Measurable functions approach for approximate solutions of Linear space-time-fractional diffusion problems

S. Soradi Zeid, A. V. Kamyad* and S. Effati

Abstract

In this paper, we study an extension of Riemann–Liouville fractional derivative for a class of Riemann integrable functions to Lebesgue measurable and integrable functions. Then we used this extension for the approximate solution of a particular fractional partial differential equation (FPDE) problems (linear space-time fractional order diffusion problems). To solve this problem, we reduce it approximately to a discrete optimization problem. Then, by using partition of measurable subsets of the domain of the original problem, we obtain some approximating solutions for it which are represented with acceptable accuracy. Indeed, by obtaining the suboptimal solutions of this optimization problem, we obtain the approximate solutions of the original problem. We show the efficiency of our approach by solving some numerical examples.

Keywords: Riemann–Liouville derivative; Fractional differential equation; Fractional partial differential equation; Lebesgue measurable and integrable function.

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1 Introduction

Fractional calculus is a generalization of classical calculus, which introduces derivatives and integrals of fractional order. Major reviews on the concepts and history of fractional calculus can be found in the book of [21].

Fractional differential equations (FDEs) have been found useful and applicable in science and engineering, such as physics, control theory, biology, finance, biomechanics, and electrochemical processes (see [2, 27] for more details). Most of these FDEs do not have analytic solutions; so there are some papers dealing with their numerical solutions, for example, predictor-corrector method [29], the Adomian decomposition [18], and the variational iteration method [19]. Podlubny in [22, 23] used the matrix expression to unify the formula, where the fractional derivatives are derived from a finite difference. Liu also did some interesting works on the numerical approach for the fractional differential equations in [17, 25] and a method based on collocation using spline functions given by [4].

On the other side, different models using FPDEs have been proposed in [3, 10], and there has been significant interest in developing numerical schemes for their solution. The great difficulty to obtain the numerical solutions of such problems is their solutions, which often are represented in terms of Mittag–Leffler functions, where these functions are in the form of series and their computation is not so easy. One method for solving the FPDE is pursued in the recent paper of [11]. They transformed this partial differential equation into a system of ordinary differential equations, which is then solved by using backward differentiation formulas. In another very recent paper, [26] proposed a finite difference method for the fractional diffusion-wave equation. [15] considered an implicit numerical scheme for fractional diffusion equation. The authors of [8] proposed a method for the solution of time fractional order partial differential equations by converting it into a nonlinear programming problem. [16] constructed an efficient spectral method for the numerical approximation of the space-time fractional diffusion equations, and [5] developed a finite element method for the time fractional Fokker–Planck equation. We refer the interested readers to see [12] for more information.

Based on the above review, we observe that the numerical methods for the FPDEs are abundant and when the function is only integrable are very limited. As far as we searched the related literature there were no reports on similar results as we are investigating in this paper. In this work we focus on a novel applicable approach with a better computational cost based on optimization problem. We consider a class of fractional convection-diffusion equations with measurable (or Lebesgue integrable) functions on the real line $\mathbb{R}$ and obtain a numerical approximation for the Riemann–Liouville derivative of Lebesgue measurable functions. Using the approximated functions whenever needed by a joint application of minimization the total error, we transform the original FPDE into a discrete optimization problem. By ob-

taining the optimal solutions of this problem, we obtain the approximate solution of the original problem. The results are more accurate and more useful than the ones introduced in [7] and should be very interesting for applications in sciences and engineering.

The discussion in the rest of paper will be as follows: in the next section, we introduce several definitions for different types of fractional derivatives of Lebesgue (or Riemann) integrable functions and the notation used in the numerical approximation for FPDE. The case with the FPDEs is displayed in section 3. Furthermore, in this section, we design an efficient approach to approximate the Riemann–Liouville fractional derivative and use it in our numerical approach. Finally, we will give some numerical examples in Section 4. Conclusions are included in the last section.

2 Preliminaries

First of all, let us introduce the properties of measurable functions.

2.1 Riemann–Liouville fractional extension to Lebesgue integral

Assume that (X, M, μ) is a fix measure space [6].

Definition 1. *If (X, M) is a measurable space and $A \subset X$, then the characteristic function χ_A of A is defined by:*

$$\chi_A(t) = \begin{cases} 1, & t \in A, \\ 0, & t \notin A. \end{cases} \tag{1}$$

It is easily checked that χ_A is measurable if and only if $A \in M$. A simple function on X is a finite linear combination, with complex coefficients, of characteristic functions of sets in M. Equivalently, $u : X \to \mathbb{C}$ is simple function if f is measurable and the range f is a finite subset of $\mathbb{C}$. Indeed, we have

$$u = \sum_{i=1}^{n} \gamma_i \chi_{A_i},$$

where $A_i = u^{-1}(\gamma_i)$, $1 \leq i \leq n$, and range$(u) = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$.

Theorem 1. *If $u : X \to [0, \infty)$ is measurable, there is a monotone increasing sequence $\{u_n\}$ of simple functions such that $0 \leq u_1 \leq u_2 \leq \cdots \leq u$, $u_n \to u$ pointwise, and $u_n \to u$ uniformly on any set on which u is bounded.*

Proof. See p. 47 of [6].   □

Theorem 2. *Let $u_n : X \to [0, \infty)$ be an increasing sequence of measurable functions on X and for a.e $x \in X$, $u(x) = \lim_{n \to \infty} u_n(x)$. Then:*

$$\int_X u = \lim_{n \to \infty} \int_X u_n.$$

Proof. See p. 76 of [24].   □

Theorem 3. *If $u_n : X \to [0, \infty)$ is a finite or infinite sequence of measurable functions on X and $u(x) = \sum_n u_n(x)$ for a.e $x \in X$, then $\int_X u = \sum_n \int_X u_n$.*

Proof. See p. 51 of [6].   □

In the special case when the μ is Lebesgue measure on $\mathbb{R}$, the integral, which we have developed, is called the Lebesgue integral. At this point it is appropriate to study the relation between the Lebesgue integrals and the Riemann integrals on $\mathbb{R}$.

Theorem 4. *Let u be a bounded real-valued function on $[a, b]$.*

  (a) *If u is Riemann integrable, then u is Lebesgue measurable and $\int_a^b u = \int_{[a,b]} u$.*

  (b) *u is Riemann integrable if and only if $\{x \in [a, b] : u$ is discontinuous at $x\}$ has Lebesgue measure zero.*

Proof. For more details of this theorem, see p. 57 of [6].   □

We shall generally use the notation $\int_a^b u(t)dt$ for Lebesgue integrals. Now, to compute the Lebesgue integral of u, we define the following partition on $[0, 1]$.

Definition 2. *We define the regular partition P_n on $[0, 1]$ by:*

$$P_n = \left\{ 0 = \delta_0, \delta_1, \cdots, \delta_n = 1 \right\},$$

where $\delta_0 < \delta_1 < \cdots < \delta_n$. Here, according to the regularization of the partitions for each $i = 0, 1, \ldots, n$, we have $\delta_i = \dfrac{i}{n}$ and the partition norm is defined as:

$$\|P_n\|_\infty = \max_i \left\{ |\delta_i - \delta_{i-1}| \right\} = \frac{1}{n}.$$

In the following, we introduce the definitions of fractional integral and derivatives of Lebesgue measurable and integrable functions that can be seen as a generalization of the classical derivative.

2.2 Fractional calculus

There are several different ways to define the fractional derivatives, and the most commonly used fractional derivatives are the Grunwald–Letnikov derivative, the Riemann Liouville derivative, and the Caputo derivative. For the definitions of fractional derivatives and some of their applications, see [14].

Let $[a, b] \in \mathbb{R}$. We will denote the space of all measurable and Lebesgue integrable real functions defined on $[a, b]$ by $L_1[a, b]$; that is,

$$L_1[a, b] = \left\{ u : \ \|u\|_{L_1[a,b]} = \int_a^b |u(t)| dt < \infty \right\},$$

and the space of all measurable and essential bounded real functions defined on $[a, b]$ is denoted by $L^\infty[a, b]$.

Definition 3. *[13]. The fractional integral operator of order $\alpha > 0$ of a function $u(.) \in L_1([a, b], \mathbb{R}^n)$, is defined as:*

$$_aI_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} u(\tau) d\tau, \tag{2}$$

where Γ is the classical gamma function and the integral of (2) exists and is in the sense of Riemann integral.

We may claim that the fractional integral (2) is a linear operator on the space of Lebesgue measurable functions; that is, if $f, g \in L_1[a, b]$ and C is a constant, then:

$$_aI_t^\alpha(Cf + g) = C \ _aI_t^\alpha(f) + \ _aI_t^\alpha(g).$$

Proposition 1. *The fractional integral operator $_aI_t^\alpha$, $0 < \alpha < 1$, is bounded or equivalently continuous in $L_1[a, b]$:*

$$\| \ _aI_t^\alpha(u)\|_{L_1[a,b]} \leq P_\alpha \|u\|_{L_1[a,b]}, \qquad P_\alpha = \frac{(b - a)^\alpha}{\Gamma(\alpha + 1)}.$$

Proof. See its proof in [13]. $\square$

Definition 4. *Let $u(.) \in L_1[a, b]$. The Riemann–Liouville fractional derivative (RLFD) of order $\alpha > 0$ is defined as:*

$$_aD_t^\alpha u(t) = \frac{d^m}{dt^m} \ _aI_t^{m-\alpha} u(t) = \frac{1}{\Gamma(m - \alpha)} \frac{d^m}{dt^m} \int_a^t (t - \tau)^{m-\alpha-1} u(\tau) d\tau, \tag{3}$$

where $m - 1 < \alpha \leq m$ and $m \in \mathbb{N}$.

The above fractional derivative is equal to the fractional derivative of Riemann-integrable functions just on the compact subsets of $\mathbb{R}$. On the other hand, there are some functions that are only measurable and not Riemann

integrable but its derivations compute from (3). In this case, consider the following function:

$$f(t) = \begin{cases} 1, & t \in \mathbb{Q} \cap [0,1], \\ -1, & t \in \mathbb{Q}^c \cap [0,1]. \end{cases} \tag{4}$$

We know that $f(.)$ is not Riemann integrable function but its Lebesgue integrable. Now, we want to compute the fractional derivative of $f(.)$, for $0 < \alpha < 1$, as follows:

$$\begin{aligned} {}_0D_t^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \int_{\mathbb{Q} \cap [0,t]} (t-\tau)^{-\alpha} d\tau - \int_{\mathbb{Q}^c \cap [0,t]} (t-\tau)^{-\alpha} d\tau \right\}. \end{aligned}$$

As, $m([0,t]) = m([0,t] \cap \mathbb{Q}] + m([0,t] \cap \mathbb{Q}^c) = t$, we will have

$$ {}_0D_t^\alpha f(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{\mathbb{Q}^c \cap [0,t]} (t-\tau)^{-\alpha} d\tau. $$

According to Theorem 4, the above statement is equivalent to Riemann integral and the result of this computation has been shown in Figure 1 for $\alpha = 0.5, 0.8, 0.9, 0.95$.
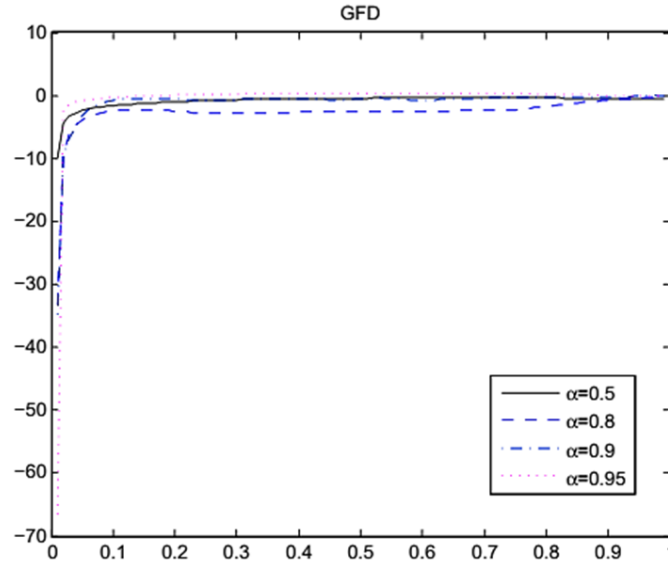

Figure 1: Fractional derivative of function (4) with different values of α

The following theorem help us to apply a fractional integral over a fractional derivative.

Theorem 5. *Let $\alpha > 0$.*

(1) $_aD_t^n \, _aD_t^\alpha u(t) = \, _aD_t^{n+\alpha}u(t)$, $n \in \mathbb{N}$, *where $\mathbb{N}$ is the set of natural numbers.*

(2) *The equality* $_aD_t^\alpha \, _aI_t^\alpha u(t) = u(t)$ *holds for every* $u(.) \in L_1(0,1)$.

(3) *For* $u(.) \in L_1(0,1)$, $n = [\alpha] + 1$, *if* $_aI_t^{n-\alpha}u \in AC^{n-1}[0,1]$, *then*

$$_aI_t^\alpha \, _aD_t^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \left(\frac{d^{n-k-1}}{dt^{n-k-1}} \, _aI_t^{n-\alpha}u \right)(a),$$

in which $AC[a,b]$ denotes the space of absolutely continuous real functions on $[a,b]$, and

$$AC^n[a,b] = \left\{ u : [a,b] \to \mathbb{R} : \frac{d^{n-1}u(t)}{dt^{n-1}} \in AC[a,b] \right\}.$$

In particular, if $0 < \alpha \leq 1$ and $_aD_t^\alpha u(t) \in L_1[a,b]$, then

$$_aI_t^\alpha \, _aD_t^\alpha u(t) = u(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \left(_aI_t^{1-\alpha}u(a) \right).$$

Proof. See pp. 241–256 of [13]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, we introduce the definitions of Caputo fractional derivative as follows.

Definition 5. *[13]. Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $u \in AC^n[a,b]$. The Caputo derivative of order $\alpha > 0$ is defined as*

$$_a^C D_t^\alpha u(t) = \, _aI_t^{n-\alpha} \frac{d^n}{dt^n} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \frac{d^n u(\tau)}{d\tau^n} d\tau.$$

In the space of the functions belonging to $AC^n[a,b]$ the following relation between Riemann–Liouville and Caputo derivative holds [9].

Theorem 6. *For $u \in AC^n[a,b]$, $n = \lceil \alpha \rceil$, the Riemann–Liouville derivative of order α of u exists almost everywhere and it can be written as*

$$_aD_t^\alpha u(t) = \, _a^C D_t^\alpha u(t) + \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} u^{(k)}(a). \tag{5}$$

In order to simplify our discussion, throughout this paper, we will consider the Riemann–Liouville definition, since most useful tools have been

established by using the Riemann–Liouville definition. It is worthwhile to note, by virtue of (5), that for the homogeneous condition considered here the Riemann–Liouville definition coincides with the Caputo version.

3 Numerical approximation of FPDEs

Consider the space-time fractional convection-diffusion equation in one dimension as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^\beta u(x,t)}{\partial x^\beta} = f(x,t), \quad x,t \in [0,1], \alpha \in (0,1], \ \beta \in (1,2], \quad (6)$$

subject to the following initial and boundary conditions:

$$u(x,0) = g(x), \qquad x \in [0,1], \tag{7}$$

$$u(0,t) = u(1,t) = 0, \qquad t \in [0,1], \tag{8}$$

where g is given a smooth function and f is a measurable function on $[0,1]$. Here, $\frac{\partial^\alpha}{\partial t^\alpha}$ is defined as the RLFD of order α and for all $x \in [0,1]$ it is given by:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t u(x,s)(t-s)^{-\alpha} ds \quad t \in [0,1], \tag{9}$$

and for all $t \in [0,1]$, $\frac{\partial^\beta}{\partial x^\beta}$ is the RLFD of order β that is given by:

$$\frac{\partial^\beta}{\partial x^\beta} u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{d^2}{dx^2} \int_0^x u(\xi,t)(x-\xi)^{1-\beta} d\xi \quad x \in [0,1]. \tag{10}$$

Now, we assume that the integrals in equations (9) and (10) are well-defined. Also, suppose that the function u satisfies the Carathéodory conditions with respect to $L_1[0,1]$, which means, a non-negative measurable function is positive on a set of positive measure or, equivalently, the following conditions hold:

(C_1) For each $x \in [0,1]$, the mapping $t \mapsto u(x,t)$ is Riemann integrable.

(C_2) For a.e $t \in [0,1]$, the mapping $x \mapsto u(x,t)$ is continuous on $[0,1]$.

(C_3) Let $u(x,t) : [0,1] \times [0,1] \to [0,\infty)$ be a bounded function a.e on $L_1[0,1]$.

A standard assumption of continuity of the inhomogeneous term can be replaced with that of satisfying the Carathéodory conditions with respect to $L_1[0,1]$.

With specific α and β in equation (6), when $\alpha = 1$ and $\beta = 2$, we have the following classical diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \qquad x,t \in [0,1]. \tag{11}$$

In fact the time derivative and space derivative of integer order in (11) can be obtained by taking the limit $\alpha \to 1^-$ in (9) and $\beta \to 2^-$ in (10).

In the following, we present a numerical approximation for the Riemann–Liouville derivative of Lebesgue measurable functions such that these operators simplify to the classical RLFI, and Riemann–Liouville fractional derivatives (RLFD) and also the numerical method that gives an approximate solution to the fractional convection-diffusion equations with measurable (or Lebesgue integrable) functions on real line $\mathbb{R}$, studied in this paper.

In the analysis of the numerical approach that follows, we will assume that this space-time fractional convection diffusion equation has a unique and sufficiently smooth solution. To establish the numerical approximation, and in order to simplify the notation, we consider

$$\Upsilon_\alpha^{x,u}(t) = \int_0^t u(x,s)(t-s)^{-\alpha} ds, \qquad 0 < \alpha < 1. \tag{12}$$

$$\Lambda_\beta^{t,u}(x) = \int_0^x u(\xi,t)(x-\xi)^{1-\beta} d\xi, \qquad 1 < \beta < 2. \tag{13}$$

Let $t_n = n\Delta t$, $n = 0,1,\ldots,K$, where $\Delta t := \frac{1}{K}$ is the regular partition of time interval and $x_j = j\Delta x$, $j = 0,1,\ldots,N$ and $\Delta x := \frac{1}{N}$ is the regular partition of space interval and K and N are two positive natural numbers. To motivate the construction of the scheme, we now define two operators that will be focus on them. We use the following approximation at t_n for all $0 \leq n \leq K$,

$$\left(\frac{d}{dt}\Upsilon_\alpha^{x,u}\right)(t_n) \simeq \frac{1}{\Delta t}\left[\Upsilon_\alpha^{x,u}(t_n) - \Upsilon_\alpha^{x,u}(t_{n-1})\right], \tag{14}$$

and the following approximation at x_j, for all $0 \leq j \leq N$,

$$\left(\frac{d^2}{dx^2}\Lambda_\beta^{t,u}\right)(x_j) \simeq \frac{1}{\Delta x^2}\left[\Lambda_\beta^{t,u}(x_{j-1}) - 2\Lambda_\beta^{t,u}(x_j) + \Lambda_\beta^{t,u}(x_{j+1})\right]. \tag{15}$$

For each t_n and x_j, respectively, we need to calculate $\Upsilon_\alpha^{x,u}(t_n)$ and $\Lambda_\beta^{t,u}(x_j)$. So, first, we compute (12) by approximating $u(x,s)$ at the fixed constant x by simple functions. Therefore, for each $s \in [0,1]$, we define $U(x,s)$ as an approximation of $u(x,s)$ at the fixed constant x, such that

$$U(x,s) = \sum_{i=1}^K u(x,t_i)\chi_{A_i}(s), \tag{16}$$

in which A_i is a sequence of disjoint sets in $[0,1]$ and $t_i = \frac{i}{K}$, $i = 1,2,\ldots,K$. Now, we prove the uniformly convergence theorem of our approach.

Theorem 7. *Let $u(x,t)$ satisfy conditions $(C_1) - (C_3)$, and let $U(x,t)$ be defined in (16) for the fixed constant x. If K tends to infinity, then $U(x,t)$ tends to $u(x,t)$ uniformly on $[0,1]$.*

Proof. The proof is obtained from Theorem 1. $\qquad\square$

This theorem means where we choose K as a sufficiently big number, closeness $U(x,t)$ to $u(x,t)$, at the fixed constant x, is independent of $t \in [0,1]$. So, by using the approximation (16) for $u(x,s)$ and suppose that $s \in [0,t_n]$, we will have

$$I_\alpha(t_n) = \int_0^{t_n} \sum_{i=1}^n u(x,t_i)\chi_{A_i}(s)(t_n - s)^{-\alpha}ds,$$

that is an approximation to $\Upsilon_\alpha^{x,u}(t_n)$. So,

$$I_\alpha(t_n) = \sum_{i=1}^n u(x,t_i)\int_0^{t_n}(t_n-s)^{-\alpha}\chi_{A_i}(s)ds = \sum_{i=1}^n u(x,t_i)\int_{[0,t_n]\cap A_i}(t_n-s)^{-\alpha}ds.$$

We denote $[0,t_n] \cap A_i$ with $A_{i,n}$, and note that every bounded Riemann integrable function defined on a bounded interval is Lebesgue integrable and the two integrals are the same (Theorem 4). Then we have

$$I_\alpha(t_n) = \frac{1}{1-\alpha}\sum_{i=1}^n u(x,t_i)\Big[-(t_n-s)^{1-\alpha}\Big]\Big|_{A_{i,n}} = \frac{1}{1-\alpha}\sum_{i=1}^n u(x,t_i)W_{i,n}^{(\alpha)},$$

$$(17)$$

where

$$W_{i,n}^{(\alpha)} = \Big[-(t_n-s)^{1-\alpha}\Big]\Big|_{A_{i,n}}$$

and $t_i = \frac{i}{K}$ for $i = 1, 2, \ldots, n$.

Finally an approximation for (14) is given by $\frac{1}{\Delta t}[I_\alpha(t_n) - I_\alpha(t_{n-1})]$; that is,

$$\Big(\frac{d}{dt}\Upsilon_\alpha^{x,u}\Big)(t_n) \simeq \frac{1}{(1-\alpha)\Delta t}\sum_{i=1}^n u(x,t_i)\Big(W_{i,n}^{(\alpha)} - W_{i,n-1}^{(\alpha)}\Big).$$

Denote U_j^n as an approximation to the value $u(x_j,t_n)$ on grid point (x_j,t_n). Then we take the following approximation for time fractional derivative (9) appeared in problem (6) for the fixed instant $x_j = j\Delta x \in [0,1]$,

$$\delta_t^\alpha U_j^n = \begin{cases} \dfrac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^n u(x_j,t_i)q_{i,n}^{(\alpha)}, & 0 < \alpha < 1, \\[2ex] \dfrac{u(x_j,t_n) - u(x_j,t_{n-1})}{\Delta t}, & \alpha = 1, \end{cases}$$

$$(18)$$

in which $q_{i,n}^{(\alpha)} = W_{i,n}^{(\alpha)} - W_{i,n-1}^{(\alpha)}$, and then we have

$$q_{i,n}^{(\alpha)} = \begin{cases} (n-i-1)^{1-\alpha} - 2(n-i)^{1-\alpha} + (n-i+1)^{1-\alpha} , & i \leq n-1, \\ 1, & i = n. \end{cases} \tag{19}$$

Incidentally, we find that $|q_{i,n}^{(\alpha)}|$ are bounded for all $\alpha \in [0,1]$ and all $i \geq 1$, as proven in the following Lemma.

Lemma 1. *For all $\alpha \in (0,1]$ and all $i \geq 1$, it holds $|q_{i,n}^{(\alpha)}| \leq c$, where c is dependent on n and α.*

Proof. First, for $\alpha = 1$ a direct calculation shows that $q_{i,n}^{(\alpha)}$ is a constant, for all $i \geq 1$. Now we prove the lemma for $\alpha \in (0,1)$. It can be verified that

$$A_{i,n-1} \subset A_{i,n} \subseteq [0, t_n] \Longrightarrow m(A_{i,n-1}) < m(A_{i,n}) \leq t_n.$$

In addition, It is well known that if $A \subseteq B$ and f is a measurable function on A and B, then $\int_A f \leq \int_B f$, and by consequence, we have

$$0 \leq W_{i,n-1}^{(\alpha)} \leq W_{i,n}^{(\alpha)} \leq t_n^{2-\alpha}.$$

So the proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2. *Let $U(t) = \sum_{i=1}^{K} u(t_i) \chi_{A_i}(s)$, as explained in (16), be a sum of non-negative monotone increasing of simple functions that approximates the function $u(t) \in L_1[0,1]$. Then $\delta_t^\alpha(U(t))$, which is achieved by (18), tends to $D_{0,t}^\alpha u(t)$ as K tends to infinity.*

Proof. Using Lemma 1, and taking into account the fact that the phrase $(t-s)^{-\alpha}$, $0 < \alpha < 1$, is a continuous and integrable function and $\chi_{A_i}(s)$ is a measurable function on $[0,t]$, and then according to the convergence theorems of measurable functions (see Theorem 2 and Theorem 7 of this paper), the proof is obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Similarly, we approximate $u(\xi, t)$ at the fixed instant t by simple functions

$$\begin{cases} U(\xi, t) = \sum_{i=1}^{N} u(x_i, t) \chi_{B_i}(\xi), \\ \cup_{i=1}^{N} B_i = [0,1], B_i \cap B_j = \emptyset; i \neq j, x_i = \frac{i}{N}, \end{cases} \tag{20}$$

where $B_{i,j} = [0, x_j] \cap B_i$, $B_i = [\frac{i-1}{N}, \frac{i}{N}]$, and $x_j = \frac{j}{N}$. Now, we can approximate the space fractional derivative (10) for the fixed instant $t_n = n\Delta t \in [0,1]$ to form

$$\delta_x^\beta U_j^n = \begin{cases} \dfrac{(\Delta x)^{-\beta}}{\Gamma(3-\beta)} \sum_{i=1}^{j+1} u(x_i, t_n) q_{i,j}^{(\beta)}, & 1 < \beta < 2, \\ \dfrac{u(x_{j-1}, t_n) - 2u(x_j, t_n) + u(x_{j+1}, t_n)}{\Delta x^2}, & \beta = 2, \end{cases} \tag{21}$$

in which $q_{i,j}^{(\beta)} = W_{i,j-1}^{(\beta)} - 2W_{i,j}^{(\beta)} + W_{i,j+1}^{(\beta)}$ and $W_{i,j}^{(\beta)} = \left[-(x_j - s)^{2-\beta} \right]\Big|_{B_{i,j}}$,
and we have

$$
q_{i,j}^{(\beta)} = \begin{cases}
(j-i+2)^{2-\beta} - 3(j-i+1)^{2-\beta} + 3(j-i)^{2-\beta} - (j-i-1)^{2-\beta}, & i \leq j-1, \\
-3 + 2^{2-\beta}, & i = j, \\
1, & i = j+1.
\end{cases}
\tag{22}
$$

Now, for obtaining the solution of space-time fractional convection-diffusion equation (6)–(8), we apply (18) and (21) to (6) and reach the following numerical method,

$$
\delta_t^\alpha U_j^n - \delta_x^\beta U_j^n = f(x_j, t_n),
\tag{23}
$$

$$
U_j^0 = g(x_j), \ U_0^n = U_N^n = 0, \quad j = 1, \ldots, N, \ n = 1, \ldots, K,
\tag{24}
$$

So, if $u(x,t)$ is a solution of system (6)-(8), equivalently it is a solution of system of equations (23)–(24). Hence, we mention the following main theorem of this section.

Theorem 8. *The truncation error for the numerical approximation (23)–(24) is of order $O(\Delta x)^{2-\beta} + O(\Delta t)^{1-\alpha}$.*

Proof. Let $u(x,t)$ be a solution to the fractional partial differential equation (6) in which satisfies the conditions (7) and (8). Then we have

$$
\frac{\partial^\alpha}{\partial t^\alpha} u(x, t_n) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \Upsilon_\alpha^{x,u}(t_n)
$$

$$
= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Delta t} \left(\Upsilon_\alpha^{x,u}(t_n) - \Upsilon_\alpha^{x,u}(t_{n-1}) \right) + \epsilon_1(t_n),
\tag{25}
$$

in which $\epsilon_1(t_n) = O(\Delta t)$. Let us define the error $E(t)$, such that,

$$
\Upsilon_\alpha^{x,u}(t_n) - \Upsilon_\alpha^{x,u}(t_{n-1}) = I_\alpha(t_n) - I_\alpha(t_{n-1}) + E_1(t).
\tag{26}
$$

So, we have

$$
\frac{\partial^\alpha}{\partial t^\alpha} u(x, t_n) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Delta t} \left(I_\alpha(t_n) - I_\alpha(t_{n-1}) \right) + \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Delta t} E_1(t) + O(\Delta t).
\tag{27}
$$

Now, by using Lemma 1 we have $E_1(t) = O(\Delta t)^{2-\alpha}$. So it follows that:

$$
\frac{\partial^\alpha}{\partial t^\alpha} u(x, t_n) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Delta t} \left(I_\alpha(t_n) - I_\alpha(t_{n-1}) \right) + \frac{1}{\Gamma(1-\alpha)} O(\Delta t)^{1-\alpha} + O(\Delta t).
$$

Therefore,

$$\left(\frac{\partial^\alpha}{\partial t^\alpha}u\right)(x_j, t_n) = \delta_t^\alpha U_j^n + O(\Delta t)^{1-\alpha}. \tag{28}$$

Similarly, for the fractional derivative of order β, we have

$$\left(\frac{\partial^\beta}{\partial x^\beta}u\right)(x_j, t_n) = \delta_x^\beta U_j^n + O(\Delta x)^{2-\beta}. \tag{29}$$

Finally, by using the numerical method (23)–(24), we will have

$$\|U_j^n - u_j^n\|_\infty = O(\Delta t)^{1-\alpha} + O(\Delta x)^{2-\beta}, \tag{30}$$

and the proof is completed. $\square$

In the remainder of this section, it will be shown that equations (23)–(24) can be converted to an optimization problem. For this suggested approach we need the following theorem.

Proposition 2. *If $h : [a, b] \to [0, \infty)$ is a measurable function, the necessary and sufficient condition for $\int_a^b h = 0$ is that $h \equiv 0$ a.e., on $[a, b]$.*

Proof. See p. 51 of [6]. $\square$

So, according to Proposition 2, necessary and sufficient condition for $u(x, t)$ to be a solution of system (23)–(24) is that the optimal solution of the following problem is zero (see Theorem 1 of [1] for convergence):

$$\min \int_0^1 \int_0^1 \left| \delta_t^\alpha U(x,t) - \delta_x^\beta U(x,t) - f(x,t) \right| dt dx. \tag{31}$$

By trapezoidal method and using the ending point in any subinterval for approximating integrals and using approximations (18) and (21) for (31), we obtain the following discretized problem:

$$\min_u \frac{\Delta t \Delta x}{c} \sum_{j=1}^N \sum_{n=1}^K \left| a \sum_{i=1}^n u(x_j, t_i) q_{i,n}^{(\alpha)} - b \sum_{i=1}^{j+1} u(x_i, t_n) q_{i,j}^{(\beta)} - c f(x_j, t_n) \right|, \tag{32}$$

where $a = (\Delta x)^\beta \Gamma(3 - \beta)$, $b = (\Delta t)^\alpha \Gamma(2 - \alpha)$, $c = ab$, and $q_{i,n}^{(\alpha)}$, $q_{i,j}^{(\beta)}$ are as before. Now, we convert the problem (32) to a linear programming problem with the following change of variables (see Theorem 3 and Lemma 2 of [1] and [28] for more details):

$$\min_{u} \frac{\Delta t \Delta x}{c} \sum_{j=1}^{N} \sum_{n=1}^{K} \left(r_j^n + e_j^n \right) \tag{33}$$

$$s.t. \ \ a \sum_{i=1}^{n} u_j^i q_{i,n}^{(\alpha)} - b \sum_{i=1}^{j+1} u_i^n q_{i,j}^{(\beta)} - c f(x_j, t_n) = r_j^n - e_j^n$$

$$u_j^0 = g(x_j), \ u_0^n = u_N^n = 0,$$

$$r_j^n, e_j^n \geq 0, j = 1, \ldots, N, \ n = 1, \ldots, K.$$

Finally, by obtaining the solution of problem (33), we recognize the value of unknown variables.

4 Numerical examples

In this section, we give some numerical examples and apply the presented approximation for solving them. These test problems demonstrate the validity and efficiency of this techniques.

Example 1. We compute $_0D_t^\alpha x(t)$, with $\alpha = 0.5$, for $x(t) = t^4$ by approximation (18).

The exact formulas of the derivatives are derived from

$$_0D_t^{0.5}(t^s) = \frac{\Gamma(s+1)}{\Gamma(s+1-0.5)} t^{s-0.5},$$

Figure 2 shows the results. Since the exact solution for this problem is known, we compute the approximation error by using the maximum norm for each K. Assume that $\bar{x}(t_n)$, $n = 1, \ldots, K$, are the approximated values on the discrete time horizon $t_1, \ldots, t_K$. Then the error is given by

$$E = \max_{n} \left(|x(t_n) - \bar{x}(t_n)| \right). \tag{34}$$

In the case of $\alpha = 0.5$, with various choices of K, the maximum absolute errors are computed by equation (34) and shown in Table 1.

Example 2. Consider the following fractional differential problem:

$$_0D_t^\alpha x(t) = g(x(t), t), \tag{35}$$

where $0 < \alpha \leq 1$, $x(0) = 0$ and $g(x, t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} - x(t) + t^2 - t$.
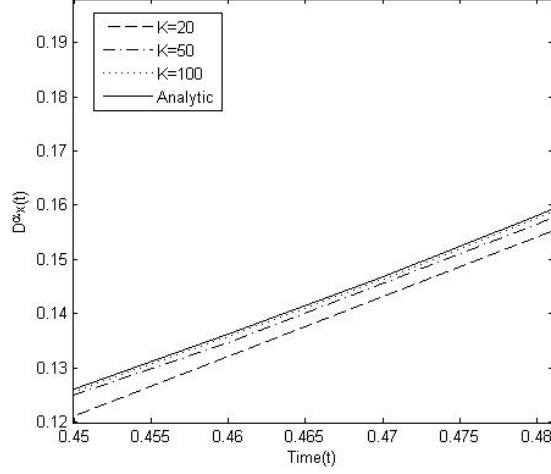
Figure 2: Analytic solution and numerical approximation (18), with various choices of K, for Example 1

Table 1: Maximum absolute error for Example 1

K	E_{appro}
5	2.65757×10^{-3}
10	6.47115×10^{-4}
20	4.09595×10^{-4}
40	9.77932×10^{-5}
50	2.27352×10^{-5}
100	5.04375×10^{-6}

The exact solution of this equation is $x(t) = t^2 - t$. We can rewrite the equation (35) as follows:

$$_0D_t^\alpha x(t) - g(x(t), t) = 0. \tag{36}$$

Also, we have from equation (36) that

$$\left| _0D_t^\alpha x(t) - g(x(t), t) \right| = 0. \tag{37}$$

By using again Proposition 2, we conclude that if $x(t)$ is a solution of equation (37) with the initial condition $x(0) = 0$, equivalently it is a solution of the following optimization problem:

$$\min \int_0^1 \left| _0D_t^\alpha x(t) - g(x(t), t) \right| dt. \tag{38}$$

Then by discretization the integral as before an using equation (18) for approximate of $_0D_t^\alpha x(t)$, we simplify obtained problem (38) as follows:

$$\min_{x_i} \sum_{i=1}^{N} \sum_{k=1}^{K} \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \Big| \sum_{p=1}^{k} x_i(t_p) q_{p,k}^{(\alpha)} \tag{39}$$
$$- (\Delta t)^\alpha \Big(\frac{2t_k^{2-\alpha}}{2-\alpha} - t_k^{1-\alpha} - \Gamma(2-\alpha)(x_i(t_k) - t_k^2 + t_k) \Big) \Big|,$$

where t_p and $q_{p,k}^{(\alpha)}$ are the same as before. Now, for solving the minimum problem (39), we can change it to a linear programming problem like (33). In Figures 3 and 4 we compare the exact solution with numerical approximation (39) for two different values of K, $N = 100$, and $\alpha = 0.5$.
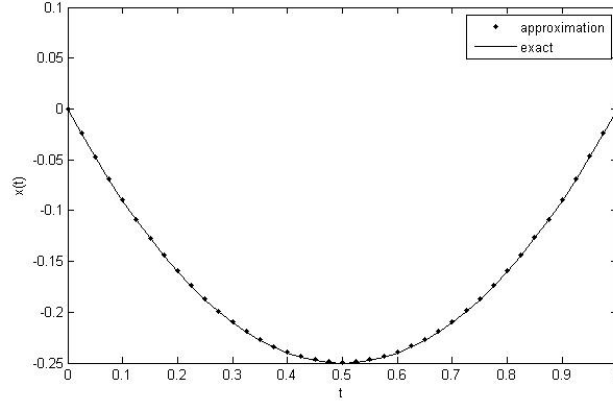

Figure 3: Analytic solution and numerical approximation (39) for Example 2 for $K = 40$

Table 2 shows the exact solution and the approximate solution for equation (35) by using problem (39) for $K = N = 100$ and $\alpha = 0.5, 0.99$. The results compare well with those obtained in [20]. From numerical results we can indicate that the solution of FDE approaches to the solution of integer order differential equation, whenever α approaches to its integer value.

Example 3. We consider the space-time fractional diffusion equation (6) with the following initial and boundary conditions:

$$u(x, 0) = 0, \qquad 0 < x < 1,$$

$$u(0, t) = u(1, t) = 0, \qquad t \in [0, 1],$$

where

$$f(x, t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} sin(2\pi x) + 4\pi^2 t^2 sin(2\pi x).$$
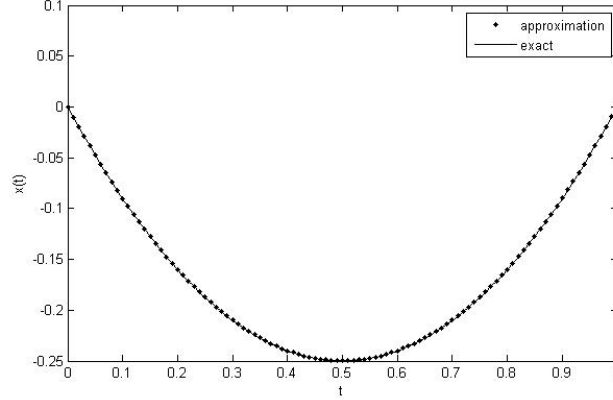
Figure 4: Analytic solution and numerical approximation (39) for Example 2 for $K = 100$

Table 2: Numerical values for Example 2 when $\alpha = 0.5, 0.99$

t	$x_{appro}(\alpha = 0.5)$	x_{exact}	$x_{appro}(\alpha = 0.99)$	x_{exact}
0.0	0.000000	0.000000	0.000000	0.000000
0.1	-0.089978	-0.090000	-0.089586	-0.090000
0.2	-0.159889	-0.160000	-0.159688	-0.160000
0.3	-0.209891	-0.210000	-0.209707	-0.210000
0.4	-0.239974	-0.240000	-0.239787	-0.240000
0.5	-0.249896	-0.250000	-0.249738	-0.250000
0.6	-0.239998	-0.240000	-0.239795	-0.240000
0.7	-0.199879	-0.210000	-0.209830	-0.210000
0.8	-0.160109	-0.160000	-0.159897	-0.160000
0.9	-0.096390	-0.090000	-0.100098	-0.090000

The exact solution of this problem when $\beta = 2$ is given by $u(x,t) = t^2 sin(2\pi x)$. The spatial and temporal meshes are taken uniform; that is, $\Delta t = \frac{1}{K}$, $\Delta x = \frac{1}{N}$. From the Table 3 and Figures 5, 6, and 7, we get that the approximate analytical solution is consistent with the exact solution when $\alpha \to 1$, $\beta \to 2$ and different values for K and N. Therefore, the validity of our numerical methods is confirmed.

Now, consider the vector $U(\Delta x) = (U_0, U_1, \cdots, U_N)$, where U_j is the approximate solution, for $x_j = j\Delta x$, $j = 0, 1, \ldots, N$ at a certain time t, and $u(\Delta x) = (u(x_0, t), \ldots, u(x_N, t))$, where u is the exact solution. The absolute error is defined as follows:

$$\|u(\Delta x) - U(\Delta x)\|_\infty = \max_{0 \leq j \leq N} |u(x_j, t) - U_j|. \tag{40}$$

Table 3: Maximum absolute error (40) for the numerical scheme of Example 3 at $t = 1$ for different values of α and $\beta = 2$ with various choices of Δt and Δx

	Δt	Δx	E_{appro}
	1/5	1/5	0.00847
$\alpha = 0.5$	1/10	1/10	0.00431
	1/15	1/15	5.32982×10^{-4}
	1/20	1/20	7.05965×10^{-5}
	1/5	1/5	0.00480
$\alpha = 0.9$	1/10	1/10	1.64153×10^{-4}
	1/15	1/15	7.19164×10^{-5}
	1/20	1/20	7.39623×10^{-6}
	1/5	1/5	0.00536
$\alpha = 0.95$	1/10	1/10	1.00857×10^{-4}
	1/15	1/15	3.38148×10^{-5}
	1/20	1/20	2.41435×10^{-6}
	1/5	1/5	0.00568
$\alpha = 0.98$	1/10	1/10	2.31643×10^{-4}
	1/15	1/15	4.62727×10^{-5}
	1/20	1/20	5.70822×10^{-6}
	1/5	1/5	0.00579
$\alpha = 0.99$	1/10	1/10	1.14488×10^{-4}
	1/15	1/15	9.19805×10^{-5}
	1/20	1/20	2.82570×10^{-6}

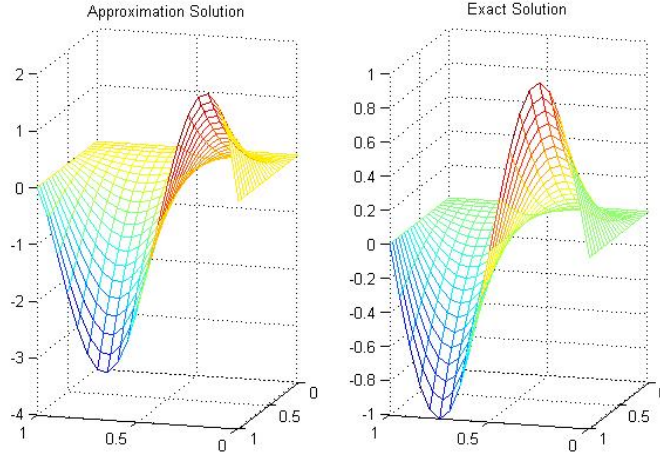

Figure 5: The evolution of $u(x, t)$ for anomalous diffusion coefficients $\alpha = 0.5$, $\beta = 2$ for Example 3
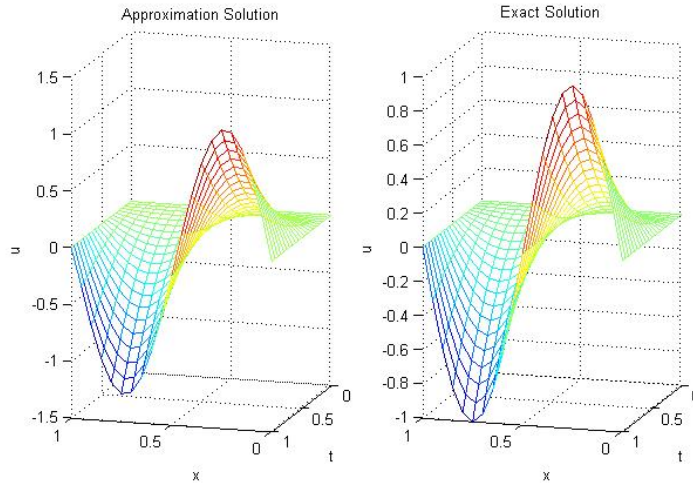
Figure 6: The evolution of $u(x,t)$ for anomalous diffusion coefficients $\alpha = 0.9$, $\beta = 2$ for Example 3
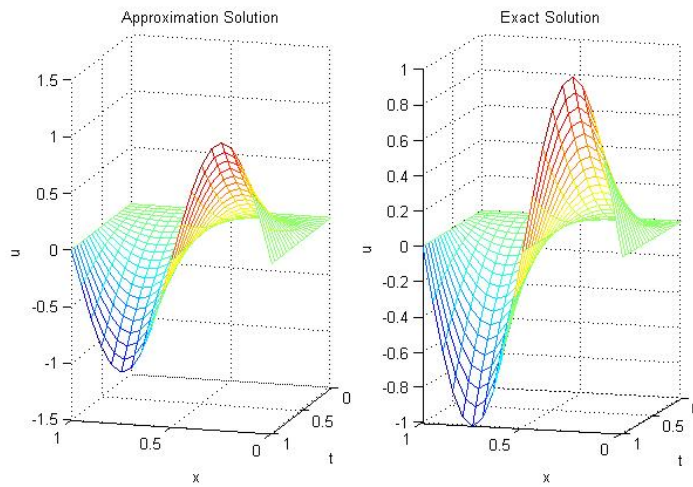

Figure 7: The evolution of $u(x,t)$ for anomalous diffusion coefficients $\alpha = 0.98$, $\beta = 2$ for Example 3

Table 4 presents a comparison between the absolute error of our method and the method presented by Neville et al. (2011) in [7] with different space and time steps. It should be noted that our numerical results, quickly converging to the exact solution with a lower division, are more accurate than with the results obtained in [16].

Table 4: Comparision of absolute error for the numerical scheme of Example 3 at $\alpha = 0.5$ and $\beta = 2$ with various choices of K, N

K	N	our method	K	N	estimated in [7]
5	5	0.000360	4	100	0.000926
10	10	0.000133	8	100	0.000337
15	15	5.12841×10^{-6}	16	100	0.000120
20	20	8.33727×10^{-6}	32	100	4.184925×10^{-5}

Example 4. we consider the following space-time fractional diffusion problem:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\Gamma(2.8)x}{2}\frac{\partial^\beta u(x,t)}{\partial x^\beta} = q(x,t) - x^{0.8}\frac{\partial u}{\partial x}, \qquad (x,t) \in [0,1] \times [0,1] \tag{41}$$

with the initial condition:

$$u(x,0) = x^2(1-x), \qquad x \in [0,1], \tag{42}$$

and the boundary condition:

$$u(0,t) = u(1,t) = 0, \qquad t > 0, \tag{43}$$

where

$$q(x,t) = \frac{2x^2(1-x)^{\frac{1}{2}}}{\Gamma(2.2)} + 0.2x^{1.8}(1+t^2).$$

The exact solution of to this FPDE for $\alpha = 0.8$ and $\beta = 1.5$ is given by $u(x,t) = x^2(1-x)(1+t^2)$. Table 5 shows the absolute errors at time $t = 1$, between the analytical solution and the numerical solution obtained by applying the method discussed in this paper and the method presented in [30]. It should be noted that our results are very closely identical with other results.

Table 5: Comparision of absolute error for the numerical scheme of Example 4 for $\alpha = 0.8$ and $\beta = 1.5$ at $t = 1$ with various choices of K and N

K	N	our method	estimated in [30]
10	10	1.23×10^{-3}	1.26×10^{-3}
20	20	5.86×10^{-4}	6.74×10^{-4}
40	40	2.87×10^{-4}	3.48×10^{-4}
80	80	8.3×10^{-5}	8.6×10^{-5}

5 Conclusion

FDEs and FPDEs have caught considerable attentions due to their various applications. Since there is no systematic method to derive the exact solution of these equations, in this paper we partly solved a class of FPDE and obtained some approximating solution for it by provide an efficient approach based on measurable functions. For this purpose, at first, we present an approximation for the fractional derivatives of measurable functions. Then, by using this approximation and application of minimization the total error, we transform the original FPDE into a discrete optimization problem such that the optimal solutions of this problem is the approximate solution of the original problem. The suggested method represents a unifying approach for the solution of partial differential equations of fractional order.

From the numerical examples and the results that are compared with the exact solutions and with the other methods, it is shown that, as the number of discrete points, K and N, in proposed approach of this paper was increased, the solutions converged to the exact solutions. In addition, as the value of α approaches one and β approaches two, the numerical solutions for both the FDE and FPDE approach the analytical solutions for $\alpha = 1$ and $\beta = 2$. Since the proposed approximation of this paper is based on the minimization of total error, it is clear from the results that there is no difference between exact and approximate solution in point to point case.

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حل تقریبی مساله خطی زمان-مکان کسری با استفاده از توابع اندازه‌پذیر

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چکیده : در این مقاله، مشتق کسری ریمان-لیوویل را برای توابع انتگرال‌پذیر لبگ توسیع خواهیم داد. سپس این تقریب را برای حل رده‌ای خاص از معادلات دیفرانسیل با مشتقات زمان-مکان کسری به کار خواهیم برد. برای این منظور، مساله فوق را به یک مساله بهینه‌سازی گسسته تبدیل می‌کنیم به گونه‌ای که با یافتن جواب این مساله بهینه‌سازی، تقریب مناسبی از جواب مساله اصلی به دست خواهد آمد. کارآمدی روش را در مثال‌های ارائه شده خواهید دید.

کلمات کلیدی : مشتق ریمان-لیوویل؛ معادله دیفرانسیل کسری؛ معادله دیفرانسیل با مشتقات جزئی کسری؛ توابع انتگرال‌پذیر و اندازه‌پذیر لبگ.