in } \mathfrak{D}_2, \\ \eta_d(y,\ s) = E_{d\gamma}, \qquad\qquad (y,\ s) \in \mathfrak{D}_1 \quad \text{for} \quad d = 1, 2, \\ \eta_d(0,\ s) = 0, \quad \eta_d(1,\ s) = -\vartheta_d(1,\ s), \quad s \in [\gamma, 2\gamma], \end{cases}$$

$$\begin{cases} \Big(\dfrac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big)\vartheta_d = 0 \quad \text{in } \mathfrak{D}_2 \\ \vartheta_d(y,\ s) = 0, \qquad\qquad (y,\ s) \in \mathfrak{D}_1 \quad \text{for} \quad d = 1, 2, \\ \vartheta_d(0,\ s) = 0, \quad \vartheta_d(1,\ s) = -\eta_d(1,\ s), \quad s \in [\gamma, 2\gamma]. \end{cases} \tag{28}$$

Lemma 4. The local truncation error in $\mathfrak{D}_2^M$ associated with w_r satisfies

$$(w_r - W_r)(y_i,\ s_n) = \mathcal{C}\Big((M^{-1}\ln M)^2 + \Delta s^2\Big) + (y_i - y_{i-1})^2\eta_1 + \eta_2\Delta s + O(M^{-4} + \Delta s^2).$$

Proof. From Lemma 3 and (28), one can easily get

$$\begin{aligned} \Big(2D_s^- + \mathfrak{L}_\varepsilon\Big)((w_r - W_r)(y_i,\ s_n) = {} & C\big((M^{-1}\ln M)^2 + \Delta s\big) \\ & + h_i^2\left(\Big(\frac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big) - (2D_s^- + \mathfrak{L}_\varepsilon)\right)\eta_1 \\ & + \Delta s\left(\Big(\frac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big) - (2D_s^- + \mathfrak{L}_\varepsilon)\right)\eta_2 \\ & + O(h^4 + \Delta s^4). \end{aligned} \tag{29}$$

Using the derivative bounds of η_d and from the Taylor's expansion, it follows that for $d = 1, 2$,

$$\left| h_i^2\left(\Big(\frac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big) - (2D_s^- + \mathfrak{L}_\varepsilon)\right)\eta_1 + \Delta s\left(\Big(\frac{\partial}{\partial s} + \mathfrak{L}_{\varepsilon,y}\Big) - (2D_s^- + \mathfrak{L}_\varepsilon)\right)\eta_2 \right|$$

$$\leq \mathcal{C}\Big(h^4 + \Delta s^4\Big). \tag{30}$$

Therefore, from (29) and (30), we have

$$\Big(2D_s^- + \mathfrak{L}_\varepsilon\Big)(w_r - W_r)(y_i,\ s_n) \leq \mathcal{C}\Big((M^{-1}\ln M)^2 + \Delta s^2 + \big(h^4 + \Delta s^4\big)\Big),$$

and, by applying barrier functions and the discrete maximum principle as done in [22], we obtain the following bound:

$$\Big|(w_r - W_r)(y_i,\ s_n)\Big| \leq \mathcal{C}\Big((M^{-1}\ln M)^2 + \Delta s^2\Big) + h_i^2\eta_1 + \eta_2\Delta s^2 + O(h^4 + \Delta s^4).$$

□

Lemma 5. The error associated with W_r after extrapolation satisfies

$$(w_r - W_{r\,extp})(y_i,\ s_n) \leq \mathcal{C}\Big(M^{-4} + \Delta s^4\Big) \quad \text{for} \quad (y_i,\ s_n)\ \mathfrak{D}_2^M.$$

Proof. From Lemma 4 on the fine mesh $\mathfrak{D}_2^{2N}$, we have

$$(\overline{W_r} - w_r)(y_i,\ s_n) = \mathcal{C}\Big((2M)^{-2}(\ln 2M)^2 + \frac{\Delta s^2}{4}\Big) + \frac{h_i^2}{4}\eta_1 + \eta_2\frac{\Delta s^2}{4} + O(M^{-4} + \Delta s^4). \tag{31}$$

From the extrapolation formula (15), we can write

$$\begin{aligned}(w_r - W_{r\,extp})(y_i,\ s_n) &= w_r(y_i,\ s_n) - \big(\frac{1}{3}(4\overline{W_r} - W_r)(y_i,\ s_n)\big)\\ &= -\frac{1}{3}\Big(4(\overline{W_r} - w_r) + (W_r - w_r)\Big)(y_i,\ s_n).\end{aligned}$$

By using Lemma 4 and (31), we obtain

$$\begin{aligned}-\frac{1}{3}\Big(4(\overline{W_r} - w_r) + (W_r - w_r)\Big)(y_i,\ s_n) =& \frac{1}{3}\Bigg(-C\Big((M^{-1}\ln 2M)^2 + \Delta s^2\Big)\\ &- h_i^2\eta_1 - \eta_2\Delta s^2\\ &+ C\Big((M^{-1}\ln M)^2 + \Delta s^2\Big)\\ &+ h_i^2\eta_1 + \eta_2\Delta s^2\Bigg) + O(M^{-4} + \Delta s^4)\\ =& O(M^{-4} + \Delta s^4),\end{aligned}$$

which is our desired bound. □

Now, we define the function $F_d = F_{dl} + F_{dr}$, $d = 1, 2$, by the following IBVPs:

$$
\begin{cases}
\Big(\dfrac{\partial}{\partial s}+\mathfrak{L}_{\varepsilon,y}\Big)F_{1l}=\dfrac{\rho_0\varepsilon}{3}\dfrac{\partial^4 w_s(y,\ s)}{\partial y^4}, & (y,\ s) \text{ in } \mathfrak{D}_l=(0,\xi)\times(0,\mathcal{S}],\\
\Big(\dfrac{\partial}{\partial s}+\mathfrak{L}_{\varepsilon,y}\Big)F_{1r}=\dfrac{\rho_0\varepsilon}{3}\dfrac{\partial^4 w_s(y,\ s)}{\partial y^4}, & (y,\ s) \text{ in } \mathfrak{D}_r=(1-\xi,1)\times(0,\mathcal{S}],\\
F_{1l}(y,\ s)=0, & (y,\ s)\in[0,\xi]\times(-\gamma,0),\\
F_{1r}(y,\ s)=0, & (y,\ s)\in[1-\xi,1]\times(-\gamma,0),\\
F_{1l}(0,\ s)=F_{1l}(\xi,\ s)=0, & s\in(0,\mathcal{S}],\\
F_{1r}(1-\xi,\ s)=F_{1r}(1,\ s)=0, & s\in(0,\mathcal{S}],
\end{cases}
\tag{32}
$$

$$
\begin{cases}
\Big(\dfrac{\partial}{\partial s}+\mathfrak{L}_{\varepsilon,y}\Big)F_{2l}=\dfrac{1}{12}\dfrac{\partial^3 w_s(y,\ s)}{\partial s^3}, & (y,\ s) \text{ in } \mathfrak{D}_l=(0,\xi)\times(0,\mathcal{S}],\\
\Big(\dfrac{\partial}{\partial s}+\mathfrak{L}_{\varepsilon,y}\Big)F_{2r}=\dfrac{1}{12}\dfrac{\partial^3 w_s(y,\ s)}{\partial s^3}, & (y,\ s) \text{ in } \mathfrak{D}_r=(1-\xi,1)\times(0,\mathcal{S}],\\
F_{2l}(y,\ s)=0, & (y,\ s)\in[0,\xi]\times(-\gamma,0),\\
F_{2r}(y,\ s)=0, & (y,\ s)\in[1-\xi,1]\times(-\gamma,0),\\
F_{2l}(0,\ s)=F_{1l}(\xi,\ s)=0, & s\in(0,\mathcal{S}],\\
F_{2r}(1-\xi,\ s)=F_{1r}(1,\ s)=0, & s\in(0,\mathcal{S}],
\end{cases}
\tag{33}
$$

The solution F_d, $d=1,2$, to (32) and (33) satisfies the following bounds:

$$
\left|\frac{\partial^{l+m}F_d}{\partial y^l\partial s^m}\right|_\infty \le \mathcal{C}\varepsilon^{-l/2}\Big(\exp(-y\sqrt{\beta/\varepsilon})+\exp(-(1-y)\sqrt{\beta/\varepsilon})\Big), \quad (y,\ s)\in\mathfrak{D}.
$$

In this context, by considering $s\in(\gamma,2\gamma)$, therefore (32) and (33) reduces to

$$
\begin{cases}
\Big(\dfrac{\partial}{\partial s}+\mathfrak{L}_{\varepsilon,y}\Big)F_{1l}=\dfrac{\rho_0\varepsilon}{3}\dfrac{\partial^4 w_s(y,\ s)}{\partial y^4}, & (y,\ s) \text{ in } \mathfrak{D}_l=(0,\xi)\times(\gamma,2\gamma),\\
\Big(\dfrac{\partial}{\partial s}+\mathbb{L}_{\varepsilon,y}\Big)F_{1r}=\dfrac{\rho_0\varepsilon}{3}\dfrac{\partial^4 w_s(y,\ s)}{\partial y^4}, & (y,\ s) \text{ in } D_r=(1-\xi,1)\times(\gamma,2\gamma),\\
F_{1l}(y,\ s)=F_{1l\gamma}(y,\ s), & (y,\ s)\in[0,\xi]\times(0,\gamma),\\
F_{1r}(y,\ s)=F_{1r\gamma}(y,\ s), & (y,\ s)\in[1-\xi,1]\times(0,\gamma),\\
F_{1l}(0,\ s)=F_{1l}(\xi,t)=0, & s\in(\gamma,2\gamma),\\
F_{1r}(1-\xi,t)=F_{1r}(1,\ s)=0, & s\in(\gamma,2\gamma),
\end{cases}
\tag{34}
$$

$$\begin{cases}\Big(\dfrac{\partial}{\partial s}+\mathfrak{L}_{\varepsilon,x}\Big)F_{2l}=\dfrac{1}{12}\dfrac{\partial^3 w_s(y,\ s)}{\partial s^3}, & (y,\ s) \text{ in } \mathfrak{D}_l=(0,\xi)\times(\gamma,2\gamma),\\ \Big(\dfrac{\partial}{\partial s}+\mathfrak{L}_{\varepsilon,y}\Big)F_{2r}=\dfrac{1}{12}\dfrac{\partial^3 w_s(y,\ s)}{\partial s^3}, & (y,\ s) \text{ in } \mathfrak{D}_r=(1-\xi,1)\times(\gamma,2\gamma),\\ F_{2l}(y,\ s)=F_{2l\gamma}(y,\ s), & (y,\ s)\in[0,\xi]\times(0,\gamma),\\ F_{2r}(y,\ s)=F_{2r\gamma}(y,\ s), & (y,\ s)\in[1-\xi,1]\times(0,\gamma),\\ F_{2l}(0,\ s)=F_{2l}(\xi,\ s)=0, & s\in(\gamma,2\gamma),\\ F_{2r}(1-\xi,\ s)=F_{1r}(1,\ s)=0, & s\in(\gamma,2\gamma),\end{cases} \tag{35}$$

where $F_{kl}(\cdot,\cdot)$, $k=1,2$, are the analytic solutions in $[0,\xi]\times(0,\gamma)$ and $F_{kr}(\cdot,\cdot)$, $k=1,2$, are the analytic solutions in $[1-\xi,1]\times(0,\gamma)$.

Lemma 6. For $(y_i,\ s_n)$ $\mathfrak{D}_2^M$, the error associated with W_s satisfies

$$(W_s-w_s)(y_i,\ s_n)=(M^{-1}\ln M)^2F_1(y_i,\ s_n)+\Delta s^2F_2(y_i,\ s_n)+O\Big(M^{-4}\ln^4 M+\Delta s^4\Big).$$

Proof. Write $W_s=W_{sl}+W_{sr}$, where W_{sl} and W_{sr} are defined in (25). Now $W_s-w_s=(W_{sl}-w_{sl})+(W_{sr}-w_{sr})$, where the error $W_{sl}-w_{sl}$ is related to a layer on $\mathfrak{D}_l$ and $W_{sr}-w_{sr}$ on $\mathfrak{D}_r$. These errors can be estimated separately. First, we estimate $W_{sl}-w_{sl}$, from (22) and difference equation (25) and follow the similar procedure of Lemma 4. □

Next, we can estimate the error $W_{sr}-w_{sr}$.

Lemma 7. For $(y_i,\ s_n)$ ${\mathfrak{D}_2}^M$, the error associated to W_s after extrapolation satisfies

$$\Big|(w_s-W_{sextp})(y_i,\ s_n)\Big|\le\mathcal{C}\Big(M^{-4}\ln^4 M+\Delta s^4\Big).$$

Proof. From Lemma 6, we get

$$\begin{aligned}(W_s-w_s)= &\quad (M^{-1}\ln M)^2F_1(y_i,\ s_n)+\Delta s^2F_2(y_i,\ s_n)\\ &+O\Big(M^{-4}\ln^4 N+\Delta s^4\Big)\quad \text{for}\ \ (y_i,\ s_n)\ \mathfrak{D}_2^M,\end{aligned} \tag{36}$$

and

$$\begin{aligned}(w_s-\overline{W_s})=&((2M)^{-1}\ln 2M)^2F_1(y_i,\ s_n)+\frac{\Delta s^2}{4}F_2(y_i,\ s_n)\\ &+O\Big(M^{-4}\ln^4 M+\Delta s^4\Big)\end{aligned} \tag{37}$$

for $(y_i,\ s_n)\in\mathfrak{D}_2^{2M}$. Eliminating the terms $O(M^{-2})$ and Δs^2 from (36) and (37), the required result is achieved. □

Theorem 6 (Error after extrapolation in $\mathfrak{D}_2^M$)**.** Let Z_{extp} be the extrapolated solution (by technique (15)) for solving (11) on ${\mathfrak{D}_2}^M$ and $\mathfrak{D}_2^{2M}$. Let z be the solution to (1). Assume that $\varepsilon<M^{-2}$. Then we have the following error bound associated with Z_{extp} :

$$\Big|z(y_i,\ s_n) - Z_{extp}(y_i,\ s_n)\Big| \le \mathcal{C}\Big(M^{-4}\ln^4 M + \Delta s^4\Big) \quad \text{for} \quad 1 \le i \le M-1.$$

Proof. We have

$$\begin{aligned}\Big|z(y_i,\ s_n) - Z_{extp}(y_i,\ s_n)\Big| \le& \Big|w_r(y_i,\ s_n) - W_{r\,extp}(y_i,\ s_n)\Big| \\ &+ \Big|w_s(y_i,\ s_n) - W_{s\,extp}(y_i,\ s_n)\Big|\end{aligned}$$

for all $(y_i,\ s_n) \in \mathfrak{D}_2^M$. Combining Lemmas 4 and 7, we can get the required result. □

Remark 1. The error bound on B-S-mesh in D_2^M is

$$\Big|z(y_i,\ s_n) - Z_{extp}(y_i,\ s_n)\Big| \le \mathcal{C}\Big(M^{-4} + \Delta s^4\Big) \quad \text{for} \quad 1 \le i \le M-1.$$

Remark 2.(Error after extrapolation in $\mathfrak{D}^M$) Let Z_{extp} be the extrapolated solution (by technique (15)) for solving (11) on ${\mathfrak{D}_2}^M$ and $\mathfrak{D}_2^{2M}$. Let z be the solution to (1). Then we have the following error bound on S-mesh:

$$\Big|z(y_i,\ s_n) - Z_{extp}(y_i,\ s_n)\Big| \le \mathcal{C}\Big(M^{-4}\ln^4 M + \Delta s^2\Big) \quad \text{for} \quad 1 \le i \le M-1,$$

similarly, on B-S-mesh

$$\Big|z(y_i,\ s_n) - Z_{extp}(y_i,\ s_n)\Big| \le \mathcal{C}\Big(M^{-4} + \Delta s^4\Big) \quad \text{for} \quad 1 \le i \le M-1.$$

5 Numerical experiments

The proposed scheme (10) is tested on two test problems in this section. In this section, all the numerical results are obtained using MATLAB software in 64 GB RAM workstation.

Example 1. Consider

$$\begin{cases} z_s - \varepsilon z_{yy} + 0.5z = -2e^{-1}z(y,\ s-1) + \mathfrak{f}(y,s), \quad (x,\ t) \in (0,\ 1) \times (0,\ 2], \\ z(y,\ s) = e^{-s+y/\sqrt{\varepsilon}} + e^{-s+(1-y)/\sqrt{\varepsilon}}, \quad (y,\ s) \in [0,\ 1] \times [-1,\ 0], \\ z(0,\ s) = e^{-s} + e^{-s+1/\sqrt{\varepsilon}}, z(1,\ s) = e^{-s+1/\sqrt{\varepsilon}} + e^{-s},\ s \in [0,\ 2]. \end{cases} \tag{38}$$

The exact solution for Example 1 is $z(y,\ s) = e^{-s+y/\sqrt{\varepsilon}} + e^{-s+(1-y)/\sqrt{\varepsilon}}$. The maximum pointwise error before and after extrapolation given by

$$E_\varepsilon^{M,\Delta s} = \max_{(y_i,\ s_n)\in\mathfrak{D}^M} |z(y_i,\ s_n) - Z^{M,\Delta s}(y_i,\ s_n)|$$

and

$$E_{\varepsilon}^{M,\Delta s} = \max_{(y_i,\ s_n)\in \mathfrak{D}^M} |z(y_i,\ s_n) - Z_{extp}^{M,\Delta s}(y_i,\ s_n)|.$$

The corresponding order of convergence is $P_{\varepsilon}^{M,\Delta s} = \log_2\left(\frac{E_{\varepsilon}^{M,\Delta s}}{E_{\varepsilon}^{2M,\Delta s/2}}\right)$. Here, $z(y_i,\ s_n)$ is the exact solution and $Z^{M,\Delta s}(y_i,\ s_n)$ and $Z_{extp}^{M,\Delta s}(y_i,\ s_n)$ are the numerical solutions before and after extrapolation, respectively.

Example 2. Consider the test problem:

$$\begin{cases} z_s - \varepsilon z_{yy} + \dfrac{(1+y)^2}{2} z = s^3 - z(y, s-1), \quad (y,\ s) \in (0,\ 1) \times (0,\ 2], \\ z(y,\ s) = 0, \quad (y,\ s) \in [0,\ 1] \times [-1,\ 0], \\ z(0, s) = 0, z(1, s) = 0,\ s \in [0,\ 2]. \end{cases} \tag{39}$$

Since the exact solution to (2) is not known, we use the idea of double mesh principle to obtain the pointwise errors and to verify the ε-uniform convergence. Define $\widetilde{Z}(y_i,\ s_n)$ as the numerical solution obtained on $\widetilde{\mathfrak{D}}^{2M} = \widetilde{\Pi}_y^{2M} \times \widetilde{\Pi}_s^{2N}$ with $2M$ mesh intervals in space and $2N$ mesh intervals in the s-direction, where $\widetilde{\Pi}_y^{2M}$ is the Shishkin-type mesh as defined Π_y^M with the fixed transition parameter. For each ε, we calculate the maximum pointwise error before and after extrapolation by $\widehat{E}_{\varepsilon}^{M,\Delta s} = \max_{(y_i,\ s_n)\in \mathfrak{D}^M} |Z^{M,\Delta s}(y_i,\ s_n) - \widetilde{Z}^{M,\Delta s}(y_i,\ s_n)|$ and $\widehat{E}_{\varepsilon}^{M,\Delta s} = \max_{(y_i,\ s_n)\in \mathfrak{D}^M} |Z_{extp}^{M,\Delta s}(y_i,\ s_n) - \widetilde{Z}_{extp}^{M,\Delta s}(y_i,\ s_n)|$, respectively. The corresponding order of convergence is obtained by $\widehat{P}_{\varepsilon}^{M,\Delta s} = \log_2\left(\frac{\widehat{E}_{\varepsilon}^{M,\Delta s}}{\widehat{E}_{\varepsilon}^{2M,\Delta s/2}}\right)$. Here, $\widetilde{Z}_{extp}^{M,\Delta s}(y_i,\ s_n)$ is the extrapolation solution obtained by the double mesh principle.

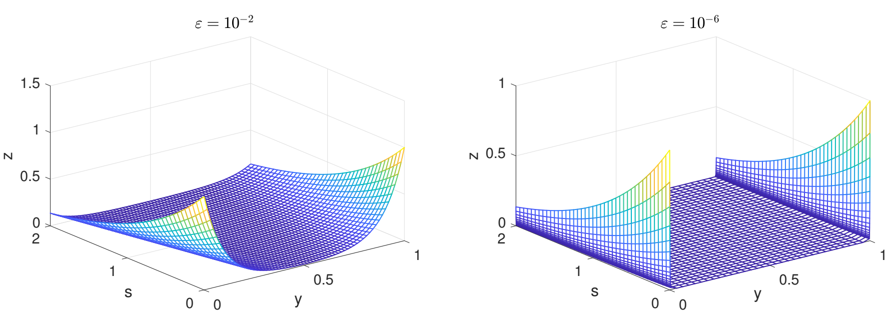


Figure 1: Surface plots of the computed solution on S-mesh for Example 1.

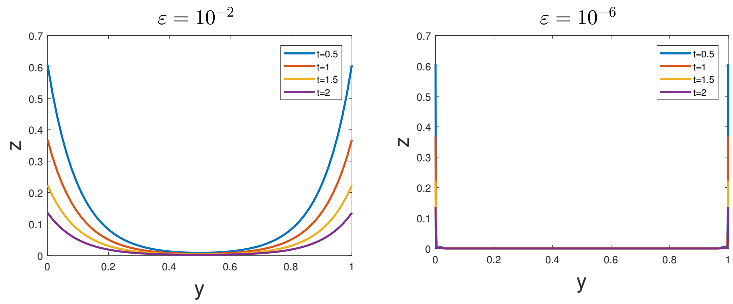
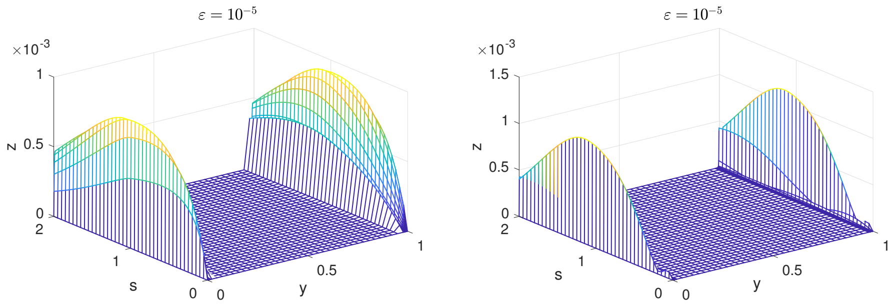


Figure 2: Solution plots of each time on S-mesh for Example 1.



(a) Before extrapolation. *(b) After extrapolation.*

Figure 3: Error graphs of the computed solution on B-S mesh for Example 1.

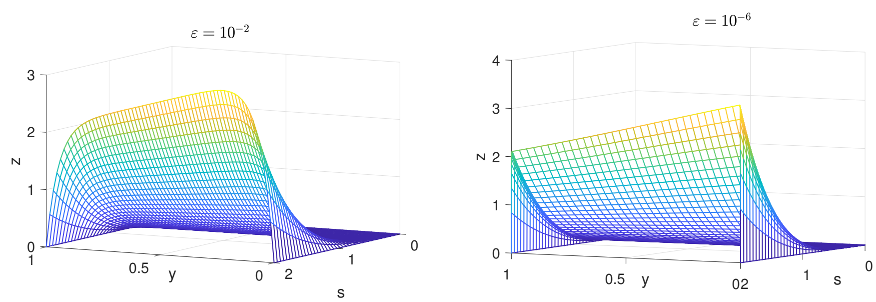
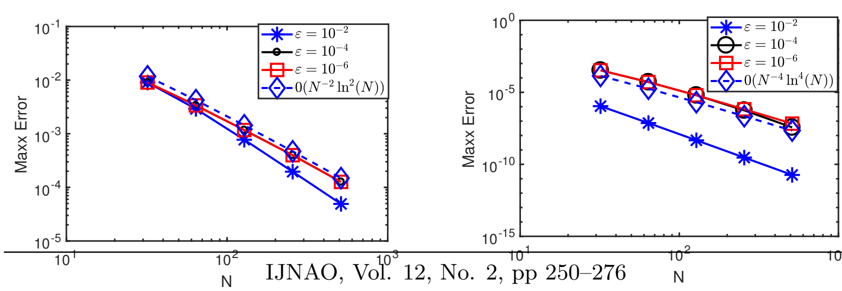


Figure 4: Surface plots of the computed solution on S-mesh for Example 2.



(a) Before extrapolation. *(b) After extrapolation.*

Figure 9: Log-log plots on B-S mesh for Example 2.

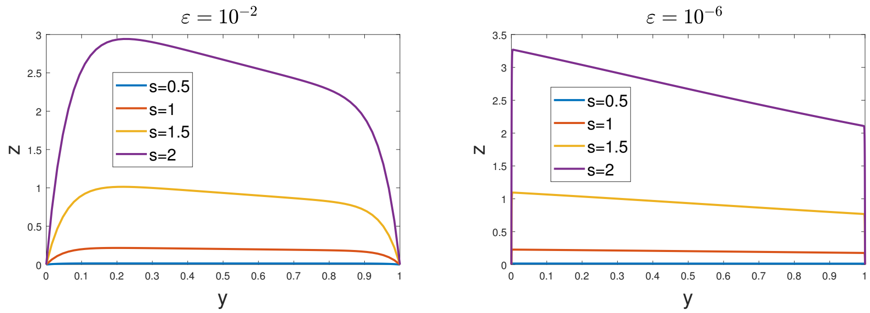
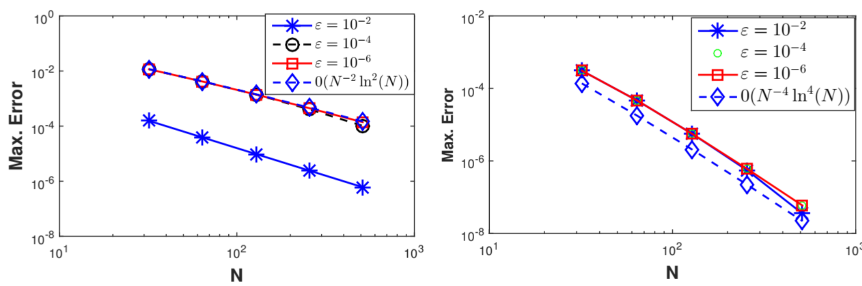
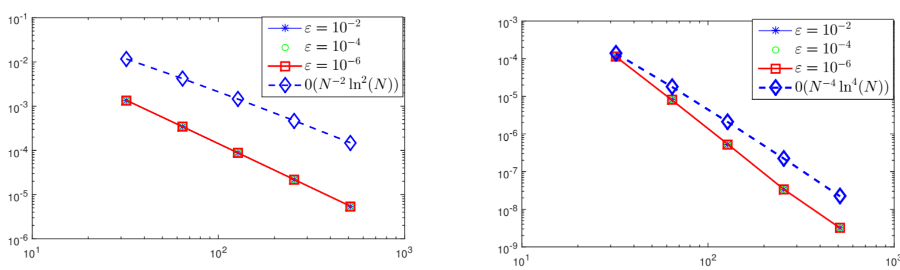


Figure 5: Solution plots on S-mesh for Example 2.



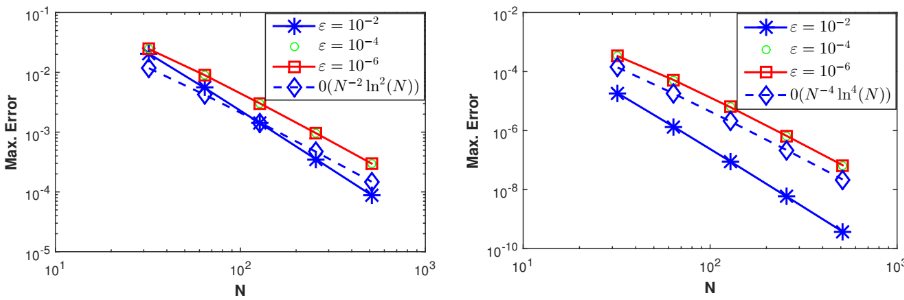
(a) Before extrapolation. *(b) After extrapolation.*

Figure 6: Log-log plots on S-mesh for Example 1.



(a) Before extrapolation. *(b) After extrapolation.*

Figure 7: Log-log plots on B-S mesh for Example 1.



(a) Before extrapolation. *(b) After extrapolation.*

Figure 8: Log-log plots on S-mesh for Example 2.

Table 1: $E_\varepsilon^{M,\Delta s}$ and $P_\varepsilon^{M,\Delta s}$ generated on S-mesh using the proposed method for Example 1.

$M/\Delta s$	Extrapolation	1e-4		1e-6	
		Singular layer	Regular layer	Singular layer	Regular layer
32/10	before	1.15e-2	6.42e-4	1.15e-2	6.42e-4
		1.47	2.13	1.47	2.13
	after	3.09e-4	1.52e-6	3.09e-4	1.52e-6
		2.72	3.56	2.72	3.56
64/40	before	4.15e-3	1.46e-4	4.15e-3	1.46e-4
		1.56	2.25	1.56	2.25
	after	1.33e-6	1.10e-7	1.33e-6	1.10e-7
		3.05	3.78	3.05	3.78
128/160	before	1.40e-3	3.07e-5	1.40e-3	3.07e-5
		1.65	2.38	1.65	2.38
	after	5.61e-6	9.33e-9	5.61e-6	9.33e-9
		3.22	3.89	3.22	3.89
256/640	before	4.46e-4	5.90e-6	4.46e-4	5.90e-6
		1.68	2.48	1.68	2.48
	after	6.01e-7	6.28e-10	6.00e-7	6.28e-10
		3.34	4.27	3.34	4.27
512/2560	before	1.38e-4	1.05e-6	1.38e-4	1.05e-6
	after	5.91e-8	3.25e-11	5.91e-8	3.25e-11

Table 2: $E_{\varepsilon}^{M,\Delta s}$ and $P_{\varepsilon}^{M,\Delta s}$ generated on B-S-mesh using the proposed method for Example 1.

ε	Extrapolation	Number of intervals M				
		32/10	64/40	128/160	256/640	512/2560
$1e-6$	before	4.9562e-3 1.8637	1.3618e-3 1.9783	3.4560e-4 1.9918	8.6895e-5 1.9965	2.1776e-5
	after	5.3442e-4 3.5744	4.4861e-5 3.8767	3.0539e-6 3.9730	1.9448e-7 3.9906	1.2234e-8
$1e-8$	before	4.9562e-3 1.8637	1.3618e-3 1.9783	3.4560e-4 1.9918	8.6895e-5 1.9965	2.1776e-5
	after	5.3442e-4 3.5744	4.4861e-5 3.8767	3.0539e-6 3.9730	1.9448e-7 3.9906	1.2234e-8
$1e-12$	before	4.9562e-3 1.8637	1.3618e-3 1.9783	3.4560e-4 1.9918	8.6895e-5 1.9965	2.1776e-5
	after	5.3442e-4 3.5744	4.4861e-5 3.8767	3.0539e-6 3.9730	1.9448e-7 3.9906	1.2234e-8

Table 3: $\widehat{E}_{\varepsilon}^{M,\Delta s}$ and $\widehat{P}_{\varepsilon}^{M,\Delta s}$ generated on S-mesh using the proposed method for Example 2.

$N/\Delta s$	Extrapolation	1e-4		1e-6		1e-8	
		Inner layer	Outer layer	Inner layer	Outer layer	Inner layer	Outer layer
32/10	before	9.021e-3 1.633	2.800e-3 2.001	9.021e-3 1.418	3.100e-3 2.015	9.021e-3 1.418	3.100e-3 2.015
	after	1.850e-5 3.788	6.976e-6 3.982	3.344e-4 2.704	7.103e-6 3.970	3.344e-4 2.704	7.103e-6 3.970
64/40	before	2.907e-3 1.911	7.068e-4 2.001	3.374e-3 1.528	7.669e-4 2.000	3.374e-3 1.528	7.669e-4 2.000
	after	1.339e-6 3.878	4.414e-7 4.06	5.132e-5 3.037	4.530e-7 3.925	5.132e-5 3.037	4.530e-7 3.925
128/160	before	7.729e-4 1.418	1.765e-4 2.002	1.169e-3 1.633	1.917e-4 2.000	1.169e-3 1.6337	1.917e-4 2.000
	after	9.108e-8 3.949	2.631e-8 4.024	6.249e-6 3.489	2.981e-8 4.011	6.249e-6 3.489	2.981e-8 4.011
256/640	before	1.959e-4 1.980	4.404e-5 2.010	3.979e-4 1.655	4.793e-5 2.000	3.979e-4 1.655	4.793e-5 2.000
	after	5.897e-9 3.974	1.617e-9 4.162	5.565e-7 3.920	1.848e-9 4.291	5.565e-7 3.920	1.848e-9 4.291
512/2560	before	4.909e-5	1.093e-5	1.263e-4	1.198e-5	1.263e-4	1.198e-5
	after	3.752e-10	1.778e-11	3.675e-8	9.440e-11	3.675e-8	9.440e-11

Table 4: $\widehat{E}_{\varepsilon}^{M,\Delta s}$ and $\widehat{P}_{\varepsilon}^{M,\Delta s}$ generated on B-S-mesh using the proposed method for Example 2.

ε	Extrapolation	Number of intervals M				
		32/10	64/40	128/160	256/640	512/2560
$1e-6$	before	1.3012e-2	3.5208e-3	9.2402e-4	2.3350e-4	5.8667e-5
		1.8859	1.9299	1.9845	1.9928	
	after	5.4243e-4	4.6020e-5	3.1547e-6	2.0324e-7	1.2801e-8
		3.5591	3.8667	3.9562	3.9889	
$1e-8$	before	1.3012e-2	3.5208e-3	9.2402e-4	2.3350e-4	5.8667e-5
		1.8859	1.9299	1.9845	1.9928	
	after	5.4243e-4	4.6020e-5	3.1547e-6	2.0324e-7	1.2801e-8
		3.5591	3.8667	3.9562	3.9889	
$1e-12$	before	1.3012e-2	3.5208e-3	9.2402e-4	2.3350e-4	5.8667e-5
		1.8859	1.9299	1.9845	1.9928	
	after	5.4243e-4	4.6020e-5	3.1547e-6	2.0324e-7	1.2801e-8
		3.5591	3.8667	3.9562	3.9889	

The surface plots are plotted for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-6}$ in Figure 1 for Example 1 and Figure 4 for Example 2 on S-mesh, respectively. Figures 2 and 5 display the solution for different values ε with respect to time for Example 1 and Example 2, respectively. From these figures, one can visualize that the boundary layers appear at $y = 0$ and $y = 1$. The error plots are given in Figure 3 for Example 1 on the B-S mesh before and after extrapolation. These graphs show that the error is less after extrapolation compared to before extrapolation. To see the numerical convergence rate graphically, $E_{\varepsilon}^{M,\Delta s}$ are plotted in the log-log scale before extrapolation in Figures 6(a), 7(a),8(a), and 9(a) and after extrapolation in Figures 6(b), 7(b),8(b), and 9(b). Moreover, $E_{\varepsilon}^{M,\Delta s}$ and $P_{\varepsilon}^{M,\Delta s}$ by the proposed scheme for Example 1 are presented in Tables 1 and 2 on S-mesh and B-S-mesh, respectively. Similar results for Example 2 are shown in Tables 3 and 4. From these tables, one can conclude that B-S-meshes are more accurate, and the rate of convergence is more compared to the S-mesh. One can notice that the numerical results are well by our theoretical findings.

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