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Research Article

A new algorithm for solving linear programming problems with bipolar fuzzy relation equation constraints

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Abstract

This paper studies the linear optimization problem subject to a system of bipolar fuzzy relation equations with the max-product composition operator. Its feasible domain is briefly characterized by its lower and upper bound, and its consistency is considered. Also, some sufficient conditions are proposed to reduce the size of the search domain of the optimal solution to the problem. Under these conditions, some equations can be deleted to compute the minimum objective value. Some sufficient conditions are then proposed which under them, one of the optimal solutions of the problem is explicitly determined and the uniqueness conditions of the optimal solution are expressed. Moreover, a modified branch-and-bound method based on a value matrix is proposed to solve the reduced problem. A new algorithm is finally designed to solve the problem based on the conditions and modified branch-and-bound method. The algorithm is compared to the methods in other papers to show its efficiency.

AMS subject classifications (2020): 90C70, 90C46, 90C26.

Keywords: Bipolar Fuzzy Relation Equation; Linear Optimization; Max-Product Composition; Modified Branch-and-Bound Method.

1 Introduction

Sanchez [31] has firstly studied Fuzzy Relation Equations (FREs) and their associated problems. Then, many researchers investigated them from a theoretical standpoint and in view of applications [26, 28, 32].

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Various approaches were designed to solve a system of FREs such as the algebraic method [21], the matrix pattern method [25], the universal algorithm [27], and the improved Lichun and Boxing's method [39]. A comprehensive review of their resolution methods has been expressed in [7] and references therein. An extended kind of FREs is a system of fuzzy Relation Inequalities (FRIs), which its solution set can be completely determined by finding its minimal solutions and maximum solution similar to FREs [16]. Their applications can be seen in the supply chain [38] and Peer-to-Peer data transmission network system [22, 36, 4].

The above FREs and FRIs are increasing in each of the variables. In some applications, for example, in an application of product public awareness in revenue management, we need variables with a bipolar characterization [9]. Some researchers introduced such bipolarity effects with the max-min composition operator in the application [9]. Li and Jin [19] showed that checking the consistency of the system of bipolar max-min FREs is NP-complete. Therefore, the resolution of the linear optimization problem with constraints of bipolar FREs will be NP-hard. Yang [37] proposed a bipolar path approach to finding the complete solution set of the system. The characterization of the solvability of bipolar max-product FREs was investigated with the standard negation [5] and the product negation [6].

The optimization of objective functions with FRE constraints is an interesting research topic [8, 14, 17, 24, 29, 30, 33, 34, 35]. Fang and Li [8] firstly studied the linear programming problem provided to a system of the max-min FREs. They proposed an algorithm based on the branch-and-bound method with the jump-tracking technique for its resolution. The algorithm was extended to the problem with the max-product composition operator in [24]. Wu, Guu, and Liu, [35] improved their approach by designing an efficient procedure. In the procedure, the branch-and-bound method was applied based on an upper bound for its optimal objective value. Hence, the procedure checks much fewer nodes to find the optimal solution with respect to Fang and Li's approach. A necessary condition has been given to find an optimal solution to the problem with the max-min [34] and the max-product [14] composition operator. Then, the necessary condition was extended for fuzzy relation programming problem with the max-strict-t-norm composition [33]. Three rules were presented to simplify the process of finding the optimal solution based on the necessary condition [34]. Li and Fang [17] investigated the resolution and optimization of a system of FREs with Sup-T composition, and they generalized the most known results in literature and provided a unified framework for the resolution and optimization of the Sup-T equation. Recently, fuzzy relation programming was extended to optimize separable functions in [15]. Different kinds of fuzzy relation programming have appeared like the max-min fuzzy relation programming problem with the addition-min FRIs [4], linear optimization with the addition-min FRIs [12], and fuzzy relation lexicographic programming [40]. Recently, Zhou et al. [41] considered the problem of optimizing a nonlinear objective function subject to a system of bipolar FREs with the max-Lukasiewicz triangular norm composition. They equivalently converted the problem to a 0-1 mixed nonlinear integer programming problem. It is NP-hard, and its resolution has high computational complexity. To increase the efficiency of algorithms, researchers focused on special classes of the bipolar fuzzy relation programming problems. Two important classes from the problem are linear [9, 20, 23, 3] and geometric [1, 2] with bipolar FRE constraints. For the first time, Freson, De Baets, and De Meyer [9] formulated the system of bipolar max-min FREs. They obtained the solution set of each of its equations. Using this point, they determined the solution set of a system of bipolar max-min FREs. This set can be characterized by a finite set of maximal and minimal solution pairs. They then studied the linear optimization problem subject to the system with a potential application of product public awareness in revenue management. They also found its optimal solutions based on the structure of the solution set of the system. Li and Liu [20] considered the problem with the max-Lukasiewicz t-norm. They converted the problem into a 0-1 integer linear optimization problem and solved it by using integer optimization techniques. However, the techniques may involve a high computational complexity. To overcome the point, Liu, Lur, and Wu [23] used the useful property that each component of an optimal solution can either be the corresponding component of lower or upper bound value and they proposed a simple value matrix with some simplification rules to reduce the dimensions of the original problem. To improve computational efficiency, some other rules were presented for more reduction of the dimensions of the problem with the max-parametric Hamacher composition operator in [3]. With regard to the above points, a modified branch-and-bound method was designed for the resolution of the problem in [3].

In this paper, the problem with the max-product operator is investigated. Its simplification procedures are completely different from the rules in [23, 3]. The motivations of this paper are the answers to the following questions:
1- How do we can detect and remove the redundancy constraints in the bipolar FRE system?
2- What are the sufficient conditions of optimality for a feasible solution to the original problem?
3- What are the sufficient conditions for the uniqueness of the optimal solution to the original problem?
4- How can we design an efficient algorithm to solve the original problem with respect to the answers to questions 1 and 2?
To find the answers to the above questions, two characteristic matrices are defined based on the components of lower and upper bound vector. Some sufficient conditions are presented to remove some equations. Moreover, some sufficient conditions are proposed which under them, one of the optimal solutions of the problem is determined. Then, the sufficient conditions for the uniqueness of the optimal solution are expressed. Furthermore, a modified branch-and-bound method based on a value matrix is designed to solve the

reduced problem. A new algorithm is proposed to solve the original problem based on the conditions and modified branch-and-bound method.

The structure of this paper is organized as follows: Section 2 introduces the linear optimization problem subject to bipolar max-product FREs. We also investigate the characterizations of its feasible domain. In Section 3, the optimality conditions are presented for a feasible solution to the problem and some theorems are given to simplify and reduce the problem. Section 4 proposes an algorithm to solve the problem. Some numerical examples are presented to illustrate the algorithm in Section 5. A comparative study is done with other methods to show the efficiency of the algorithm in Section 6. Finally, conclusions are given in Section 7.

2 Linear programming problem with bipolar max-product FREs

This section is divided into two subsections. In the first subsection, the linear programming problem subject to bipolar FREs is formulated. The characterizations of its feasible domain are finally illustrated in the second subsection.

2.1 Formulation of the problem

Let $A^+ = (a^+_{ij})$ and $A^- = (a^-_{ij})$ be two $m \times n$ fuzzy relation matrices with $0 \leq a^+_{ij}, a^-_{ij} \leq 1$ for each $i \in I = \{1, 2, \dots, m\}$ and $j \in J = \{1, 2, \dots, n\}$. Also, assume that $b = (b_1, \dots, b_m)^T \in [0, 1]^m$ and that $c = (c_1, \dots, c_n)$ is a vector of cost coefficients, where $c_j \geq 0$ for each $j \in J$. In this paper, the following programming problem is considered:

$$\min \qquad Z(x) = \sum_{j=1}^{n} c_j x_j, \tag{1}$$

$$\text{s.t.} \qquad A^+ \circ x \vee A^- \circ \neg x = b, \tag{2}$$

where $x = (x_1, \dots, x_n)^T \in [0, 1]^n$ is the vector of decision variables to be determined and $\neg x$ denotes the negation of x, that is, $\neg x = (1 - x_1, \dots, 1 - x_n)^T$. The operator of "$\circ$" represents the max-$T_p$ composition, where T_p denotes the product operator. Moreover, $S(A^+, A^-, b) = \{x \in [0, 1]^n \mid A^+ \circ x \vee A^- \circ \neg x = b\}$, which consists of a set of solution vectors $x \in [0, 1]^n$ such that

$$\max_{j \in J} \max\left\{a^+_{ij}.x_j, a^-_{ij}.(1 - x_j)\right\} = b_i \quad \text{ for all } i \in I. \tag{3}$$

Problem (1)–(2) with the real cost coefficients can be converted to a problem with nonnegative cost coefficients in a similar method to Subsection 3.3 in [9]. Hence, without loss of generality, we assume that $c_j \geq 0$ for each $j \in J$.

2.2 The structure of the feasible domain of problem (1)–(2)

A system of bipolar max-T_p FREs $A^+ \circ x \vee A^- \circ \neg x = b$ is called consistent if its solution set, that is, $S(A^+, A^-, b)$, is nonempty. Otherwise, it is inconsistent. Now, we focus on the system of bipolar max-T_p FREs (3), when $S(A^+, A^-, b) \neq \emptyset$.

Lemma 1. A vector $x \in [0,1]^n$ is a solution for the system of bipolar max-T_p FREs (3) if and only if $\max\left\{a^+_{ij}.x_j, a^-_{ij}.(1-x_j)\right\} \leq b_i$ for all $i \in I$ and $j \in J$, and for each $i \in I$, there exists an index $j_i \in J$ such that $\max\left\{a^+_{ij_i}.x_{j_i}, a^-_{ij_i}.(1-x_{j_i})\right\} = b_i$.

Proof. It is obvious. □

Remark 1. For any a^+_{ij}, a^-_{ij}, and b_i with $i \in I$ and $j \in J$, it is assumed that if $a^-_{ij} = 0$, then we define $\max\{1 - \frac{b_i}{a^-_{ij}}, 0\} = 0$. Also, if $a^+_{ij} = 0$, then we define $\min\{\frac{b_i}{a^+_{ij}}, 1\} = 1$.

Lemma 2. For any a^+_{ij}, a^-_{ij}, and b_i with $i \in I$ and $j \in J$, the inequality of $\max\left\{a^+_{ij}.x_j, a^-_{ij}.(1-x_j)\right\} \leq b_i$ holds if and only if $\max\{1 - \frac{b_i}{a^-_{ij}}, 0\} \leq x_j \leq \min\{\frac{b_i}{a^+_{ij}}, 1\}$. Especially, if $\max\left\{a^+_{ij}.x_j, a^-_{ij}.(1-x_j)\right\} \leq b_i = 0$, then at least one of two statements $a^+_{ij} = 0$ or $a^-_{ij} = 0$ holds. According to Remark 1, $\max\left\{a^+_{ij}.x_j, a^-_{ij}.(1-x_j)\right\} \leq b_i = 0$ if and only if $\max\{1 - \frac{b_i}{a^-_{ij}}, 0\} \leq x_j \leq \min\{\frac{b_i}{a^+_{ij}}, 1\}$.

Proof. The proof will be divided into four cases as follows:

Case 1. $a^+_{ij} \neq 0$ and $a^-_{ij} \neq 0$.

Case 2. $a^+_{ij} = a^-_{ij} = 0$.

Case 3. $a^+_{ij} = 0$ and $a^-_{ij} \neq 0$.

Case 4. $a^+_{ij} \neq 0$ and $a^-_{ij} = 0$.

Case 1. It is obvious that the inequality $\max\left\{a^+_{ij}.x_j, a^-_{ij}.(1-x_j)\right\} \leq b_i$ holds if and only if $a^+_{ij}.x_j \leq b_i$ and $a^-_{ij}.(1-x_j) \leq b_i$. If $b_i \neq 0$, then the recent inequalities can equivalently be rewritten as $1-\frac{b_i}{a^-_{ij}} \leq x_j \leq \frac{b_i}{a^+_{ij}}$. On the other hand, according to the assumption, we have $0 \leq x_j \leq 1$. The inequalities are equivalently concluded $\max\{1-\frac{b_i}{a^-_{ij}}, 0\} \leq x_j \leq \min\{\frac{b_i}{a^+_{ij}}, 1\}$. Whenever $b_i = 0$, the inequalities imply $x_j \leq 0$ and $x_j \geq 1$ which do not hold for any $x_j \in [0,1]$.

Case 2. Since $a^+_{ij} = a^-_{ij} = 0$ and $b_i \geq 0$, the inequality of $\max\{a^+_{ij}.x_j, a^-_{ij}.(1-x_j)\} \leq b_i$ holds if and only if $x_j \in [0,1]$. On the other hand, we have $\max\{1-\frac{b_i}{a^-_{ij}}, 0\} = 0$ and $\min\{\frac{b_i}{a^+_{ij}}, 1\} = 1$ with regard to Remark 1. Consequently, the result is also true in this case.

Case 3. Since $a^+_{ij} = 0$, then $\min\{\frac{b_i}{a^+_{ij}}, 1\} = 1$ with regard to Remark 1. We now have the following subcases: 1. $b_i = 0$ and 2. $b_i \neq 0$. In the first subcase, we have $\max\{1-\frac{b_i}{a^-_{ij}}, 0\} = 1$, that is, $\max\{1-\frac{b_i}{a^-_{ij}}, 0\} \leq x_j \leq \min\{\frac{b_i}{a^+_{ij}}, 1\}$ implies that $x_j = 1$. On the other hand, the inequality $\max\left\{a^+_{ij}.x_j, a^-_{ij}.(1-x_j)\right\} \leq b_i = 0$ holds if and only if $a^-_{ij}.(1-x_j) = 0$. This implies that $x_j = 1$. In the second subcase, the inequality of $\max\left\{a^+_{ij}.x_j, a^-_{ij}.(1-x_j)\right\} \leq b_i$ holds if and only if $a^-_{ij}.(1-x_j) \leq b_i$, which is equivalent to $\max\{1-\frac{b_i}{a^-_{ij}}, 0\} \leq x_j$. On the other hand, we have $0 \leq x_j \leq 1$. This implies that $\max\{1-\frac{b_i}{a^-_{ij}}, 0\} \leq x_j \leq 1 = \min\{\frac{b_i}{a^+_{ij}}, 1\}$ with regard to Remark 1. Hence, the result is true in both subcases. □

Whenever $S(A^+, A^-, b) \neq \emptyset$, the lower and upper bound on the solution set for the system of equations (3) can be determined using the following lemma.

Lemma 3. [1] The vector of $\check{x} = (\check{x}_1, \ldots, \check{x}_n)^T$ is the lower bound on the solution set of equations (3), where $\check{x}_j = \underset{i\in I}{max}\{1-\frac{b_i}{a^-_{ij}} \mid a^-_{ij} > b_i\}$ for each $j \in J$. Also, the vector of $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^T$ is the upper bound on the solution set of equations (3), where $\hat{x}_j = \underset{i\in I}{min}\{\frac{b_i}{a^+_{ij}} \mid a^+_{ij} > b_i\}$, for each $j \in J$. Furthermore, if there exists $j \in J$ such that for each $i \in I$, $a^-_{ij} \leq b_i$, then $\check{x}_j = 0$ and if there exists $j \in J$ such that for each $i \in I$, $a^+_{ij} \leq b_i$, then $\hat{x}_j = 1$, that is, $max\ \emptyset = 0$ and $min\ \emptyset = 1$ are defined.

Lemma 4. [1] Assume that $S(A^+, A^-, b) \neq \emptyset$ and that its lower and upper bound are $\check{x}$ and $\hat{x}$, respectively. If there exists $j_0 \in J$ such that $\check{x}_{j_0} = \hat{x}_{j_0}$, then $x_{j_0} = \check{x}_{j_0} = \hat{x}_{j_0}$ for all $x \in S(A^+, A^-, b)$. Also, the solution set of system (2) (or (3)) is the same to the following system:

$$\begin{cases} \max\limits_{j\in J-\{j_0\}} \max\left\{a^+_{ij}.x_j, a^-_{ij}.(1-x_j)\right\} = b_i & \text{for all } i \in I - \overline{I}, \\ \check{x}_j \leq x_j \leq \hat{x}_j & \text{for all } j \in J-\{j_0\};\ x_{j_0} = \check{x}_{j_0} = \hat{x}_{j_0}, \end{cases}$$

where $\check{x}_j$ and $\hat{x}_j$ for all $j \in J$ are defined on the basis of system (2) (or (3)) and $\overline{I} = \left\{ i \in I | \max\{a^+_{ij_0}.x_{j_0}, a^-_{ij_0}.(1 - x_{j_0})\} = b_i \right\}$ and $\overline{I} \neq \emptyset$.

We first consider the special case of $b_i = 0$ for $i \in I$. Since $\max\left\{a^+_{ij}.x_j, a^-_{ij}.(1 - x_j)\right\}$ is always nonnegative, the following inequality $\max\{a^+_{ij}.x_j, a^-_{ij}.(1 - x_j)\} \leq b_i$ can be converted into the equation $\max\left\{a^+_{ij}.x_j, a^-_{ij}.(1 - x_j)\right\} = b_i$. We now express the following lemma.

Lemma 5. Let $b_i = 0$ for $i \in I$. A vector x is a solution for ith equation of the system (3) if and only if $\max\{1 - \frac{b_i}{a^-_{ij}}, 0\} \leq x_j \leq \min\{\frac{b_i}{a^+_{ij}}, 1\}$ for each $j \in J$.

Proof. Considering Remark 1, the proof can be easily obtained by the proofs of Lemmas 1 and 2. □

Lemma 6. [1] Suppose that $S(A^+, A^-, b) \neq \emptyset$, and that its lower and upper are $\check{x}_j$ and $\hat{x}_j$, respectively. Then the solution set of system (2) (or (3)) is the same to the following system:

$$\begin{cases} \max\limits_{j \in J} \max\{a^+_{ij}.x_j, a^-_{ij}.(1 - x_j)\} = b_i & \text{for all } i \in I - I_0, \\ \check{x}_j \leq x_j \leq \hat{x}_j & \text{for all } j \in J, \end{cases}$$

where $I_0 = \{i \in I | b_i = 0\}$. Also, $\check{x}_j$ and $\hat{x}_j$, for all $j \in J$, are defined on the basis of system (2) (or (3)).

Furthermore, if $b_i = 0$ for each $i \in I$, then we can easily obtain the solution set of equations (3) and an optimal solution for problem (1)–(2). These points are expressed in the following corollary.

Corollary 1. If $b = 0$ in the constraint part of problem (1)–(2), then we have
$S(A^+, A^-, b) = \{x \mid \check{x} \leq x \leq \hat{x}\}$ and $x^* = \check{x}$ is an optimal solution to problem (1)-(2).

Proof. It can be easily seen that $S(A^+, A^-, b) = \{x \mid \check{x} \leq x \leq \hat{x}\}$ with regard to Lemmas 3 and 5. The proof is completed by showing that $x^* = \check{x}$ is an optimal solution of problem (1)–(2). Since $\check{x} \in S(A^+, A^-, b)$, $c \geq 0$, and the problem is minimization, then $\check{x}$ is an optimal solution of problem (1)–(2). □

Without loss of generality, from now on, we will assume that $\check{x}_j < \hat{x}_j$, for each $j \in J$, and $b_i > 0$, for each $i \in I$.

Remark 2. In this paper, if $a^+_{ij} = a^-_{ij} = 0$, then we define $\frac{a^+_{ij}.a^-_{ij}}{a^+_{ij} + a^-_{ij}} = 0$.

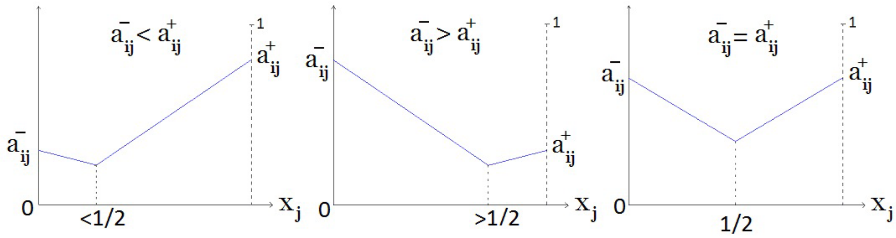

Figure 1: Illustration of function $f(x_j) = max\{a_{ij}^+.x_j, a_{ij}^-.(1-x_j)\}$

Lemma 7. For any a_{ij}^+, a_{ij}^-, and b_i with $i \in I$ and $j \in J$, the equation of $\max\left\{a_{ij}^+.x_j, a_{ij}^-.(1-x_j)\right\} = b_i$ has a solution if and only if $\frac{a_{ij}^+.a_{ij}^-}{a_{ij}^+ + a_{ij}^-} \leq b_i \leq \max\{a_{ij}^+, a_{ij}^-\}$. Also, its solution set is determined with regard to the following cases:

Case 1: If $a_{ij}^- < b_i \leq a_{ij}^+$, then $S(a_{ij}^+, a_{ij}^-, b_i) = \left\{\min\left(\frac{b_i}{a_{ij}^+}, 1\right)\right\}$;

Case 2: If $a_{ij}^+ < b_i \leq a_{ij}^-$, then $S(a_{ij}^+, a_{ij}^-, b_i) = \left\{\max\left(1 - \frac{b_i}{a_{ij}^-}, 0\right)\right\}$;

Case 3: If $\frac{a_{ij}^+.a_{ij}^-}{a_{ij}^+ + a_{ij}^-} \leq b_i \leq \min\left\{a_{ij}^+, a_{ij}^-\right\}$, then

$$S(a_{ij}^+, a_{ij}^-, b_i) = \left\{\max\left(1 - \frac{b_i}{a_{ij}^-}, 0\right), \min\left(\frac{b_i}{a_{ij}^+}, 1\right)\right\}.$$

Proof. For given $a_{ij}^+, a_{ij}^- \in [0,1]$, and $b_i > 0$, the range of the function of $\max\left\{a_{ij}^+.x_j, a_{ij}^-.(1-x_j)\right\}$ and the solution set of $S(a_{ij}^+, a_{ij}^-, b_i)$ can be observed from Figure 1 and determined, easily. □

Note that the vectors of $\check{x}$ and $\hat{x}$ are only the lower and upper bound on the solution set of equations (3), respectively. They are not necessarily feasible solutions to the system of equations (3). Moreover, we have $S(A^+, A^-, b) \subseteq \{x \mid \check{x} \leq x \leq \hat{x}\}$. Considering Lemmas 3 and 7, each equation in system (3) can be satisfied by $\check{x}_j$ or $\hat{x}_j$. In order to store these facts, the characteristic matrices of Q^+ and Q^- are defined below.

Definition 1. Define two characteristic matrices $Q^+ = (q_{ij}^+)_{m\times n}$ and $Q^- = (q_{ij}^-)_{m\times n}$ such that for each $i \in I$ and $j \in J$, we have

$$q_{ij}^+ = \begin{cases} 1 & \text{if } \ a_{ij}^+.\hat{x}_j = b_i, \\ 0 & \text{otherwise,} \end{cases} \quad \textit{and} \quad q_{ij}^- = \begin{cases} 1 & \text{if } \ a_{ij}^-.(1-\check{x}_j) = b_i, \\ 0 & \text{otherwise.} \end{cases}$$

Also, a series of index sets is defined as follows.

Definition 2. ***(i)*** [1] For the matrix Q^+, define

$$I_j^+(x) = \{i \in I \mid x_j = \hat{x}_j \textit{ and } q_{ij}^+ = 1\} \textit{ and } J_i^+(x) = \{j \in J \mid x_j = \hat{x}_j \textit{ and } q_{ij}^+ = 1\}.$$

Also, for the matrix Q^-, define

$$I_j^-(x) = \{i \in I \mid x_j = \check{x}_j \textit{ and } q_{ij}^- = 1\} \textit{ and } J_i^-(x) = \{j \in J \mid x_j = \check{x}_j \textit{ and } q_{ij}^- = 1\},$$

for each $i \in I$ *and* $j \in J$. Furthermore, let $I_j(x) = I_j^+(x) \cup I_j^-(x)$, for each $j \in J$.
(ii) Let $I_j^+ = I_j^+(\hat{x})$, $J_i^+ = J_i^+(\hat{x})$, let $I_j^- = I_j^-(\check{x})$, and let $J_i^- = J_i^-(\check{x})$, for each $i \in I$ and $j \in J$.

Considering Lemmas 2 and 7 and the above concepts, we present a necessary and sufficient condition for the solution of system (3) (or (2)).

Theorem 1. A vector $x \in [0,1]^n$ is a solution for the system of bipolar max-T_p FREs $A^+ \circ x \vee A^- \circ \neg x = b$ if and only if $\check{x} \le x \le \hat{x}$ and $\bigcup_{j \in J} I_j(x) = I$.

Proof. With regard to Lemma 1, we show that $\check{x} \le x \le \hat{x}$ and $\bigcup_{j \in J} I_j(x) = I$ if and only if $\max\left\{a_{ij}^+.x_j, a_{ij}^-.(1-x_j)\right\} \le b_i$, for all $i \in I$ and $j \in J$, and for each $i \in I$ there exists an index $j_i \in J$ such that $\max\left\{a_{ij_i}^+.x_{j_i}, a_{ij_i}^-.(1-x_{j_i})\right\} = b_i$. Considering Lemmas 2 and 3, all the inequalities $\max\left\{a_{ij}^+.x_j, a_{ij}^-.(1-x_j)\right\} \le b_i$, for $i \in I$ and $j \in J$, hold if and only if $\check{x} \le x \le \hat{x}$. We now focus on proving $\bigcup_{j \in J} I_j(x) = I$. The equality is true if and only if for each $i \in I$, there exists an index $j_i \in J$ such that $\max\left\{a_{ij_i}^+.x_{j_i}, a_{ij_i}^-.(1-x_{j_i})\right\} = b_i$. Since $I_j(x) = I_j^+(x) \cup I_j^-(x)$, for each $j \in J$, we have $\bigcup_{j \in J} I_j(x) = I$ if and only if $\bigcup_{j \in J} (I_j^+(x) \cup I_j^-(x)) = I$. Moreover, we have $\bigcup_{j \in J} (I_j^+(x) \cup I_j^-(x)) = I$ if and only if for each $i \in I$ there exists an index $j_i \in J$ such that $(x_{j_i} = \hat{x}_{j_i} \ \&\ q_{ij_i}^+ = 1)$ or $(x_{j_i} = \check{x}_{j_i} \ \&\ q_{ij_i}^- = 1)$, with regard to Definition 2. By Definition 1 and Lemma 7, it can easily be seen that for each $i \in I$, there exists an index $j_i \in J$ such that $(x_{j_i} = \hat{x}_{j_i} \ \&\ q_{ij_i}^+ = 1)$ or $(x_{j_i} = \check{x}_{j_i} \ \&\ q_{ij_i}^- = 1)$ if and only if $\max\left\{a_{ij_i}^+.x_{j_i}, a_{ij_i}^-.(1-x_{j_i})\right\} = b_i$. $\square$

For each $j \in J$, label the values of $\hat{x}_j$ and $\check{x}_j$ with boolean variables of y_j and $\neg y_j$, respectively. The following theorem is used to determine the consistency of system $A^+ \circ x \vee A^- \circ \neg x = b$.

Theorem 2. A system of bipolar max-T_p FREs $A^+ \circ x \vee A^- \circ \neg x = b$ is consistent if and only if its characteristic boolean formula $C = \bigwedge_{i \in I} C_i$ is well-defined and satisfiable, where $C_i = \bigvee_{j \in J_i^+} y_j \vee \bigvee_{j \in J_i^-} \neg y_j$.

Proof. With regard to Definitions 1 and 2, the proof is similar to the proof of Theorem 2.5 in [18] □

Theorem 3. Let $\check{x}$ and $\hat{x}$ be the lower and upper bound, respectively. Then

$$I_j(x) \subseteq I_j^+ \cup I_j^-, \quad \text{for all } x \in S(A^+, A^-, b), \quad \text{for all } j \in J.$$

Exactly, we have $I_j(x)=I_j^+$ (when $x_j = \hat{x}_j$) or $I_j(x)=I_j^-$ (when $x_j = \check{x}_j$) or $I_j(x) = \emptyset$ (when $\check{x}_j < x_j < \hat{x}_j$).

Proof. The proof is obtained from Definitions 1 and 2. □

We are now ready to present some theorems to simplify problem (1)–(2) in the next section.

3 Some optimality sufficient conditions for problem (1)–(2)

This section studies some optimality conditions for problem (1)–(2). One of its optimal solutions is found under the conditions. Moreover, some sufficient conditions are proposed to guarantee its uniqueness, and a closed form is proposed to determine it. In this section, it is assumed that $S(A^+, A^-, b) \neq \emptyset$. The following lemma presents a useful property of the optimal solution of problem (1)–(2).

Lemma 8. Consider the optimization problem of (1)–(2). Then there exists an optimal solution $x^* = (x_1^*, \ldots, x_n^*)^T$ such that for each $j \in J$ either $x_j^* = \hat{x}_j$ or $x_j^* = \check{x}_j$.

Proof. The proof is similar to the proof of Lemma 4 in [20]. □

Some theorems are first presented to reduce the search domain of the optimal solution of problem (1)–(2). The dimensions of the matrices of Q^+ and Q^- can be reduced using these theorems.

Theorem 4. Let $T_{i_1} = \{i \in I \setminus \{i_1\} \mid J_i^+ \supseteq J_{i_1}^+ \ \& \ J_i^- \supseteq J_{i_1}^-\}$, for $i_1 \in I$. If for vector x, where $\check{x} \leq x \leq \hat{x}$, $\underset{j \in J}{max}\ max\{T_p(a_{ij}^+, x_j), T_P(a_{ij}^-, (1 - x_j))\} = b_i$ holds for $i = i_1$, then for each $i \in T_{i_1}$, we have

$$\underset{j \in J}{max}\ max\{T_p(a_{ij}^+, x_j), T_P(a_{ij}^-, (1 - x_j))\} = b_i.$$

Proof. If for the vector x, where $\check{x} \leq x \leq \hat{x}$, the following equality $\underset{j \in J}{max}\ max\{T_p(a_{ij}^+, x_j), T_P(a_{ij}^-, (1 - x_j))\} = b_i$ holds for $i = i_1$, then there exist some $j_1 \in J$ such that $\max\{T_p(a_{i_1 j_1}^+, x_{j_1}), T_P(a_{i_1 j_1}^-, (1 - x_{j_1}))\} = b_{i_1}$.

Since these equalities hold only at the values of $\check{x}_{j_1}$, for $j_1 \in J^-_{i_1}(x)$, or $\hat{x}_{j_1}$, for $j_1 \in J^+_{i_1}(x)$, then $j_1 \in J^+_{i_1}(x) \cup J^-_{i_1}(x) \neq \emptyset$. Also, with regard to Definition 2, $J^+_{i_1}(x) \subseteq J^+_{i_1}$ and $J^-_{i_1}(x) \subseteq J^-_{i_1}$. On the other hand, for each $i \in T_{i_1}$, we have $J^+_i \supseteq J^+_{i_1}$ and $J^-_i \supseteq J^-_{i_1}$. Therefore, for each $i \in T_{i_1}$, $J^+_{i_1}(x) \subseteq J^+_i$ and $J^-_{i_1}(x) \subseteq J^-_i$ because $J^+_{i_1}(x) \subseteq J^+_{i_1}$, $J^-_{i_1}(x) \subseteq J^-_{i_1}$, $J^+_{i_1} \subseteq J^+_i$, and $J^-_{i_1} \subseteq J^-_i$, for each $i \in T_{i_1}$. Since $j_1 \in J^+_{i_1}(x) \cup J^-_{i_1}(x)$, $J^+_{i_1}(x) \subseteq J^+_i$, and $J^-_{i_1}(x) \subseteq J^-_i$, for each $i \in T_{i_1}$, then it is concluded that ($j_1 \in J^+_{i_1}(x)$ and $j_1 \in J^+_i$) or ($j_1 \in J^-_{i_1}(x)$ and $j_1 \in J^-_i$). So, we can write ($x_{j_1} = \hat{x}_{j_1}$ and $q^+_{ij_1} = 1$) or ($x_{j_1} = \check{x}_{j_1}$ and $q^-_{ij_1} = 1$), for each $i \in T_{i_1}$. With regard to Definition 1, $T_p(a^+_{ij_1}, x_{j_1}) = b_i$ or $T_P(a^-_{ij_1}, (1 - x_{j_1})) = b_i$, for each $i \in T_{i_1}$. Thus, the equality $\max\{T_p(a^+_{ij_1}, x_{j_1}), T_P(a^-_{ij_1}, (1 - x_{j_1}))\} = b_i$ holds true for each $i \in T_{i_1}$. Since $\check{x} \leq x \leq \hat{x}$ and $\max\{T_p(a^+_{ij_1}, x_{j_1}), T_P(a^-_{ij_1}, (1 - x_{j_1}))\} = b_i$, for each $i \in T_{i_1}$, then the equality $\underset{j\in J}{max}\ max\{T_p(a^+_{ij}, x_j), T_P(a^-_{ij}, (1 - x_j))\} = b_i$ holds true for each $i \in T_{i_1}$. □

The following corollary is a direct result of Theorem 4.

Corollary 2. Under the conditions of Theorem 4, all the equations with numbers $i \in T_{i_1}$ can be removed from the matrices of Q^+ and Q^- once $\check{x}$ and $\hat{x}$ have been obtained.

In the next theorem, an equivalent system with system (2) is presented.

Theorem 5. A binary vector $u \in \{0, 1\}^n$ induces a solution $x = \check{x} + V u$ for the system of bipolar max-T_p FREs $A^+ \circ x \vee A^- \circ \neg x = b$ if and only if $(Q^+ - Q^-)u + Q^- e_n \geq e_m$, where V=$diag(\hat{x} - \check{x})$ and e_k is a k-dimensional vector with the unit components.

Proof. The proof is similar to the proof of Theorem 3 in [20]. □

Lemma 9. If there exists a pair $i \in I$ and $k \in J$ such that q^+_{ik}=$q^-_{ik} = 1$, then the reduced system of $(Q^{+'} - Q^{-'})u + Q^{-'} e_n \geq e'_{m-1}$ and the system of $(Q^+ - Q^-)u + Q^- e_n \geq e_m$ have the same solution set, where the matrices of $Q^{+'}$, $Q^{-'}$, and e'_{m-1} are obtained by removing the ith row of the matrices of Q^+, Q^-, and e_m, respectively.

Proof. Let $S(Q^+, Q^-, u) = \{u \in \{0, 1\}^n \mid (Q^+ - Q^-)u + Q^- e_n \geq e_m\}$ and $S(Q^{+'}, Q^{-'}, u) = \{u \in \{0, 1\}^n \mid (Q^{+'} - Q^{-'})u + Q^{-'} e_n \geq e'_{m-1}\}$, where $Q^{+'}$, $Q^{-'}$, and e'_{m-1} are obtained from Q^+, Q^-, and e_m by removing their row of i, respectively. We will now show $S(Q^+, Q^-, u) = S(Q^{+'}, Q^{-'}, u)$. Consider the ith equation of system (2). Since q^+_{ik}=$q^-_{ik} = 1$ and $u_k \in \{0, 1\}$, the ith inequality of $(Q^+ - Q^-)u + Q^- e_n \geq e_m$, or equivalently $Q^+ u + Q^-(e_n - u) \geq e_m$, is automatically satisfied. Therefore, $S(Q^+, Q^-, u) = S(Q^{+'}, Q^{-'}, u)$. □

Theorem 6. If there exists a pair $i \in I$ and $k \in J$ such that q^+_{ik}=q^-_{ik}=1, then we can remove the ith row in the computation of the minimum objective value.

Proof. It is obvious from Lemma 9 and Theorem 4 in [20]. □

Now, some sufficient conditions are presented to determine one of the optimal solutions of problem (1)–(2). First of all, we express the following definition.

Definition 3. Define two index sets I_1 and I_2 as follows:

$$I_1 = \bigcup_{j\in J} I_j^- \qquad \textit{and} \qquad I_2 = I \setminus I_1.$$

Corollary 3. If $I_2 = \emptyset$, then $x^* = \check{x}$ is an optimal solution of problem (1)–(2).

Proof. If $\bigcup_{j\in J} I_j^- = I$, then $\check{x}$ is a feasible solution for the system of equations (2). Hence, we can assign $\check{x}_j$ to x_j^* due to $c_j \geq 0$, for each $j \in J$. □

If the vector of $\check{x}$ is a feasible solution of problem (1)–(2), then $\check{x}$ is an optimal solution due to $c_j \geq 0$, for each $j \in J$. Otherwise, we have to construct a solution x^* with elements $x_j^* = \check{x}_j$ or $x_j^* = \hat{x}_j$ such that $\bigcup_{j\in J} I_j(x^*) = I$ and the value of $Z(x^*)$ is the minimum objective value. With regard to these points, we present the following theorems.

Theorem 7. If there exists an index $k \in \bigcap_{i\in I_2} J_i^+$ such that the following conditions are satisfied

1. for all $j \in \bigcup_{i\in I_2} J_i^+$, $c_k(\hat{x}_k - \check{x}_k) \leq c_j(\hat{x}_j - \check{x}_j)$ and
2. $I_k^- \setminus I_k^+ \subseteq \bigcup_{\substack{j\in J\\ j\neq k}} I_j^-$,

then there exists an optimal solution $x^* = (x_j^*)_{j\in J}$ for problem (1)–(2) as follows:

$$x_j^* = \begin{cases} \hat{x}_k & \text{if } j = k, \\ \\ \check{x}_j & \text{otherwise,} \end{cases} \qquad \text{for all } j \in J. \tag{4}$$

Proof. We show that x^* is an optimal solution of problem (1)–(2). It is enough to show that *i)* $x^* \in S(A^+, A^-, b)$ and *ii)* $Z(x^*) \leq Z(x)$, for each $x \in S(A^+, A^-, b)$.

i) With regard to the structure of vector x^* and $I_k^- \setminus I_k^+ \subseteq \bigcup_{\substack{j\in J\\ j\neq k}} I_j^-$, the following equalities hold:

$$\bigcup_{j\in J} I_j(x^*) = (\bigcup_{\substack{j\in J\\ j\neq k}} I_j^-) \cup I_k^+ = (\bigcup_{j\in J} I_j^-) \cup I_k^+. \tag{5}$$

On the other hand, $k \in \bigcap_{i\in I_2} J_i^+$ implies that $I_k^+ \supseteq I_2 = I \setminus \bigcup_{j\in J} I_j^-$. Therefore, we have

$$I_k^+ \cup (\bigcup_{j\in J} I_j^-) = I. \tag{6}$$

With regard to the expressions of (5) and (6), it is concluded that $\bigcup_{j\in J} I_j(x^*) = I$. Since $\check{x} \le x^* \le \hat{x}$ and $\bigcup_{j\in J} I_j(x^*) = I$, the vector x^* is a feasible solution for the system of equations (3) with regard to Theorem 1.

ii) For each $x \in S(A^+, A^-, b)$ and $x \ne x^*$, we have $x_k = \hat{x}_k$ or there exists an index $j \in \bigcup_{i\in I_2} J_i^+ \setminus \{k\}$ such that $x_j = \hat{x}_j$.

If $x_k = \hat{x}_k$, then there exists an index $j' \in J \setminus \{k\}$ such that $x_{j'} = \hat{x}_{j'}$ due to $x \ne x^*$. Hence, we have $Z(x^*) \le Z(x^*) + c_{j'}(\hat{x}_{j'} - \check{x}_{j'}) \le Z(x)$. If there exists an index $j \in \bigcup_{i\in I_2} J_i^+ \setminus \{k\}$ such that $x_j = \hat{x}_j$, then $Z(x) \ge Z(x^*)$ with regard to the condition 1 in Theorem 7. □

Note that the optimal solution introduced in the relation (4) is not necessarily unique with regard to condition 1 in Theorem 7 and $c_j \ge 0$, for each $j \in J$. Considering Theorem 7, some sufficient conditions are expressed in Lemma 10 that under them, problem (1)–(2) has a unique optimal solution and the optimal solution is explicitly determined.

Lemma 10. If for each $j \in J \setminus \bigcup_{i\in I_2} J_i^+$, $c_j > 0$ and there exists an index $k \in \bigcap_{i\in I_2} J_i^+$ such that the conditions

1. for all $j \in \bigcup_{i\in I_2} J_i^+ \setminus \{k\}$, $c_k(\hat{x}_k - \check{x}_k) < c_j(\hat{x}_j - \check{x}_j)$, and
2. $I_k^- \setminus I_k^+ \subseteq \bigcup_{\substack{j\in J\\ j\ne k}} I_j^-$

are satisfied, then the optimization problem of (1)–(2) has a unique optimal solution of $x^* = (x_j^*)_{j\in J}$ as relation (4).

Proof. Since the assumptions of Theorem 7 hold, problem (1)–(2) has an optimal solution as the relation (4). We now show its uniqueness. By the assumptions of $\check{x}_j < \hat{x}_j$ and $c_j \ge 0$, for each $j \in J$, the condition 1 in Lemma 10 implies that $c_j > 0$, for each $j \in \bigcup_{i\in I_2} J_i^+ \setminus \{k\}$. This point implies that $c_j > 0$, for each $j \in J \setminus \{k\}$, with regard to the assumption for all $j \in J \setminus \bigcup_{i\in I_2} J_i^+$, $c_j > 0$. For each $x \in S(A^+, A^-, b)$ and $x \ne x^*$, we have $x_k = \hat{x}_k$ or there exists an index $j \in \bigcup_{i\in I_2} J_i^+ \setminus \{k\}$ such that $x_j = \hat{x}_j$. If $x_k = \hat{x}_k$, then there exists an index $j' \in J \setminus \{k\}$ such that $x_{j'} = \hat{x}_{j'}$ due to $x \ne x^*$. Hence, we have $Z(x^*) < Z(x^*) + c_{j'}(\hat{x}_{j'} - \check{x}_{j'}) \le Z(x)$ due to the condition 1. If there exists $j \in \bigcup_{i\in I_2} J_i^+ \setminus \{k\}$ such that $x_j = \hat{x}_j$, then $Z(x) > Z(x^*)$ due to $c_j > 0$, for each $j \in J \setminus \{k\}$. This shows the uniqueness of optimal solution x^* for problem (1)–(2). □

The lemmas, theorems, and corollaries of this section are firstly applied to reduce the size of the original problem. If all of the components of the optimal solution of problem (1)–(2) were not determined, then we have to solve the reduced problem. To do this, we will explain the modified branch-and-bound method to find the rest of its components in the next section.

4 A procedure for the resolution of problem (1)–(2)

We first define a simple value matrix in this section. Applying the value matrix and some points, the branch-and-bound method with the jump-tracking technique is modified to solve problem (1)–(2). Finally, an algorithm is proposed for the resolution of problem (1)–(2).

4.1 Modified branch-and-bound method

In this subsection, we rewrite the objective function (1) as follows:

$$Z(x) = \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} c_j \check{x}_j + \sum_{j=1}^{n} c_j \check{x}_j = \sum_{j=1}^{n} c_j (x_j - \check{x}_j) + \sum_{j=1}^{n} c_j \check{x}_j.$$

Now, the optimal solutions of problem (1)–(2) are the same with the optimal solutions of the following problem:

$$\min \quad \bar{Z}(x) = \sum_{j=1}^{n} c_j (x_j - \check{x}_j), \tag{7}$$

$$\text{s.t.} \quad A^+ \circ x \vee A^- \circ \neg x = b, \tag{8}$$

$$x \in [0,1]^n. \tag{9}$$

It is necessary to recall that $Z(x^*) = \bar{Z}(x^*) + \sum_{j=1}^{n} c_j \check{x}_j$. Therefore, we try to find the optimal solution of problem (7)–(9). Since $\check{x} \leq x \leq \hat{x}$, for each $x \in S(A^+, A^-, b)$ and $c \geq 0$, we have $\bar{Z}(x) \geq 0$ and $Z(x) \geq \sum_{j \in J} c_j \check{x}_j$ for each $x \in S(A^+, A^-, b)$. As it was stated before, if $\check{x} \in S(A^+, A^-, b)$, then we can set $\check{x}$ as an optimal solution. Otherwise, we have $\check{x} \notin S(A^+, A^-, b)$. In order to find the optimal solution of problem (1)–(2) (or (7)–(9)) according to Lemma 8, we have to set some $\hat{x}_j$ instead of $\check{x}_j$, $j \in J$, in the vector $\check{x}$ such that the new vector $\check{x}^*$ satisfy all equations (8)–(9) and minimize the objective function (7). To do this, we first express a useful property of the objective function (7).

Proposition 1. Let $x = (x_j)_{j \in J}$ be a vector in $S(A^+, A^-, b)$ where $x_t = \check{x}_t$ and $x_k = \check{x}_k$ for two indices t and k such that $t \neq k$ and vectors $\check{x}$ and $\hat{x}$ are lower and upper bounds of system (8)–(9), respectively. Construct two new vectors $x' = (x'_j)_{j \in J}$ and $x'' = (x''_j)_{j \in J}$ such that $x'_t = \hat{x}_t$ and $x'_j = x_j$, for each $j \in J \setminus \{t\}$, and $x''_k = \hat{x}_k$ and $x''_j = x_j$, for each $j \in J \setminus \{k\}$. If $c_k(\hat{x}_k - \check{x}_k) < c_t(\hat{x}_t - \check{x}_t)$, then $\bar{Z}(x'') < \bar{Z}(x')$.

Proof. It is obvious. □

It is necessary to recall the importance of the form of problem (7)–(9) with respect to problem (1)–(2). With regard to Proposition 1, if $c_k\hat{x}_k > c_t\hat{x}_t$, then we could not conclude that $Z(x'') > Z(x')$ or $Z(x'') < Z(x')$, but problem (7)–(9) gives us more information. If $c_k(\hat{x}_k - \check{x}_k) < c_t(\hat{x}_t - \check{x}_t)$, then we have $\bar{Z}(x'') < \bar{Z}(x')$. We use the useful property of the objective function (7) to present a modified branch-and-bound method to solve problem (1)–(2). First of all, we consider the following remark.

Remark 3. Rearrange the rows of matrices Q^+ and Q^- such that the first $|I_2|$ rows in these matrices are the rows $i \in I_2$. More precisely, transfer all rows $i \in I_2$ to the top $|I_2|$ of the rows of the matrices Q^+ and Q^-.

According to Lemma 8, there exists an optimal solution $x^* = (x^*_j)_{j \in J}$ such that either $x^*_j = \hat{x}_j$ or $x^*_j = \check{x}_j$ for each $j \in J$. Hence, it is concluded that $I_j(x^*){=}I^+_j$ or $I_j(x^*){=}I^-_j$, for each $j \in J$, with regard to Theorem 3. Since $I_j(x^*){=}I^+_j$ or $I_j(x^*){=}I^-_j$, for each $j \in J$, we hereinafter focus on these columns. To do this, the following value matrix is defined based on problem (7)–(9).

Definition 4. Define the value matrix of $M{=}(m_{ij})_{m \times 2n}$, where

$$m_{i,2j-1} = \begin{cases} c_j(\hat{x}_j - \check{x}_j) & \text{if } \ q^+_{ij} = 1, \\ \infty & \text{otherwise,} \end{cases} \qquad \textit{and} \qquad m_{i,2j} = \begin{cases} 0 & \text{if } \ q^-_{ij} = 1, \\ \infty & \text{otherwise,} \end{cases}$$

for each $i \in I$ and $j \in J$.

We will employ the branch-and-bound method with the jump-tracking technique to solve problem (1)–(2) using the value matrix M. Since $\hat{x}_j$ and $\check{x}_j$, for each $j \in J$, cannot be selected simultaneously along a branch, we should modify the branch-and-bound method. We consider three modifications on this method similar to [1] as follows:

1. If we choose $\hat{x}_j$ (or $\check{x}_j$) to branch from one node to another node, then we never use $\check{x}_j$ (or $\hat{x}_j$) to branch further on the current node.

2. Under the stated conditions below, we cannot branch further on Node k.

2.1. We have reached to the last row of the matrix M.

2.2. The selected variables along Node 0 to Node k together with $\check{x}_j$, for each $j \in J \setminus J_k$, satisfy all the equations, where

$J_k = \{j \in J | x_j$ has been selected along the branches from Node 0 to Node k$\}$.

2.3. We do not have any candidate for satisfying an equation with regard to modification 1.

3. If we cannot branch further on Node k under the conditions 2.1 and 2.2, then we assign $\check{x}_j$ to x_j for each $j \in J \setminus J_k$.

Note that if we cannot branch further on Node k with the value of Z_k under the conditions 2.1 and 2.2, then Z_k represents the objective value of problem (7)–(9) for the obtained vector x along Node 0 to Node k. Then the total of Z_k along the branches from Node 0 to Node k can be calculated as follows:

$$Total\ Z_k = Z_k + \sum_{j \in J} c_j \check{x}_j, \tag{10}$$

where total Z_k represents the objective value of problem (1)–(2) for the obtained vector x.

In problem (7)–(9), each equation i, for $i \in I_1$, of its constraints can be satisfied by $\check{x}_j$, for $j \in J_i^-$, or $\hat{x}_j$, for $j \in J_i^+$. In this case, the ith equation may be satisfied without imposing any extra cost to the objective function. On the other hand, each equation i, for $i \in I_2$, can only be satisfied by $\hat{x}_j$, for $j \in J_i^+$, that is, we have to expend extra cost for satisfying each equation i, for $i \in I_2$. Since each equation i, for $i \in I_2$, is satisfied with extra cost, we start the modified branch-and-bound method from row(s) $i \in I_2$. In this case, the ith equation, for $i \in I_1$, may be satisfied with the expended cost for $i \in I_2$ or without imposing any extra cost to the objective function. Hence, we expect that the visited nodes of the modified branch-and-bound method can be decreased when we consider Remark 3.

We are now ready to design an algorithm to solve problem (1)–(2) based on the obtained results up to now.

4.2 An algorithm for the resolution of problem (1)–(2)

Algorithm 1. Consider the optimization problem (1)–(2).

Step 1. Compute the lower and upper bound of $\check{x}$ and $\hat{x}$ applying Lemma 3.

Step 2. If $b = 0$ and $\check{x}_j \leq \hat{x}_j$, for each $j \in J$, then $S(A^+, A^-, b) = \{x \mid \check{x} \leq x \leq \hat{x}\}$ and $x^* = \check{x}$ is an optimal solution of problem (1)–(2) with regard to Corollary 1 and stop!

Step 3. If $\check{x}_j < \hat{x}_j$, for each $j \in J$, and $b_i > 0$, for each $i \in I$, then go to Step 4. Otherwise, use Lemmas 4 and 6.

Step 4. Compute the matrices of Q^+ and Q^-, the index sets of I_j^+ and I_j^-, for each $j \in J$, and the index sets of J_i^+ and J_i^-, for each $i \in I$ using Definitions 1 and 2.

Step 5. Check the consistency of bipolar max-T_p FREs (2) using Theorem 2. If it is inconsistent, then stop! Otherwise, go to Step 6.

Step 6. Perform the process of problem reduction as follows:

6.1. Compute two index sets I_1 and I_2 using Definition 3.

6.2. If $I_2 = \emptyset$, then $x^* = \check{x}$ is an optimal solution of problem (1)–(2) with regard to Corollary 3 and stop!

6.3. If the conditions of Lemma 10 are satisfied, then the unique optimal solution x^* of problem (1)–(2) can be obtained by relation (4) and stop!

6.4. Check the conditions of Theorem 7. If the conditions are satisfied, then there exists an optimal solution x^* according to relation (4) and stop!

6.5. Check the conditions of Theorem 4. If the conditions are satisfied, then remove all the equations with numbers $i \in T_{i_1}$ from the matrices of Q^+ and Q^- with regard to Corollary 2.

6.6. If there exists a pair $i \in I$ and $k \in J$ such that $q_{ik}^+ = q_{ik}^- = 1$, then we can remove the row of i in the computation of the minimum objective value with regard to Theorem 6.

Step 7. If $Q^+ = Q^- = \emptyset$, then assign $\check{x}_j$ to x_j^* and go to Step 10.

Step 8. Rearrange the rows of the matrices Q^+ and Q^- according to Remark 3. Also, generate the value matrix of M using Definition 4.

Step 9. Employ the modified branch-and-bound method with the jump-tracking technique on the matrix M to solve the optimization problem of (1)–(2).

Step 10. Produce the optimal solution and the optimal value of problem (1)–(2). End.

5 Numerical examples

We now illustrate Algorithm 1 by the following examples.

Example 1. Consider the following optimization problem:

$$\min \quad x_1 + 3x_2 + 2x_3 + 5x_4 + 8x_5 + 7x_6, \tag{11}$$

$$\text{s.t.} \quad A^+ \circ x \vee A^- \circ \neg x = b, \tag{12}$$

$$x \in [0,1]^6,$$

where

$$A^{+}=\begin{pmatrix}0.4&0.1&0.25&0.29&0.18&0.07\\0.32&0.16&0.23&0.1&0.2&0.48\\0.08&0.05&0.02&0.1&0.03&0.06\\0.15&0.3&0.17&0.2&0.14&0.05\\0.09&0.2&0.03&0.04&0.15&0.24\\0.49&0.58&0.6&0.37&0.75&0.54\end{pmatrix},$$

$$A^{-}=\begin{pmatrix}0.21&0.4&0.19&0.27&0.12&0.04\\0.15&0.32&0.8&0.14&0.22&0.2\\0.1&0.08&0.05&0.18&0.07&0.06\\0.17&0.04&0.15&0.07&0.3&0.2\\0.11&0.1&0.03&0.14&0.14&0.07\\0.24&0.52&0.47&0.5&0.33&0.25\end{pmatrix},$$

$b = (0.3, 0.24, 0.09, 0.18, 0.12, 0.6)^T$, and $x = (x_1, x_2, x_3, x_4, x_5, x_6)^T$. Now, we apply Algorithm 1 to solve the optimization problem of (11)–(12).

Step 1. The lower and upper bound of $\check{x}$ and $\hat{x}$ are as follows:
$\check{x} = (0.1, 0.25, 0.7, 0.5, 0.4, 0.1)^T$ and $\hat{x} = (0.75, 0.6, 1, 0.9, 0.8, 0.5)^T$.

Step 2. Since the conditions of Corollary 1 do not hold, we go to Step 3.

Step 3. In this example $\check{x}_j < \hat{x}_j$, for each $j \in J$ and $b_i > 0$, for each $i \in I$. Therefore, we go to Step 4.

Step 4. Applying Definition 1, the matrices of Q^+ and Q^- are obtained as follows:

$$Q^{+}=\begin{array}{c}\\1\\2\\3\\4\\5\\6\end{array}\begin{array}{c}1\,2\,3\,4\,5\,6\\\begin{pmatrix}1&0&0&0&0&0\\1&0&0&0&0&1\\0&0&0&1&0&0\\0&1&0&1&0&0\\0&1&0&0&1&1\\0&0&1&0&1&0\end{pmatrix}\end{array}\ and\ Q^{-}=\begin{array}{c}\\1\\2\\3\\4\\5\\6\end{array}\begin{array}{c}1\,2\,3\,4\,5\,6\\\begin{pmatrix}0&1&0&0&0&0\\0&1&1&0&0&0\\1&0&0&1&0&0\\0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\end{array}.$$

Also, the index sets of I_j^+ and I_j^-, for all $j \in J$ can be computed as follows:
$I_1^+ = \{1,2\}$, $I_2^+ = \{4,5\}$, $I_3^+ = \{6\}$, $I_4^+ = \{3,4\}$, $I_5^+ = \{5,6\}$, $I_6^+ = \{2,5\}$, $I_1^- = \{3\}$, $I_2^- = \{1,2\}$, $I_3^- = \{2\}$, $I_4^- = \{3\}$, $I_5^- = \{4\}$, and $I_6^- = \{4\}$.

Moreover, we can compute the index sets of J_i^+ and J_i^-, for all $i \in I$, as follows:
$J_1^+ = \{1\}$, $J_2^+ = \{1,6\}$, $J_3^+ = \{4\}$, $J_4^+ = \{2,4\}$, $J_5^+ = \{2,5,6\}$, $J_6^+ = \{3,5\}$, $J_1^- = \{2\}$, $J_2^- = \{2,3\}$, $J_3^- = \{1,4\}$, $J_4^- = \{5,6\}$, and $J_5^- = J_6^- = \emptyset$.

Step 5. The bipolar max-T_p FREs of $A^+ \circ x \vee A^- \circ \neg x = b$ is consistent according to Theorem 2. So, we go to Step 6.

Step 6. Perform the process of problem reduction as follows:

6.1. Two index sets I_1 and I_2 are as follows: $I_1 = \{1,2,3,4\}$ and $I_2 = \{5,6\}$.

Since the optimization problem of (11)–(12) cannot be reduced by Steps 6.2–6.4, we go to Step 6.5.

6.5. In this example, $T_1 = \{2\}$, that is, $J_1^+ \subseteq J_2^+$ and $J_1^- \subseteq J_2^-$. Applying Corollary 2, the second row of two matrices Q^+ and Q^- can be removed.

6.6. Since $q_{34}^+ = q_{34}^- = 1$, the third equation can be eliminated from our consideration with regard to Theorem 6.

The matrices of Q^+ and Q^- cannot be reduced further. So, we go to Step 7.

Step 7. Since $Q^+ \neq \emptyset$ and $Q^- \neq \emptyset$, we go to Step 8.

Step 8. With regard to Remark 3, the matrices of Q^+ and Q^- can be updated as follows:

$$Q^+ = \begin{array}{c} \\ 5 \\ 6 \\ 1 \\ 4 \end{array} \begin{array}{c} 1\,2\,3\,4\,5\,6 \\ \begin{pmatrix} 0&1&0&0&1&1 \\ 0&0&1&0&1&0 \\ 1&0&0&0&0&0 \\ 0&1&0&1&0&0 \end{pmatrix} \end{array} \; and \; Q^- = \begin{array}{c} \\ 5 \\ 6 \\ 1 \\ 4 \end{array} \begin{array}{c} 1\,2\,3\,4\,5\,6 \\ \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&1 \end{pmatrix} \end{array}$$

For the updated matrices of Q^+ and Q^-, the value matrix of M can be generated as follows:

$$M = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} \infty\ \infty & 1.05\ \infty & \infty\ \infty & \infty\ \infty & 3.2\ \infty & 2.8\ \infty \\ \infty\ \infty & \infty\ \infty & 0.6\ \infty & \infty\ \infty & 3.2\ \infty & \infty\ \infty \\ 0.65\ \infty & \infty\ \ 0 & \infty\ \infty & \infty\ \infty & \infty\ \infty & \infty\ \infty \\ \infty\ \infty & 1.05\ \infty & \infty\ \infty & 2\ \ \infty & \infty\ \ 0 & \infty\ \ 0 \end{pmatrix} \end{array}.$$

Step 9. We are now ready to use the modified branch-and-bound method with the jump-tracking technique on the matrix M. We begin with the first equation, that is, $i = 1$. The set of $\{\hat{x}_2, \hat{x}_5, \hat{x}_6\}$ introduces three candidates to satisfy the first equation. Therefore, we have to branch from Node 0 in Figure 2. If we select $\hat{x}_2$ (Node 1), then the value of Z_1 is 1.05. Note that $\check{x}_2$ cannot be used for further branching on Node 1. Also, if we select $\hat{x}_5$ ($\hat{x}_6$), then we have $Z_2 = 3.2$ ($Z_3 = 2.8$). Furthermore, we never use $\check{x}_5$ ($\check{x}_6$) to branch further on Node 2 (Node 3). Here, Node 1 is selected to branch further because of the least objective value.

Move to the second row of the matrix M. Since the set of $\{\hat{x}_3, \hat{x}_5\}$ contains two candidates to satisfy the second equation, we have two branches from Node 1 as it has been illustrated in Figure 2. If $\hat{x}_3$ (Node 4) is selected, then the value of Z_4 is 1.65. If $\hat{x}_5$ (Node 5) is considered, then we have $Z_5 = 4.25$. Now, we can branch further on four Nodes 2, 3, 4, and 5 but Node 4 is chosen with regard to the least objective value.

Move to the third row of the matrix M. The set of $\{\hat{x}_1, \check{x}_2\}$ contains two candidates to satisfy the third equation, but $\check{x}_2$ cannot be used to branch further on Node 4 because $\hat{x}_2$ has been chosen along Node 0 to Node 4 (modification 1). Therefore, the set of $\{\hat{x}_1\}$ contains the only candidate to

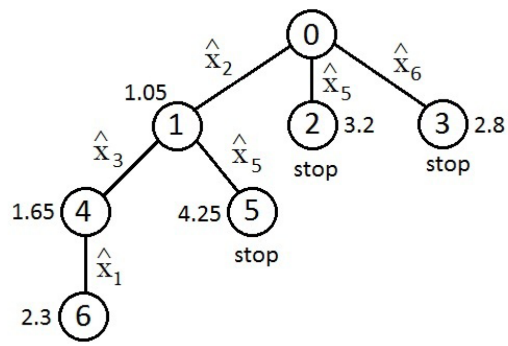

Figure 2: The modified branch-and-bound method

satisfy the third equation here. If $\hat{x}_1$ (Node 6) is selected, then it is concluded that $Z_6 = 2.3$.

Since $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$ together with $\check{x}_4$, $\check{x}_5$, and $\check{x}_6$ satisfy all the equations, we do not branch further on Node 6 with regard to modification 2.2. Therefore, considering modification 3, the vector $x{=}(\hat{x}_1, \hat{x}_2, \hat{x}_3, \check{x}_4, \check{x}_5, \check{x}_6)^T$ is a solution with the objective value of 2.3 for the equivalent problem. Since the value of Z_6 is less than Z_2, Z_3, and Z_5, we can stop further branching on all Nodes 2, 3, and 5. Thus, the total of Z_6 can be computed as follows: $Total\ Z_6{=}Z_6 + \sum_{j=1}^{6} c_j \check{x}_j = 2.3 + 8.65 = 10.95$. Every node is stopped further branching. Hence, the optimal solution can be obtained from Node 6, that is, $x_1^* = 0.75$, $x_2^* = 0.6$, $x_3^* = 1$, $x_4^* = 0.5$, $x_5^* = 0.4$, and $x_6^* = 0.1$ with the total of $Z_6 = 10.95$.

Step 10. The optimal objective value is 10.95 and the optimal solution is $x^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)^T = (0.75, 0.6, 1, 0.5, 0.4, 0.1)^T$.
If we do not apply Remark 3 to solve this example, we need to consider 25 nodes. Considering Remark 3, we need only six nodes to solve this example.

Example 2. Consider the following optimization problem:

$$\begin{aligned} \min \quad & 2x_1 + 5x_2 + 3x_3 + 4x_4 + x_5 + 6x_6, && (13)\\ \text{s.t.} \quad & A^+ \circ x \vee A^- \circ \neg x = b, && (14)\\ & x \in [0,1]^6, \end{aligned}$$

where

$$A^+ = \begin{pmatrix} 0.7 & 0.3 & 0.96 & 0.66 & 0.8 & 0.05 \\ 0.31 & 0.54 & 0.34 & 0.3 & 0.36 & 0.45 \\ 0.06 & 1 & 0.04 & 0.625 & 0.03 & 0.08 \\ 0.15 & 0.15 & 0.19 & 0.225 & 0.16 & 0.03 \\ 0.6 & 0.08 & 0.07 & 0.06 & 0.13 & 0.4 \\ 0.3 & 0.05 & 0.09 & 0.07 & 0.11 & 0.19 \end{pmatrix},$$

$$A^- = \begin{pmatrix} 0.11 & 0.4 & 0.19 & 0.27 & 0.12 & 0.04 \\ 0.14 & 0.24 & 0.09 & 0.14 & 0.22 & 0.2 \\ 0.1 & 0.08 & 0.05 & 0.18 & 0.8 & 0.8 \\ 0.2 & 0.04 & 0.15 & 0.3 & 0.11 & 0.05 \\ 0.11 & 0.4 & 0.32 & 0.14 & 0.14 & 0.07 \\ 0.03 & 0.2 & 0.06 & 0.02 & 0.04 & 0.07 \end{pmatrix},$$

$b = (0.6, 0.27, 0.5, 0.18, 0.24, 0.12)^T$, and $x = (x_1, x_2, x_3, x_4, x_5, x_6)^T$. Now, we apply Algorithm 1 to solve the optimization problem of (13)–(14).

Step 1. The lower and upper bound of $\check{x}$ and $\hat{x}$ are as follows:
$\check{x} = (0.1, 0.4, 0.25, 0.4, 0.375, 0.375)^T$ and $\hat{x} = (0.4, 0.5, 0.625, 0.8, 0.75, 0.6)^T$.

Step 2. Since the conditions of Corollary 1 do not hold, we go to Step 3.

Step 3. In this example, $\check{x}_j < \hat{x}_j$, for each $j \in J$ and $b_i > 0$, for each $i \in I$. Therefore, we go to Step 4.

Step 4. Applying Definition 1, the matrices of Q^+ and Q^- are obtained as follows:

$$Q^+ = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \; and \; Q^- = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Also, the index sets of I_j^+ and I_j^-, for all $j \in J$ can be computed as follows:
$I_1^+ = \{5,6\}$, $I_2^+ = \{2,3\}$, $I_3^+ = \{1\}$, $I_4^+ = \{3,4\}$, $I_5^+ = \{1,2\}$, $I_6^+ = \{2,5\}$, $I_1^- = \{4\}$, $I_2^- = \{5,6\}$, $I_3^- = \{5\}$, $I_4^- = \{4\}$, $I_5^- = \{3\}$, and $I_6^- = \{3\}$.

Moreover, we can compute the index sets of J_i^+ and J_i^-, for all $i \in I$, as follows:
$J_1^+ = \{3,5\}$, $J_2^+ = \{2,5,6\}$, $J_3^+ = \{2,4\}$, $J_4^+ = \{4\}$, $J_5^+ = \{1,6\}$, $J_6^+ = \{1\}$, $J_1^- = J_2^- = \emptyset$, $J_3^- = \{5,6\}$, $J_4^- = \{1,4\}$, $J_5^- = \{2,3\}$, and $J_6^- = \{2\}$.

Step 5. The bipolar max-T_p FREs of $A^+ \circ x \vee A^- \circ \neg x = b$ is consistent according to Theorem 2. So, we go to Step 6.

Step 6. Perform the process of problem reduction as follows:

6.1. Two index sets I_1 and I_2 are as follows: $I_1 = \{3,4,5,6\}$ and $I_2 = \{1,2\}$.

6.2. The condition of Substep 2 is not satisfied for this problem.
6.3. The conditions of this substep hold for this problem. Since for each $j \in J - \bigcup_{i\in I_2} = \{1,\dots,6\} - (J_1^+ \bigcup J_2^+) = \{1,4\}$, we have $c_1 = 2 > 0$ and $c_4 = 4 > 0$. Also, for $k \in \bigcap_{i\in I_2} = \{5\}$, the following conditions are satisfied:
1. for all $j \in \bigcup_{i\in I_2} J_i^+ - \{5\} = \{2,3,6\}$, we have $c_5(\hat{x}_5 - \check{x}_5) = 0.225 < c_j(\hat{x}_j - \check{x}_j)$.
2. $I_5^- - I_5^+ = \{3\} \subseteq \bigcup_{j\in J, j\neq 5} I_j^- = \{3,4,5,6\}$.
Therefore, according to Lemma 10, the unique optimal solution of the problem is as follows: $x^* = (0.1, 0.4, 0.25, 0.4, 0.75, 0.375)$ with the optimal objective value $Z^* = 7.55$

6 Comparison of the proposed algorithm with the methods in other papers

In this section, we compare the proposed algorithm with the methods in other papers [9, 20, 23, 3] to solve problem (1)–(2) with regard to the obtained results from Examples 1 and 2 in Section 5.

As it is seen in Figure 2, Algorithm 1 solves the problem of Example 1 only in six nodes by the branch-and-bound method. Also, Algorithm 1 solves Example 2 without using the branch-and-bound method. Its optimal solution is found in Substep 6.3 by Lemma 10.

Freson, De Baets, and De Meyer [9] discussed problem (1)–(2) with the max-min composition operator. They designed an algorithm to solve the problem. Its Step 1 checks the necessary condition (46) in [9]. Its computational cost is $O(mn)$. Step 2 constructs vectors g^+ and s^- taking supremum and infimum from m maximal solutions which its computational cost is $O(mn)$. Step 3 generates all elements of set B and keeps those that satisfy constraints (30) in [9]. To generate all the elements of B according to relation (48) in [9], we firstly need to produce the set $(\{0,1\} \bigcup \{b_i, \bar{b}_i | i = 1,\dots,m\})^n$. Its computational cost is $O((2m+2)^n)$. Then we must check whether each its element belongs to $[s^-, g^+]$ or not? Its computational cost is $O(n(2m+2)^n)$. Now, we must check whether each element B satisfies constraints (30) in [9] or not? The computational cost of this work is $O(mn)$ for each element of the set B. If we check all the elements of set B, then its computational cost is $O(mn(2m+2)^n)$. So, the computational cost of Step 3 is $T_3 = O(n(2m+2)^n + mn(2m+2)^n) = O(mn(2m+2)^n)$. Step 4 selects the elements in $B \bigcap D$ with the highest value for the objective function. To do this, we must check the objective function for $|B \bigcap D|$ elements which its computational cost is $O(|B \bigcap D|) \leq O((2m+2)^n)$. Therefore, the computational complexity of the given algorithm in [9] is $T_F = O(mn(2m+2)^n)$. For an instance of the problem with the dimensions $m = n = 6$ like Examples 1 and 2, its computational complexity is $T_F = O(6\times 6\times (2\times 6+2)^6) = 271063296 \times O(1)$. This point implies that $(2 \times 6 + 2)^6 = 7529536$ elements are produced and its feasibility is checked in m equations of n-variable. Then the optimizer is found by computing the objective function values in the feasible vectors.

In [20], problem (1)–(2) is discussed with the max-Lukasiewicz t-norm composition. Li and Liu [20] directly converted the problem to a 0-1 integer linear optimization problem without its simplification and reduction. If we apply their method to solve the problem of Example 1, we should consider the following 0-1 integer programming problem:

$$\begin{aligned} Z = min \quad & 8.65 + 0.65u_1 + 1.05u_2 + 0.6u_3 + 2u_4 + 3.2u_5 + 2.8u_6, \\ s.t. \quad & \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} \geq \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \\ & u \in \{0,1\}^6. \end{aligned}$$

If we apply Algorithm 1 of this paper for Example 1, we should solve the following problem with smaller dimensions:

$$\begin{aligned} Z = min \quad & 8.65 + 0.65u_1 + 1.05u_2 + 0.6u_3 + 2u_4 + 3.2u_5 + 2.8u_6, \\ s.t. \quad & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \\ & u \in \{0,1\}^6. \end{aligned}$$

If we apply Li and Liu's method to solve the problem of Example 2, we should consider the following 0-1 integer programming problem:

$$\begin{aligned} Z = min \quad & 7.175 + 0.6u_1 + 0.5u_2 + 1.125u_3 + 1.6u_4 + 0.375u_5 + 1.35u_6, \\ s.t. \quad & \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \\ & u \in \{0,1\}^6. \end{aligned}$$

If we apply Algorithm 1 of this paper for Example 2, we directly obtain the unique optimal solution of the problem of Example 2 without using the branch-and-bound method and 0-1 integer programming problem.

In [3], problem (1)–(2) was discussed with the max-Hamacher t-norm composition with some rules to simplify the problem. The rules are completely different from the proposed procedures of this paper for simplification. The rules in [3] are not applicable for the problem in Example 1. Hence, if we use the given algorithm in [3] for Example 1, the branch-and-bound method should be applied on the matrix M as follows:

$$M = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.65 & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 0.65 & \infty & \infty & 0 & \infty & 0 & \infty & \infty & \infty & \infty & 2.8 & \infty \\ \infty & 0 & \infty & \infty & \infty & \infty & 2 & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & 1.05 & \infty & \infty & \infty & 2 & \infty & \infty & 0 & \infty & 0 \\ \infty & \infty & 1.05 & \infty & \infty & \infty & \infty & \infty & 3.2 & \infty & 2.8 & \infty \\ \infty & \infty & \infty & \infty & 0.6 & \infty & \infty & \infty & 3.2 & \infty & \infty & \infty \end{pmatrix} \end{array}. \tag{15}$$

If we use Algorithm 1 for Example 1, the branch-and-bound method should be applied on the matrix M as follows:

$$M = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} \infty & \infty & 1.05 & \infty & \infty & \infty & \infty & \infty & 3.2 & \infty & 2.8 & \infty \\ \infty & \infty & \infty & \infty & 0.6 & \infty & \infty & \infty & 3.2 & \infty & \infty & \infty \\ 0.65 & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 1.05 & \infty & \infty & \infty & 2 & \infty & \infty & 0 & \infty & 0 \end{pmatrix} \end{array}. \tag{16}$$

Also, the rules in [3] are not applicable for the problem in Example 2. Hence, if we use the given algorithm in [3] for Example 2, the branch-and-bound method should be applied on the matrix M as follows:

$$M = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} \infty & \infty & \infty & \infty & 1.125 & \infty & \infty & \infty & 0.375 & \infty & \infty & \infty \\ \infty & \infty & 0.5 & \infty & \infty & \infty & \infty & \infty & 0.375 & \infty & 1.35 & \infty \\ \infty & \infty & 0.5 & \infty & \infty & \infty & 1.6 & \infty & \infty & 0 & \infty & 0 \\ \infty & 0 & \infty & \infty & \infty & \infty & 1.6 & 0 & \infty & \infty & \infty & \infty \\ 0.6 & \infty & \infty & 0 & \infty & 0 & \infty & \infty & \infty & \infty & 1.35 & \infty \\ 0.6 & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \end{pmatrix} \end{array}. \tag{17}$$

If we use Algorithm 1 for Example 2, its optimal solution is directly found in Substep 6.3 without using the branch-and-bound method and $M = \emptyset$.

In [23], problem (1)–(2) was discussed with the max-Lukasiewicz composition with some rules to simplify the problem. The rules are completely different from the proposed procedures of this paper for simplification. If we apply the method in [23] for the problem of Example 1 with the max-product composition, rule 3 in [23] can be used for its simplification. Applying the equivalent form of the rule 3 for problem (1)–(2), the forth row of the matrix M in the relation (15) is removed and the branch-and-bound method should be employed on the 0-1 integer programming equivalent to the following value matrix:

$$M = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccccccccc} 0.65 & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 0.65 & \infty & \infty & 0 & \infty & 0 & \infty & \infty & \infty & \infty & 2.8 & \infty \\ \infty & 0 & \infty & \infty & \infty & \infty & 2 & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & 1.05 & \infty & \infty & \infty & \infty & \infty & 3.2 & \infty & 2.8 & \infty \\ \infty & \infty & \infty & \infty & 0.6 & \infty & \infty & \infty & 3.2 & \infty & \infty & \infty \end{array}\right) \end{array}. \tag{18}$$

If we use Algorithm 1 for Example 1, the branch-and-bound method should be applied on the matrix M in the relation (16) and considering Remark 3, we need only six nodes to solve this example. If we apply the method in [23] for the problem of Example 2 with the max-product composition, rule 3 in [23] can be used for its simplification. Applying the equivalent form of rule 3 in [23], the third row of the matrix M in relation (17) is removed and the branch-and-bound method should be employed on the 0-1 integer programming equivalent to the following value matrix:

$$M = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccccccccc} \infty & \infty & \infty & \infty & 1.125 & \infty & \infty & \infty & 0.375 & \infty & \infty & \infty \\ \infty & \infty & 0.5 & \infty & \infty & \infty & \infty & \infty & 0.375 & \infty & 1.35 & \infty \\ \infty & 0 & \infty & \infty & \infty & \infty & 1.6 & 0 & \infty & \infty & \infty & \infty \\ 0.6 & \infty & \infty & 0 & \infty & 0 & \infty & \infty & \infty & \infty & 1.35 & \infty \\ 0.6 & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \end{array}\right) \end{array}. \tag{19}$$

If we use Algorithm 1 for Example 2, its optimal solution is directly found in Substep 6.3 without using the branch-and-bound method and $M = \emptyset$.

The other preferences of the proposed algorithm with respect to the presented algorithms in [9, 20, 23, 3] for the resolution of problem (1)–(2) are as follows. The proposed algorithm introduces two new classes of problem (1)–(2) with the max-product composition operator, which can directly be solved only by Substep 6.3 or Substep 6.4. The optimal solution of these classes of the problem satisfying conditions of Substep 6.3 or Substep 6.4 can be obtained by the relation (4) without applying the branch-and-bound method or using rules repeatedly. The classes have not been introduced in [9, 20, 23, 3]. Other proposed rules in Step 6 of Algorithm 1 are different from the given rules in [23, 3]. In [9, 20], the authors have not used the rules of simplification to reduce the original problem. Since the algorithms in [20, 23, 3] and the proposed algorithm are based on the branch-and-bound algorithm, the algorithms are convergent. The algorithm in [9] checks all the possible feasible solutions to find the optimal solution.

7 Conclusions and future works

The linear optimization problem with the bipolar max-product FREs was studied in this paper. The characterizations of its feasible domain were investigated. Some simplification operations were proposed to delete some equations. With regard to these operations, the size of the original problem was reduced. Then, some sufficient conditions were presented to determine one of the optimal solutions to the problem and its uniqueness. Moreover, a value matrix was defined based on the characteristic matrices of the feasible domain of the problem. Then, the branch-and-bound method was modified to solve the reduced problem with regard to the value matrix. An algorithm was finally designed to solve the problem and compared with other methods to show the efficiency of the proposed algorithm. In future work, the linear optimization problem will be developed by supposing fuzzy linear systems with bipolar fuzzy numbers based on references [10, 11].

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