Iranian Journal of Numerical Analysis and Optimization
Vol. 13, No. 4, 2023, pp 695–710
https://doi.org/10.22067/ijnao.2023.81548.1230
https://ijnao.um.ac.ir/

How to cite this article
Research Article

Numerical solution of fractional Bagley–Torvik equations using Lucas polynomials

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Abstract

The aim of this article is to present a new method based on Lucas polynomials and residual error function for a numerical solution of fractional Bagley–Torvik equations. Here, the approximate solution is expanded as a linear combination of Lucas polynomials, and by using the collocation method, the original problem is reduced to a system of linear equations. So, the approximate solution to the problem could be found by solving this system. Then, by using the residual error function and approximating the error function by utilizing the same approach, we achieve more accurate results. In addition, the convergence analysis of the method is investigated. Numerical examples demonstrate the validity and applicability of the method.

AMS subject classifications (2020): Primary 45D05; Secondary 42C10, 65G99.

Keywords: Fractional Bagley–Torvik equation; Caputo derivative; Lucas polynomials; Residual error function; Convergence analysis.

1 Introduction

Fractional differential equations have important rules in many fields of science and engineering. For example, in viscoelasticity [4, 3], economic growth model and finance [5, 16], biology [24], control theory [8, 14, 20, 21], dynamics of particle [29], electrical circuits [8], and vibration [25], some issues can be modeled as fractional differential equations.

Received 11 March 2021; revised 7 July 2022; accepted 1 August 2022

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The fractional Bagley–Torvik equation was originally introduced in 1983 to describe the motion of an immersed plate in a Newtonian fluid [30] as

$$m\frac{d^2}{dx^2}U(x) + 2A\sqrt{\eta r}\frac{d^{\frac{3}{2}}}{dx^{\frac{3}{2}}}U(x) + cU(x) = 0,$$

where m, and A are the mass and area of the plate, respectively, r is the fluid density, c is the spring of stiffness, and η is viscosity. The solution of the Bagley–Torvik equation has been studied by researchers for the past two decades. In [26] authors applied the Adomian decomposition method for the solution of Bagley–Torvik equation. El-Gamel and Abd-El-Hadi [9] presented the Legendre-collocation method to approximate the solution of fractional Bagley–Torvik equations. Zolfaghari et al. [34] studied an application of the enhanced homotopy perturbation method to find the approximate solution of Bagley–Torvik equation. In [32], an integral transform method is considered for solving Bagley–Torvik equation. Srivastava, Shah, and Abass [28] proposed a numerical method for studying Bagley–Torvik equations based on the Gegenbauer wavelet together with block pulse function. In [10], authors presented Chelyshkov–Tau as an effective tool for solving Bagley–Torvik equation. Cenesiz, Keskin, and Kurnaz [6] solved Bagley–Torvik equations by using the generalized Taylor collocation method. Authors of [15] utilized hybrid functions approximation, which consists of the block pulse function and Bernoulli polynomials, for the numerical solution of Bagley–Torvik equations. Zahra and Van Daele [33] used a discrete spline function and nonstandard Grunwald–Letnikov and weighted and shifted Grunwald–Letnikov difference operators to propose the solution to Bagley–Torvik equations. El-Gamel and Abd-El-Hadi [9], by using Legendre basis functions, reduced Bagley–Torvik equation to a system of linear equations and by solving this system presented a numerical solution to the Bagley–Torvik equation. Authors of [27] introduced the numerical solution of Bagley–Torvik based on reproducing kernel Hilbert space. In [31], generalized Bessel functions of the first kind are applied for the numerical solution of the fractional Bagley–Torvik equation.

The outlines of the article are as follows: In section 2, we briefly introduce the Caputo fractional derivative, Fibonacci, and Lucas polynomials and describe their properties. In section 3, we construct a numerical method for a solution of fractional Bagley–Torvik equations using Lucas polynomials and residual error function. In section 4, the convergence analysis of the proposed method is studied. The numerical results for some problems are given in section 5, and at the end, we have a brief conclusion.

2 Basic definitions and requirements

Definition 1. If $\alpha > 0$, then the Caputo fractional derivative operator of order α is defined as

$$D^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{(m-\alpha-1)}f^{(m)}(t)d_t,$$

where $m-1<\alpha\leq m$.

The Caputo derivative has linear property and

$$D^{\alpha}(c)=0, \quad c \text{ is a constant.}$$

$$D^{\alpha}(x^k) = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha} & \text{if } k\geq\lceil\alpha\rceil,\ k\in\mathbb{N}, \\ 0 & \text{if } k<\lceil\alpha\rceil,\ k\in\mathbb{N}_0, \end{cases}$$

where $\mathbb{N}_0=\{0,1,2,\ldots\}$ and $\mathbb{N}$ is the set of natural numbers.

The Fibonacci polynomials $F_n(x)$ and and Lucas polynomials $L_n(x)$ are defined by recursive relations as

$$\begin{aligned} &F_0(x)=0, \quad F_1(x)=1, \\ &F_n(x)=xF_{n-1}(x)+F_{n-2}(x), \quad n\geq 2, \end{aligned}$$

and

$$\begin{aligned} &L_0(x)=2, \quad L_1(x)=x, \\ &L_n(x)=xL_{n-1}(x)+L_{n-2}(x), \quad n\geq 2, \end{aligned}$$

respectively. Here we remark that Fibonacci and Lucas polynomials are the special case of Chebyshev polynomials (see [22]). Lucas polynomials have explicit form as

$$L_n(x)=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n}{n-i}\binom{n-i}{i}x^{n-2i}, \quad n\geq 1,$$

where $\lfloor x\rfloor$ is the largest integer less or equal to x. According to [18], the first derivative of Lucas polynomials can be evaluated using the Fibonacci polynomials as

$$L_n'(x)=nF_n(x). \tag{1}$$

Continuing this approach by repeating derivation on both sides of (1) gives

$$L_n^{(k)}(x)=nF_n^{(k-1)}(x), \quad k\geq 2.$$

If $u(x)$ is a continuous function, then we can approximate $u(x)$ by a linear combination of Lucas polynomials as

$$u(x)\approx\sum_{j=0}^{m}a_jL_j(x)=\mathbf{L}(x)\mathbf{A},$$

where $\mathbf{L} = [L_0(x), L_1(x), \ldots, L_m(x)]$ and $\mathbf{A} = [a_0, a_1, \ldots, a_m]^T$. Moreover, for the kth derivation of $u(x)$, we have the approximation

$$u^{(k)}(x) \approx \sum_{j=0}^{m} a_j L_j^{(k)}(x).$$

Therefore, as mentioned in [11], the approximation for the kth ($k \geq 2$) derivation of $u(x)$ can be formulated as

$$u^{(k)}(x) \approx n\mathbf{F}(x)D^{k-1}\mathbf{A},$$

where

$$\mathbf{F} = [F_0(x), F_1(x), \ldots, F_m(x)],$$

$$\mathbf{D}_{(m+1)\times(m+1)} = \begin{pmatrix} 0 \ldots 0 & \\ \vdots \;\; \mathbf{d} & \\ 0 & \end{pmatrix},$$

and $\mathbf{d}$ is an $m \times m$ matrix, which is defined as

$$\mathbf{d}_{i,j} = \begin{cases} i \;\; \sin\frac{(j-i)\pi}{2} & \text{if } j > i, \\ 0 & \text{if } j \leq i. \end{cases}$$

Further details about Lucas polynomials and application of Lucas polynomials for solving problems arising in engineering, such as ordinary and partial differential equations, can be found in [1, 2, 7, 12, 11, 13, 18, 19].

3 Construction of method

Consider the fractional Bagley–Torvik equation

$$A \; D^2 f(x) + B \; D^{\frac{3}{2}} f(x) + C \; f(x) = g(x), \quad x \in [0, 1], \tag{2}$$

with initial conditions

$$f(0) = f_0, \quad f'(0) = f_0',$$

or boundary conditions

$$f(0) = f_0, \quad f(1) = f_1.$$

By using the Caputo fractional derivation, (2) can be rewritten as

$$A \; D^2 f(x) + \frac{B}{\Gamma(\frac{1}{2})} \int_0^x (x-t)^{-\frac{1}{2}} f''(t) d_t + C \; f(x) = g(x). \tag{3}$$

Let the approximate estimation for the solution of (2) have the following form:

$$f(x) \approx \sum_{j=0}^{M} \alpha_j L_j(x). \tag{4}$$

Now, by collocating at the nodes $\{x_i : i = 1, \dots, M-1\}$, where $0 < x_1 < \cdots < x_{M-1} < 1$, and utilizing (3), we get

$$A\sum_{j=0}^{M} \alpha_j L_j''(x_i) + \frac{B}{\Gamma(\frac{1}{2})}\sum_{j=0}^{M} \alpha_j \int_0^{x_i} (x_i - t)^{-\frac{1}{2}} L_j''(t) d_t + C\sum_{j=0}^{M} \alpha_j L_j(x_i) = g(x_i). \tag{5}$$

Also, initial and boundary conditions lead to

$$\sum_{j=0}^{M} \alpha_j L_j(0) = f_0, \quad \sum_{j=0}^{M} \alpha_j L_j'(0) = f_0', \tag{6}$$

and

$$\sum_{j=0}^{M} \alpha_j L_j(0) = f_0, \quad \sum_{j=0}^{M} \alpha_j L_j(1) = f_1, \tag{7}$$

respectively. Hence, the combination of (5) together with (6) or (7) gives a system of linear equations as

$$\mathbf{U}\lambda = \mathbf{b},$$

where, for initial conditions,

$$\mathbf{b} = [g(x_1), \dots, g(x_{M-1}), f_0, f_0']^T,$$

$$\mathbf{U}_{i,j} = \begin{cases} AL_{j-1}''(x_i) + \frac{B}{\Gamma(\frac{1}{2})}\int_0^{x_i} (x_i - t)^{-\frac{1}{2}} L_{j-1}''(t) d_t + CL_{j-1}(x_i) & \text{if } 1 \le i \le M-1, \\ L_{j-1}(0) & \text{if } i = M, \\ L_{j-1}'(0) & \text{if } i = M+1, \end{cases}$$

and for boundary conditions,

$$\mathbf{b} = [g(x_1), \dots, g(x_{M-1}), f_0, f_1]^T,$$

$$\mathbf{U}_{i,j} = \begin{cases} AL_{j-1}''(x_i) + \frac{B}{\Gamma(\frac{1}{2})}\int_0^{x_i} (x_i - t)^{-\frac{1}{2}} L_{j-1}''(t) d_t + CL_{j-1}(x_i) & \text{if } 1 \le i \le M-1, \\ L_{j-1}(0) & \text{if } i = M, \\ L_{j-1}(1) & \text{if } i = M+1. \end{cases}$$

For example, if $M = 2$, by using Chebyshev–Gauss–Lobatto nodes, U has the following structure:

$$U_I = \begin{bmatrix} 2C & \frac{C}{2} & 2A + 2\sqrt{2}\frac{B}{\sqrt{\pi}} + 9\frac{C}{4} \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$U_B = \begin{bmatrix} 2C & \frac{C}{2} & 2A + 2\sqrt{2}\frac{B}{\sqrt{\pi}} + 9\frac{C}{4} \\ 2 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

Therefore, by solving the obtained system of linear equations, the approximate solution of the fractional Bagley–Torvik equation is determined. Here, we present a more accurate method using the residual error function [7, 17] for the solution of the fractional Bagley–Torvik equation. If we display the error of approximation (4), as

$$e(x) = f(x) - \sum_{j=0}^{M} \alpha_j L_j(x),$$

then the error function satisfies the fractional differential equation

$$A\ D^2 e(x) + B\ D^{\frac{3}{2}} e(x) + C\ e(x) = R(x), \tag{8}$$

where

$$R(x) = g(x) - A \sum_{j=0}^{M} \alpha_j L_j''(x) - \frac{B}{\Gamma(\frac{1}{2})} \sum_{j=0}^{M} \alpha_j \int_0^x (x-t)^{-\frac{1}{2}} L_j''(t) d_t \tag{9}$$

$$- C \sum_{j=0}^{M} \alpha_j L_j(x). \tag{10}$$

The above fractional differential equation is accompanied with initial conditions

$$e(0) = e'(0) = 0, \tag{11}$$

or the boundary conditions

$$e(0) = e(1) = 0. \tag{12}$$

Now, we propose the approximate solution to (8)–(12) using Lucas polynomials as

$$e(x) \approx \sum_{j=0}^{N} \beta_j L_j(x), \quad N > M.$$

By using the idea described above, we can get the an approximation for error function $e(x)$. So, we obtain a better approximation

$$f(x) \approx \sum_{j=0}^{M} \alpha_j L_j(x) + \sum_{j=0}^{N} \beta_j L_j(x)$$

for the numerical solution of the fractional Bagley–Torvik equation.

4 Convergence analysis

In this section, we argue about the convergence of the proposed method. For this aim, first, some requirements are given.

Lemma 1. [1] Assume that $f(x)$ is an infinitely differentiable at $x = 0$. Then $f(x)$ can be represented by using Lucas polynomials as

$$f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \delta_i f^{(i+2j)}}{j!(i+j)!} L_i(x),$$

where

$$\delta_i = \begin{cases} \frac{1}{2} & \text{if } i = 0, \\ 1 & \text{if } i \neq 0. \end{cases}$$

Lemma 2. [1] For every $i \geq 0$, the Lucas polynomials can be bounded as

$$|L_i(x)| \leq 2\sigma^i,$$

where σ is the golden ratio.

Theorem 1. [1] Let $f(x)$ be defined on $[0, 1]$, and there is a positive constant A such that $|f^{(i)}(0)| \leq A^i$, $i \geq 0$. Moreover, suppose that $f(x)$ has a representation

$$f(x) = \sum_{i=0}^{\infty} c_i L_i(x), \tag{13}$$

and that $e(x) = \sum_{i=M+1}^{\infty} c_i L_i(x)$ is an error of approximation function $f(x)$ by Lucas polynomials of degree M. Then

$$|c_i| \leq \frac{A^i \cosh(2A)}{i!}$$

and the series (13) is convergent and

$$|e(x)| < \frac{2e^{A\sigma} \cosh(2A)(A\sigma)^{M+1}}{(M+1)!}.$$

In the following theorem, we discuss the convergence of the presented method of the previous section.

Theorem 2. Let $f(x)$ be an infinitely differentiable at $x = 0$, and there is a constant $A > 0$ such that $|f^{(i)}(0)| \leq A^i$, $i > 0$. If $e(x)$ is defined as $e(x) = f(x) - \sum_{i=0}^{M} a_i L_i(x)$ and has a representation

$$e(x) = \sum_{i=0}^{\infty} b_i L_i(x),$$

then the proposed method has the error estimation

$$|E(x)| < \frac{2e^{\mathcal{A}\sigma}\cosh(2\mathcal{A})(\mathcal{A}\sigma)^{N+1}}{(N+1)!}.$$

Proof. According to the previous section, the approximation

$$f(x) \approx \sum_{i=0}^{M} a_i L_i(x) + \sum_{i=0}^{N} b_i L_i(x)$$

] has the error $E(x) = e(x) - \sum_{i=0}^{N} b_i L_i(x)$. Also,

$$e^{(i)}(x) = f^{(i)}(x) - \sum_{j=0}^{M} a_j L_j^{(i)}(x).$$

Since $L_j(x)$ is a polynomial of degree j, so $L_j^{(i)}(x)$ has the following representation:

$$L_j^{(i)}(x) = \begin{cases} \alpha_{j_0} + \alpha_{j_1} x + \cdots + \alpha_{j_{j-i}} x^{j-i} & \text{if } j \geq i, \\ 0 & \text{if } j < i. \end{cases}$$

Therefore

$$L_j^{(i)}(0) = \begin{cases} \alpha_{j_0} & \text{if } j \geq i, \\ 0 & \text{if } j < i. \end{cases}$$

If we set $P = \max\{|L_j^{(i)}(0)| : \ i, j = 0, \dots, M\}$, then for $i = 1, \dots, M$ by using Theorem 1, we get

$$\begin{aligned} |e^{(i)}(0)| &< A^i + \sum_{j=0}^{M} \frac{A^j \cosh(2A)P}{j!} \\ &< (A + \cosh(2A)Pe^A)^i. \end{aligned}$$

Moreover, for $i \geq M+1$, we have $|e^{(i)}(0)| \leq A^i$. If we apply Theorem 1 for the function $e(x) = f(x) - \sum_{i=0}^{M} a_i L_i(x)$, then

$$|E(x)| \leq \frac{2e^{\mathcal{A}\sigma}\cosh(2\mathcal{A})(\mathcal{A}\sigma)^{N+1}}{(N+1)!},$$

where

$$\mathcal{A} = A + \cosh(2A)Pe^{A}.$$

□

5 Numerical results

In this section, some examples are presented to show the accuracy of the proposed method. These examples consist of initial and boundary conditions. Also, to show the accuracy and validity of the proposed method, we have a comparison between our approach and a number of other methods. In computations, we utilize Chebyshev–Gauss–Lobatto nodes as collocation points, and all of the computations have been performed in MAPLE 18 software.

Example 1. Consider fractional Bagley–Torvik equation

$$D^2 f(x) + D^{\frac{3}{2}} f(x) + f(x) = x^3 + 7x + 1 + \frac{8x^{\frac{3}{2}}}{\sqrt{\pi}}$$

with the initial conditions $f(0) = 1$ and $f'(0) = 1$. This problem has the exact solution $f(x) = x^3 + x + 1$. Here we take $M = 6$ and $N = 10$. We compare the Lucas collocation method (LCM) and Lucas collocation method combined with residual error function (LCM-REF) with the Chelyshkov–Tau method [10] and Legendre collocation method [9]. Results are given in Table 1. Absolute errors of LCM and LCM-REF are listed in Table 2 and plotted in Figure 1.

Table 1: Comparisons of the presented methods for Example 1

x	Exact solution	LCM-REF	LCM	Chelyshkov–Tau [10]	Legendre collocation [9]
0.1	1.101000	1.101000	1.101000	1.101000	1.101000
0.25	1.265625	1.265625	1.265625	1.265625	1.265625
0.5	1.625000	1.625000	1.625000	1.625000	1.625000
0.75	2.171875	2.171875	2.171875	2.171875	2.171875
1	3.000000	3.000000	3.000000	3.000000	3.000002

Table 2: Absolute errors of the presented methods for Example 1

x	0.1	0.3	0.5	0.7	0.9
LCM	8.90000E-48	1.22000E-47	1.67000E-47	2.23000E-47	3.28000E-47
LCM-REF	8.72634E-48	1.06045E-47	1.19949E-47	1.30881E-47	1.39519E-47

Example 2. In this example, we study the fractional Bagley–Torvik equation

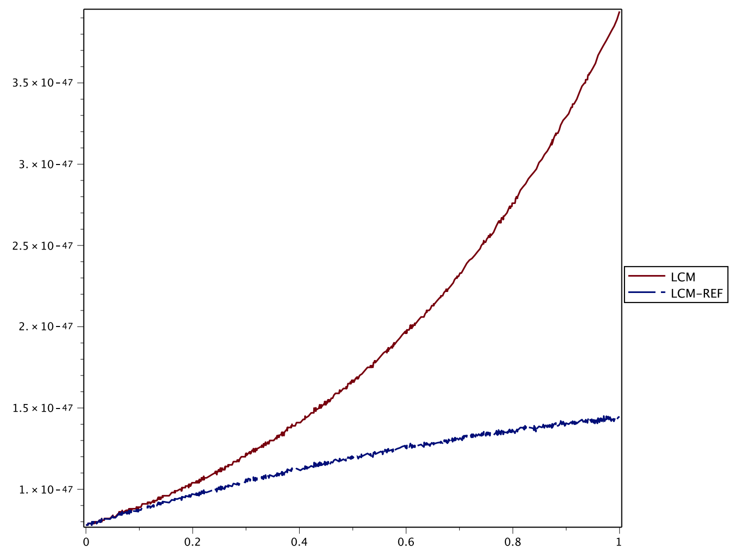

Figure 1: The plot of absolute errors of LCM and LCM-REF for Example 1

$$D^2 f(x) + \frac{8}{17} D^{\frac{3}{2}} f(x) + \frac{13}{51} f(x) = \frac{x^{-\frac{1}{2}}}{89250\sqrt{\pi}}(48p(x) + 7\sqrt{x}q(x)),$$

where

$$\begin{aligned} p(x) =& 16000x^4 - 32480x^3 + 21280x^2 - 4746x + 189, \\ q(x) =& 3250x^5 - 9425x^4 + 264880x^3 - 44, \end{aligned}$$

with the boundary conditions $f(0) = 0$, $f(1) = 0$. This problem has the exact solution

$$f(x) = x^5 - \frac{29}{10}x^4 + \frac{76}{25}x^3 - \frac{339}{250} + \frac{27}{125}x.$$

We examine the proposed method with $M = 6$, $N = 10$. In Table 3, a comparison between absolute errors of Lucas collocation method combined with residual error function (LCM-REF), Chelyshkov-Tau method [10], Harr wavelets method [23], and Bessel collocation method [31] is given. In Figure 2, the plot of the exact solution and approximate solution, which is obtained by the combination of the Lucas collocation method and residual error function, is displayed.

Example 3. Consider the fractional Bagley–Torvik equation

$$A\ D^2 f(x) + B\ D^{\frac{3}{2}} f(x) + C\ f(x) = g(x),$$

Table 3: Comparisons of LCM-REF for Example 2

x	LCM-REF	Chelyshkov–Tau [10]	Harr wavelets [23]	Bessel collocation [31]
0.1	2.69915E-48	5.92720E-14	6.49908E-7	1.0800E-2
0.2	3.21281E-48	1.18400E-13	6.35657E-7	8.9595E-3
0.3	3.72702E-48	1.77249E-13	3.71584E-7	3.7797E-3
0.4	4.23527E-48	2.35568E-13	9.48220E-7	1.4413E-7
0.5	4.71891E-48	2.17578E-13	1.59573E-6	1.0001E-3
0.6	5.20793E-48	2.92504E-13	1.05494E-6	6.6150E-8
0.7	5.69369E-48	3.82671E-13	6.34678E-7	1.2599E-3
0.8	6.20644E-48	3.82256E-13	1.88690E-6	1.2800E-3
0.9	6.54825E-48	2.90107E-13	3.13999E-6	2.0656E-8

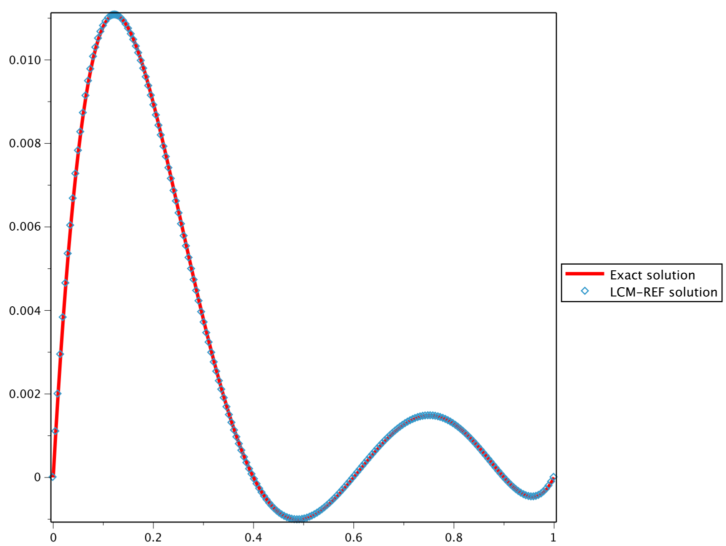

Figure 2: The plot of exact and LCM-REF solutions for Example 2

with the initial conditions $f(0) = 0,\ f'(0) = 0$. This problem has the exact solution

$$f(x) = \int_0^x G_3(x-\tau)g(\tau)d_\tau = \frac{1}{A}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\frac{C}{A})^k x^{2k+1}E^{(k)}_{\frac{1}{2},2+\frac{3k}{2}}(-\frac{B}{A}\sqrt{x}),$$

where $G_3(x)$ is three-term Green's function, which is defined as

$$G_3(x) = \frac{1}{A}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\frac{C}{A})^k x^{2k+1}E^{(k)}_{\frac{1}{2},2+\frac{3k}{2}}(-\frac{B}{A}\sqrt{x}),$$

and $E_{\lambda,\mu}$ is the Mittag-Leffler function with two parameters λ and μ, and

$$E_{\lambda,\mu}^{(k)}(y) = \sum_{j=0}^{\infty} \frac{(j+k)!y^j}{j!\Gamma(\lambda j + \lambda k + \mu)}, \quad k = 0, 1, 2, \dots .$$

Let $A = 1$, $B = \frac{1}{2}$, $C = \frac{1}{2}$, and $g(x) = 8$. For this case, we choose $M = 30$ and $N = 40$. Numerical comparisons of the proposed methods with Chelyshkov–Tau method [10], Legendre collocation method [9], and generalized Taylor collocation method [6] are listed in Table 4. In Table 5, absolute errors of LCM and LCM-REF are displayed. Figure 3 exhibits the comparison of analytical and LCM-REF solutions of this example. The plot of absolute errors of LCM and LCM-REF is illustrated in Figure 4.

Table 4: Comparisons of the presented methods for Example 3

x	Exact solution	LCM-REF	LCM	Chelyshkov-Tau [10]	Taylor-collocation [6]	Legendre collocation [9]
0.1	0.036487	0.036486	0.036483	0.036453	0.036485	0.036471
0.2	0.140639	0.140636	0.140632	0.140575	0.140634	0.140615
0.3	0.307484	0.307480	0.307473	0.307403	0.307476	0.307434
0.4	0.533284	0.533278	0.533269	0.533252	0.533271	0.533225
0.5	0.814756	0.814749	0.814739	0.814860	0.814735	0.814661
0.6	1.148837	1.148828	1.148816	1.149069	1.148805	1.148733
0.7	1.532565	1.532555	1.532541	1.532870	1.532521	1.532424
0.8	1.963029	1.963018	1.963002	1.963440	1.962974	1.962874
0.9	2.437334	2.437322	2.437305	2.437829	2.437455	2.437134

Table 5: Absolute errors of the presented methods for Example 3

x	0.1	0.3	0.5	0.7	0.9
LCM	4.23327E-6	1.16836E-5	1.82885E-5	2.41776E-5	2.93789E-5
LCM-REF	1.78503E-6	4.92605E-6	7.71291E-6	1.01973E-5	1.23905E-5

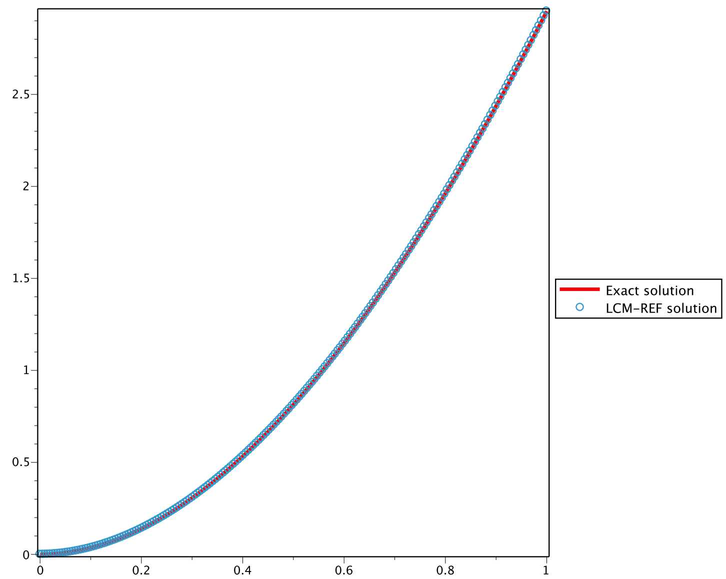

Figure 3: The plot of exact and LCM-REF solutions for Example 3

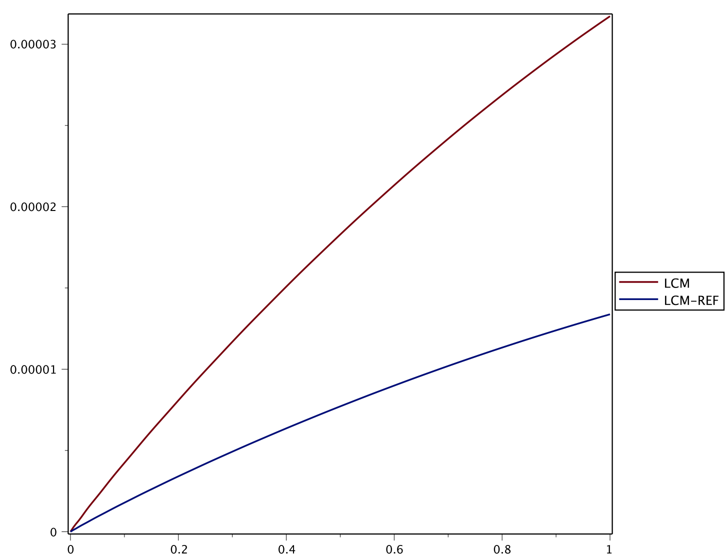

Figure 4: Absolute errors of LCM and LCM-REF for example 3

6 Conclusion

A new numerical method using Lucas polynomials was proposed to solve the fractional Bagley–Torvik equation. In this approach, we expanded the exact solution as a finite linear combination of Lucas polynomials. Then, by using Chebyshev–Gauss–Lobatto nodes as collocation points, the approximate solution was obtained. To improve the results, we applied the residual error function, and the error function was estimated by Lucas polynomials. So we can improve the results and get a more accurate approximation. Numerical tests and comparisons with other numerical methods indicated that this method is reliable and has acceptable accuracy.

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How to cite this article

Askari, M., Numerical solution of fractional Bagley–Torvik equations using Lucas polynomials. *Iran. J. Numer. Anal. Optim.*, 2023; 13(4): 695-710.
https://doi.org/10.22067/ijnao.2023.81548.1230