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Augmented Lagrangian method for finding minimum norm solution to the absolute value equation

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Abstract

In this paper, we give an algorithm to compute the minimum 1-norm solution to the absolute value equation (AVE). The augmented Lagrangian method is investigated for solving this problem. This approach leads to an unconstrained minimization problem with once differentiable convex objective function. We propose a quasi-Newton method for solving unconstrained optimization problem. Computational results show that convergence to high accuracy often occurs in just a few iterations.

Keywords: Absolute value equation; Minimum norm solution; Generalized Newton method; Augmented Lagrangian method.

1 Introduction

As shown in [1, 2, 9–11], many mathematical programming problems can be reduced to LCP which is equivalent to absolute value equation as follows [7, 8, 12]:

$$Ax - |x| = b, \tag{1}$$

where $A \in R^{n\times n}$ and $b \in R^n$ are given, and $|x|$ denotes the component-wise absolute value of vector $x \in R^n$.

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The absolute value equation (1) seems to be a useful tool in optimization since it subsumes the linear complementarity problem and thus also linear programming and convex quadratic programming.

In general, AVE may have a finite number of solutions (at most 2^n) or infinitely many solutions, if it has solutions [7]. In such cases, the selection of a particular solution may be important and a natural choice is the solution with the minimum norm [4, 5, 9].
In this paper, we consider AVE in the case where it has multiple (e.g., exponentially many) solutions; we try to compute its minimum $1-norm$ solution.

In [5], for finding minimum $2-norm$ solution of AVE, need a solution of AVE and special assumptions on A. But, in this paper to find minimum $1-norm$ solution of AVE, not only do not need a solution of AVE but also do not need any special assumptions on A.

Consider the following problem :

$$\begin{aligned} &\min_{x\in R^n} \|x\|_1 \\ \textit{subject to} \qquad &Ax-|x|=b. \end{aligned} \tag{2}$$

Motivated by the study of minimum $1-norm$ solution of the AVE formulated as a linear programming problem. To solve this linear programming problem, we suggest augmented Lagrangian method.

This paper is organized as follows. Minimum norm solution of AVE is described in Section 2. The augmented Lagrangian method is discussed in Section 3. In Section 4 compute minimum norm solution for some randomly generated AVE to demonstrate the effectiveness of our method. Concluding remarks are given in Section 5.

We now describe our notations. Let $a=[a_i]$ be a vector in R^n. By a_+ we mean a vector in R^n whose ith entry is 0 if $a_i<0$ and equals a_i if $a_i\geq 0$. By A^T we mean the transpose of matrix A, and $\nabla f(x_0)$ is the gradient of f at x_0. For two vectors x and y in the $n-$dimensional real space, x^Ty will denote the scalar product. For $x\in R^n$, $\|x\|$ and $\|x\|_1$ denote 2−norm and 1−norm respectively and $|x|$ will denote the vector in R^n of absolute values of components of x.

2 Minimum Norm Solution of AVE

In this section we first mention the cases in which the AVE has multiple solutions. Then, we examine the problem (1) to obtain its minimum norm solution.

Proposition 1. *(Existence of 2^n solutions). If $b < 0$ and $\|A\|_\infty < \gamma/2$ where $\gamma = c/d$, $c = min_i|b_i|$, $d = max_i|b_i|$, then the AVE has exactly 2^n distinct solutions, each of which has no zero components and a different sign pattern.*

Proof. See [7]. □

Consider the absolute value equation (1) for which $A = -\frac{1}{4}eye(n)$ and $b = -2ones(n,1)$, it is obvious that, it has exactly 2^n distinct solutions. Now, consider for example of an AVE that has infinite number of solutions. Consider the absolute value equation (1) for which $A = eye(n)$ and $b = zeros(n,1)$. Obviously, in this case, every x for which $x \geq 0$ is a solution of AVE. Let's again consider the problem (2). This problem can be rewritten in the equivalent form:

$$\min_{p,q\in\mathbb{R}^n} \|p-q\|_1, \quad s.t. \quad Ap - Aq - Ip - Iq = b, \qquad p,q \geq 0, \tag{3}$$

where $p = \frac{|x|+x}{2}$ and $q = \frac{|x|-x}{2}$.

Now (3) is equivalent to linear programming problem as follows:

$$\min_{X\in\mathbb{R}^n} c^T X, \quad s.t. \quad BX = b, \quad X \geq 0, \tag{4}$$

where $X = [p \quad q]^T$, $c = [e \quad e]^T$ and $B = [A - I \quad -A - I]$.

To solve the problem (4), we use augmented Lagrangian method, which is described in the next section.

3 Augmented Lagrangian Method

Now, we briefly discuss applications of augmented Lagrangian method to the linear programming subproblem (4).

In the augmented Lagrangian method, an unconstrained maximization problem is solved which gives the projection of a point on the solution set of the subproblem

$$\min_{x\in X_*} \frac{1}{2} \parallel x - \bar{x} \parallel^2, \tag{5}$$
$$X_* = \{x \in R^n : Ax = b,\, c^T x = f_*, \quad x \geq 0_n\}.$$

The Lagrange function for the problem (5) is

$$L(x,u,\alpha,\bar{x}) = \frac{1}{2} \parallel \bar{x} - x \parallel^2 - u^T(Ax - b) + \alpha\ (c^T x - f_*),$$

where $u \in R^m$, $\alpha \in R^1$ are Lagrange multipliers and $\bar{x}$ is considered as a fixed parameter. The dual problem of (5) is

$$\max_{u\in R^m}\max_{\alpha\in R^1}\min_{x\in R^n_+} L(x,u,\alpha,\bar{x}). \tag{6}$$

The optimality conditions of the inner minimization of the problem (6) is

$$\nabla L_x(x,u,\alpha,\bar{x}) = x-\bar{x}-A^Tu+\alpha c \geq 0, \tag{7}$$
$$x^\top(x-\bar{x}-A^Tu+\alpha c)=0, \qquad x\geq 0. \tag{8}$$

It follows from (7) and (8) that the solution of this minimization problem is given in the following form:

$$x = (\bar{x}+A^Tu-\alpha c)_+, \tag{9}$$

we replace x by $(\bar{x}+A^Tu-\alpha c)_+$ into $L(x,u,\alpha,\bar{x})$ and obtain the dual function

$$\Phi(u,\alpha,\bar{x}) = b^Tu - \frac{1}{2}\parallel(\bar{x}+A^Tu-\alpha c)_+\parallel^2 - \alpha f_* + \frac{1}{2}\parallel \bar{x}\parallel^2 .$$

Hence the problem (5) is reduce to its dual problem

$$\max_{u\in R^m}\max_{\alpha\in R^1}\Phi(u,\alpha,\bar{x}). \tag{10}$$

This problem is an optimization problem without any constrain and its objective function contains an unknown value f_*. By [3] there exists a positive number $\alpha_*>0$ such that, for each $\alpha>\alpha_*$, the projection x of the arbitrary vector $\bar{x}\in R^n$ onto X_* can be obtained as following:

$$x = (\bar{x}+\alpha(A^Tu(\alpha)-c))_+, \tag{11}$$

where $u(\alpha)$ is the solution of the problem (10), such that $\alpha\in R^1$, $\alpha>\alpha_*$ is fixed.

We note that, the function $\Phi(u,\alpha,\bar{x})$ is the augmented Lagrangian function for dual problem of (5) (see [3]),

$$f_* = \max_{u\in U} b^Tu, \quad U=\{u\in R^m : A^Tu\leq c\}. \tag{D}$$

The function $\Phi(u,\alpha,\bar{x})$ is piecewise quadratic, convex and just has the first derivative, but it is not twice differentiable. Suppose that s and t are arbitrary points in R^m. Then for $\nabla\Phi_u(u,\alpha,\bar{x})$ we have

$$\|\nabla\Phi_u(s,\alpha,\bar{x})-\nabla\Phi_u(t,\alpha,\bar{x})\| \leq \|A\|\|A^T\|\|s-t\|, \tag{D}$$

this means that $\nabla\Phi$ is globally Lipschitz continues with constant $K=\|A\|\|A^T\|$. Thus for this function generalized Hessian exists and is defined the $m\times m$ symmetric positive semidefinite matrix [6, 9].

Now we introduce the following iterative process :

$$u^{k+1} = arg \min_{u \in R^n} \{-b^T u + \frac{1}{2\alpha} \parallel (x^k + \alpha(A^T u - c))_+ \parallel^2, \tag{12}$$

$$x^{k+1} = (x^k + \alpha(A^T u^{k+1} - c))_+, \tag{13}$$

where u^0 and x^0 are arbitrary starting point.

Theorem 2. *Let the solution set X_* of the problem (5) be nonempty. Then, for all $\alpha > 0$ and an arbitrary initial x^0, the iterative process (12), (13) converges to $x_* \in X_*$ in finite number of step k and the primal minimum norm solution $\widehat{x}_*$ was obtained after the first iteration from above process, i.e. $k = 1$. Furthermore, $u_* = u^{k+1}$ is an exact solution of the dual problem (D).*

The proof of the finite global convergence is given in [6]

Considering the advantage of the differentiability of the objective function of problems (12) and (D) allow us to use a quadratically convergent quasi-Newton algorithm with an Armijo stepsize [6] that makes the algorithm globally convergent.

We will now present a quasi-Newton-Armijo algorithm for solving the problem (12).

Algorithm 1 Newton method with the Armijo rule
Choose any $u_0 \in R^m$ and $tol > 0$
i=0;
while $\|\nabla f(u_i)_\infty\| \geq tol$
Choose $\alpha_i = max\{s, s\delta, s\delta^2, ...\}$ *such that*
$f(u_i) - f(u_i + \alpha_i d_i) \geq -\alpha_i \mu \nabla f(u_i)^T d_i$,
where $d_i = -\nabla^2 f(u_i)^{-1} \nabla f(u_i)$, $s > 0$ *be a constant,* $\delta \in (0, 1)$ *and* $\mu \in (0, 1)$.
$u_{i+1} = u_i + \alpha_i d_i$
$i = i + 1;$
end

In this algorithm, the Hessian may be singular, thus we used a modified Newton direction as follows :

$$-(\nabla^2 f(u_i) + \delta I_m)^{-1} \nabla f(u_i),$$

where δ is a small positive number ($\delta = 10^{-4}$), and I_m is the identity matrix of order m.

4 Numerical Experiments

In this section, we present some numerical results using the previous process on various randomly generated system (see Table 1) to illustrate the performance of the proposed algorithm. The algorithm has been tested using MATLAB *7.9.0* on a *Core 2 Duo 2.53 GHz* with main memory *4 GB*.
System generator creates a random matrix A for a given n, with singular value greater than or equal to 1. Then, we choose a random vector x from a uniform distribution on [0, 100]. Finally, we computed $b = Ax - |x|$. These systems are generated using the following MATLAB code:

%Sgen: Generate random system $A * x - |x| = b$ with infinitely many solutions;
$n = input('Enter n :');$
$A = spdiags(sign((rand(n,1) - 2 * rand(n,1))) + 2, 0, n, n);$ % generates matrices in the MATLAB sparse storage organization.
$x = 10 * ((rand(n,1) - rand(n,1)));$
$b = A * x - abs(x);$

Computational results for the test problems taken from our algorithm are given in Table 1.

The first column indicates the size of matrix A and the second column headed f indicates $\|A\widetilde{x^*} - |\widetilde{x^*}| - b\|$, the third column indicates minimum 1-norm solution and the forth column indicates 1-norm solution and the final column indicates CPU time.

Table 1: Minimum norm solution of $Ax - |x| = b$.

n	f	$\|\widetilde{x^*}\|_1$	$\|x^*\|_1$	time(sec)
10	0	0	2.3057e+003	0.13
100	1.4211e-014	3.7329e+003	1.3824e+004	0.49
200	1.1369e-013	1.2783e+004	3.4507e+004	0.73
300	5.6843e-014	1.5145e+004	5.7900e+004	1.45
400	5.6843e-014	1.2150e+004	6.6566e+004	1.88
500	1.1369e-013	1.8853e+004	8.6618e+004	2.78
600	1.1369e-013	2.2635e+004	9.4418e+004	4.32
700	1.1369e-013	2.5670e+004	1.1879e+005	5.66
800	1.1369e-013	3.6042e+004	1.3374e+005	6.89
900	1.1369e-013	3.7086e+004	1.5515e+005	11.02
1000	1.1369e-013	4.4200e+004	1.6893e+005	11.38
2000	1.1369e-013	7.5278e+004	3.3016e+005	74.18
3000	1.1369e-013	1.2762e+005	4.8708e+005	205.81
4000	1.1369e-013	1.7599e+005	6.7238e+005	493.59
5000	1.1369e-013	2.1200e+005	8.2995e+005	875.87

5 Conclusion

In this paper, the augmented Lagrangian algorithm was proposed and used for solving the minimum 1-norm solution to the absolute value equation. With this idea, we obtain the problem with fewer variables and smaller in size than the nonlinear problem (3). To obtain the solution to the reduced problem, we have proposed an extension of Newton's method with the step size chosen by the Armijo rule. As the numerical results show, this algorithm has appropriate speed in most of the problems and, specifically this can be observed in problems with a large sparse matrix A.

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References

1. Cottle, R.W. and Dantzig, G. *Complementary pivot theory of mathematical programming*, Linear Algebra Appl, 1 (1968), 103-125.

2. Cottle, R.W, Pang, J.S. and Stone, R.E. *The linear complementarity problem*, Academic Press, New York, 1992.

3. Evtushenko, Yu.G., Golikov, A.I. and Mollaverdi, N. *Augmented lagrangian method for large-scale linear programming problems*, Optimization Methods and Software, 20 (2005), 515-524.

4. Kanzow, C., Qi, H. and Qi, L. *On the minimum norm solution of linear programs*, J. Optim. Theory Appl, 116 (2003), 333-345.

5. Ketabchi, S. and Moosaei, H. *Minimum norm solution to the absolute value equation in the convex case*, J. Optim. Theory Appl, 154 (2012), 1080-1087.

6. Mangasarian, O.L. *A Newton method for linear programming*, J. Optim. Theory Appl, 121 (2004), 1-18.

7. Mangasarian, O.L. and Meyer, R.R. *Absolute value equations*, Linear Algebra Appl, 419 (2006), 359-367.

8. Moosaei, H., Ketabchi, S., Noor, M., Iqbal, J. and Hooshyarbakhsh, V. *Some techniques for solving absolute value equations*, Applied Mathematics and Computation, 268 (2015), 696-705.

9. Pardalos, P.M., Ketabchi, S. and Moosaei, H. *Minimum norm solution to the positive semidefinite linear complementarity problem*, Optimization, 63(3) (2014), 359-369.

10. Pardalos, P.M. and Rosen, J.B. *Global optimization approach to the linear complementarity problem*, SIAM Journal on Scientific and Statistical Computing, 9 (1988), 341-353.

11. Pardalos, P.M. and Ye, Y. *Class of linear complementarity problems solvable in polynomial time*, Linear Algebra and its Applications, 152 (1991), 3-17.

12. Prokopyev, O. *On equivalent reformulations for absolute value equations*, Computational Optimization and Applications, 44 (3) (2009), 363-372.

روش لاگرانژ بهبود یافته برای پیدا کردن جواب با کمترین نرم- ۱ دستگاه معادلات قدر مطلق

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چکیده : دراین مقاله روش لاگرانژ بهبود یافته برای بدست آوردن جواب با کمترین نرم- ۱ دستگاه معادلات قدر مطلق (AEV) ارائه می شود. این روش منجر می شود به حل مساله مینیمم سازی نامقید با تابع هدفی که تنها یکبار مشتق پذیر است. روش شبه نیوتن را برای حل این مساله بهینه سازی نا مقید پیشنهاد می کنیم. محاسبات عددی، مبین دقت بالا در همگرایی با تعداد کم تکرار در اغلب محاسبات است.

کلمات کلیدی : دستگاه معادلات قدر مطلق؛ جواب کمترین نرم؛ روش نیوتن تعمیم یافته؛ روش لاگرانژ بهبود یافته.