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Research Article

Chebyshev pseudo-spectral method for optimal control problem of Burgers' equation

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Abstract

In this study, an indirect method is proposed based on the Chebyshev pseudo-spectral method for solving optimal control problems governed by Burgers' equation. Pseudo-spectral methods are one of the most accurate methods for solving nonlinear continuous-time problems, specially optimal control problems. By using optimality conditions, the original optimal control problem is first reduced to a system of partial differential equations with boundary conditions. Control and state functions are then approximated by interpolating polynomials. The convergence is analyzed, and some numerical examples are solved to show the efficiency and capability of the method.

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Keywords: Burgers' equation; Optimal control; Chebyshev-Gauss-Lobatto nodes.

1 Introduction

Time-dependent partial differential equations (PDEs) play an important role in various fields of application, such as fluid dynamics [4,28], electro magnetics [17,24], or heat transfers [10]. Often, these differential equations consist of

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nonlinear terms, which are challenging to solve numerically. Furthermore, in engineering applications, we often interest in an optimal solution of the considered PDE with a certain objective function. This leads to the mathematical field of PDE constrained optimization where a cost function is minimized and the PDE is considered as a constraint.

Optimal control of viscous Burgers' equation is one of the most important PDEs constraint optimization, which is taken into consideration and several papers have recently been presented in its numerical solution. Kobayashi [12] has concerned with adaptive stabilization and regulator design for a viscous Burgers' equation by nonlinear boundary control. Yilmaz and Karasozen [29] by using the high level modeling and simulation package COMSOL Multiphysics solved the optimal control of unsteady Burgers' equation without constraints and with control constraints. Also, they applied an all-at-once method for the optimal control of the unsteady Burgers' equation [30]. Moreover, in other work, they [11] transformed the optimality system for boundary controlled unsteady Burgers' equation after linearization into a biharmonic equation in the space-time domain. Smaoui et al. [23] have dealt with the sliding mode control (SMC) of the forced generalized Burgers' equation via the Karhunen–Loeve (K-L) Galerkin method. Hashemi and Werner [8] presented a finite difference scheme for the one-dimensional viscous Burgers' equation, which boundary conditions are taken as control inputs. Zeng and Zhang [31] designed a new preconditioning technique along with MINRES (minimal residual) in nonstandard inner product for the linear system of equations arising for optimal control of the unsteady Burgers' equation. Kucuk and Sadek [13] analyzed the dynamics of the forced Burgers' equation subject to Dirichlet boundary conditions by using the boundary control with the objective of minimizing the distance between the final state function and target profile along with the energy of the control. Also, they applied a robust technique for solving optimal control of coupled Burgers' equation; see [21]. Noack and Walther [16] used adjoint techniques for efficient evaluations of the gradient of the objective in gradient based on optimization algorithms. Marburger and Pinnau [15] showed the convergence of method to solve optimal control problems, where the constraints have been discretized by a particle method. Allahverdi et al. [2] discussed the efficiency of various numerical methods for the inverse design of the Burgers' equation. An augmented Lagrangian-SQP technique depending upon second-order sufficient optimality condition is analyzed in [27]. In [5], a comparison of three different numerical methods for optimal control of Burgers' equation is carried out. In [25], a Lagrangian-Newton-SQP method is presented for the solution of optimal control of Burgers' equation, where the control is restricted by pointwise lower and upper bounds. Proper orthogonal decomposition method is utilized in [14] to solve optimal control problems of the Burgers' equation. Furthermore, some controllability results for viscous Burgers' equation with distributed controls are presented in [6].

In some papers, at first the problem is discretized on one of the variables and then the Runge–Kutta method is applied to discrete the optimality conditions. To achieve good results, we have to take the number of points of discretization big enough that it can be too time consuming and has computational complexity.

The pseudo-spectral (PS) methods are one of the most accurate methods to solve continuous-time problems including ordinary differential equations (ODEs) and PDEs. Using the direct Chebyshev PS(CPS) method for optimal control problems governed by PDEs does not usually give an explicit solution. Hence, we use an indirect CPS method for solving necessary optimal conditions in an optimal control problem of Burgers' equation and we obtain an approximate optimal solution. It is the first time that the CPS method is applied by discretization of variables synchronic. By numerical examples, it can be seen that the CPS method is more effective than other methods and we can achieve the better results for the solution of optimal control problem of Burgers' equation. In fact, the error of CPS method is less than that of other methods.

The paper is organized as follows: In Section 2, we introduce the optimal control problem of Burgers' equation. In Section 3, we give the optimality conditions for optimal control problem of Burgers' equation. In Section 4, we utilize the CPS method to discretize optimality conditions. In Section 5, the convergence of the method is analyzed. Finally, the numerical examples and conclusion are given in Sections 6 and 7, respectively.

2 Optimal control problem of Burgers' equation

The distributed optimal control problem for the Burgers' equation can be stated as follows:

$$\text{Minimize} \quad J[y,u] = \frac{1}{2}\int_0^T \int_a^b (y(t,x)-z(t,x))^2 dxdt + \frac{\alpha}{2}\int_0^T \int_a^b u^2(t,x)dxdt \tag{1}$$

subject to

$$y_t(t,x) + y(t,x)y_x(t,x) - \nu y_{xx}(t,x) = \Phi(u), \; (t,x) \in Q = [0,T]\times[a,b], \tag{2}$$

$$y(t,a) = y(t,b) = 0, \quad t \in \Sigma = [0,T], \tag{3}$$

$$y(0,x) = y_0(x), \quad x \in \Omega = [a,b], \tag{4}$$

where $y(.,.)$ is the state variable, $u(.,.)$ is the control variable, $\alpha > 0$ is the regularization parameter, $\nu > 0$ denotes the viscosity parameter, and Φ is a given function. An usual selection of function Φ is as follows:

$$\Phi(u) = \begin{cases} u & u\ in\ \overline{\Omega}, \\ 0 & u\ in\ \Omega - \overline{\Omega}, \end{cases}$$

where $\overline{\Omega}$ is the set of active controls; see [5, 25, 27].

At first, optimality conditions for problem (1)–(4) are given, then we indirectly develop the CPS method to achieve an approximate optimal solution.

3 Optimality conditions

Here, we summarize the process of achieving the optimality conditions for the optimal control problem (1)–(4).
Let $L^1_{loc}(\Omega)$ be the set of all functions that are Lebesgue integrable on every compact subset of Ω and let $W^{k,2}(\Omega)$ be the linear space of all functions $w \in L^2(\Omega)$ having weak derivatives $D^\alpha w$ in $L^2(\Omega)$ for all multi-indices α of length $\mid \alpha \mid \leq k$. Assume that $C_0^\infty(\Omega)$ is the space of differentiable functions of every arbitrary order on Ω. The closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by $W_0^{k,p}(\Omega)$. Moreover, we define $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

Definition 1. Let $y \in L^1_{loc}(\Omega)$ and let some multi-index α be given. If a function $\omega \in L^1_{loc}(\Omega)$ satisfies

$$\int_\Omega y(x) D^\alpha v(x) dx = (-1)^{|\alpha|} \int_\Omega \omega(x) v(x) dx, \qquad \text{for all } v \in C_0^\infty(\Omega),$$

then ω is called the weak derivative of y (associated with α).

Let $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ be Hilbert spaces. We make use of the following Hilbert space

$$W(0,T) = \{\phi \in L^2([0,T]; V) : \phi_t \in L^2([0,T]; V^\star)\},$$

where $V^\star$ denotes the dual space of V. The inner product in the Hilbert space V is given with the natural inner product in H as

$$(\phi, \psi)_V = (\phi', \psi')_H, \quad for\ \phi, \psi \in V,$$

where ϕ' and ψ' are the derivative of ϕ and ψ, respectively.

Definition 2. Every function $y \in W(0,T)$ that satisfies

$$\begin{cases} < y_t(t), \phi >_{V^\star, V} + \nu (y_t(t), \phi)_V + (y(t) y_x(t), \phi)_H = ((f + \Phi(u))(t), \phi)_H \\ \phi \in V,\ t \in [0,T],\ (y(0), \chi)_H = (y_0, \chi), \quad \chi \in H, \end{cases}$$

where

$$< y_t(t), \phi >_{V^\star,V} = \int_\Omega \eta.\phi dx + \int_\Omega \eta'.\phi' dx, \quad \eta \in V,$$

is called a weak solution for Burgers' equation (2) and conditions (3) and (4).

In order to achieve the optimality conditions, the operator $e : X \longrightarrow Y$ is introduced by

$$e(y,u) = (e_1(y,u), e_2(y,u)) = (y_t - \nu y_{xx} + yy_x - f - \Phi(u), y(0) - y_0),$$

where $y(0) = y(0,x)$, $X = W(V) \times L^2(\bar{\Omega})$, and $Y = L^2(V) \times H$ is identified with $Y^\star = L^2(V^\star) \times H$ the dual of Y. We use $L^2(V)$ for $L^2(\Sigma; V)$. Now, the optimal control problem (1)–(4) can be transformed as the following minimization problem with equality constraints

$$\begin{aligned} &\text{minimize} \quad J(y,u) \\ &\textit{subject to} \quad e(y,u) = 0. \end{aligned}$$

Theorem 1. *Let $(y^\star, u^\star)$ be an optimal solution of (1)–(4). Then there exist Lagrange multipliers $p^\star : [0,T] \times \Omega \longrightarrow \Re$ and $\lambda^\star$ satisfying the first-order necessary optimality conditions*

$$L'(y^\star, u^\star, p^\star, \lambda^\star) = 0, \;\; e(y^\star, u^\star) = 0$$

with the Lagrangian

$$L(y,u,p,\lambda) = J(y,u) - (e_1(y,u), p)_{L^2(V^\star), L^2(V)} - (e_2(y,u), \lambda)_H.$$

Hence, first-order optimality conditions lead to the following optimality system (see [9, 26, 27])

$$\begin{cases} y_t - \nu y_{xx} + yy_x = \Phi(u), & (t,x) \in \; Q, \\ p_t + \nu p_{xx} + yp_x = y_d - y, & (t,x) \in \; Q, \\ y(t,a) = y(t,b) = 0, & t \in \; \Sigma, \\ y(0,x) = y_0, & x \in \; \Omega, \\ p(t,a) = p(t,b) = 0, & t \in \; \Sigma, \\ p(T,x) = 0, & x \in \; \Omega, \\ \alpha u + p = 0, & (t,x) \in \; Q. \end{cases} \tag{5}$$

From the last equation of system (5), we have

$$u = -\frac{1}{\alpha} p. \tag{6}$$

By using (6), we can express (5) as follows:

$$\begin{cases} y_t - \nu y_{xx} + y y_x = \Phi(-\frac{1}{\alpha} p), & (t,x) \in \ Q, \\ p_t + \nu p_{xx} + y p_x = y_d - y, & (t,x) \in \ Q, \\ y(t,a) = y(t,b) = 0, & t \in \ \Sigma, \\ p(t,a) = p(t,b) = 0, & t \in \ \Sigma, \\ y(0,x) = y_0(x), & x \in \ \Omega, \\ p(T,x) = 0, & x \in \ \Omega. \end{cases} \tag{7}$$

Now, we utilize the CPS method to solve optimality conditions (7) and obtain an approximate optimal solution for the optimal control problem (1)–(4).

4 CPS method to approximate the optimal solution

Chebyshev polynomials are in center of the CPS method, and hence we first introduce these polynomials and then demonstrate the CPS method to solve optimality conditions (7).

Chebyshev polynomials [1,18] are orthogonal polynomials, which play an important role in the theory of approximation. These polynomials are defined on $[-1,1]$ as follows:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad \dots . \tag{8}$$

These polynomials can be generated from the following recurrence relationship:

$$T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x), \quad j \geq 1, \ x \in [-1,1]. \tag{9}$$

It is possible to give an explicit expression of Chebyshev polynomials as

$$T_j(x) = \cos(j \cos^{-1} x). \tag{10}$$

It is now easy to verify that $T_N(.)$ has N zeros within the interval $(-1,1)$ as $x_k = \cos \frac{(2k-1)\pi}{2N}$, $\ k = 1,2,\dots,N$. However, in this paper, we use the CGL points on $[-1,1]$ which are as follows:

$$x_j = -\cos \frac{\pi j}{N}, \quad 0 \leq j \leq N, \tag{11}$$

where they are the roots of $(1-x^2)T'_N(x)$. For interpolating in the CPS method, the following Lagrange polynomials are utilized:

$$L_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{N} \frac{x - x_j}{x_k - x_j} = \frac{2}{N\mu_k} \sum_{j=0}^{N} \frac{1}{\mu_j} T_j(x_k) T_j(x), \quad k = 0,1,\dots,N, \ x \in [-1,1] \ ,$$

where

$$\mu_j = \begin{cases} 2, & j = 0, N, \\ 1, & 1 \le j \le N-1, \end{cases}$$

and we have

$$L_k(t_j) = \delta_{kj} = \begin{cases} 1, & j = k, \\ 0, & j \ne k. \end{cases} \tag{12}$$

To use the CPS method, the variables of system (7) are transformed to interval $[-1,1]$ by the following linear transformations:

$$\begin{cases} t = \frac{T}{2}\bar{t} + \frac{T}{2}, & t \in [0,T], \quad \bar{t} \in [-1,1], \\ x = \frac{b-a}{2}\bar{x} + \frac{b+a}{2}, & x \in [a,b], \quad \bar{x} \in [-1,1]. \end{cases} \tag{13}$$

By (13), the optimality conditions (7) can be written as follows:

$$\begin{cases} Y_{\bar{t}} = \psi_1\left(Y(\bar{t},\bar{x}), P(\bar{t},\bar{x}), Y_{\bar{x}}(\bar{t},\bar{x}), Y_{\bar{x}\bar{x}}(\bar{t},\bar{x})\right), & (\bar{t},\bar{x}) \in [-1,1]\times[-1,1], \\ P_{\bar{t}} = \psi_2\left(Y(\bar{t},\bar{x}), P(\bar{t},\bar{x}), P_{\bar{x}}(\bar{t},\bar{x}), P_{\bar{x}\bar{x}}(\bar{t},\bar{x})\right), & (\bar{t},\bar{x}) \in [-1,1]\times[-1,1], \\ Y(\bar{t},-1) = Y(\bar{t},1) = 0, & \bar{t} \in [-1,1], \\ P(\bar{t},-1) = P(\bar{t},1) = 0, & \bar{t} \in [-1,1], \\ Y(-1,\bar{x}) = Y_0(\bar{x}), & \bar{x} \in [-1,1], \\ P(1,\bar{x}) = 0, & \bar{t} \in [-1,1], \end{cases} \tag{14}$$

where

$$Y(\bar{t},\bar{x}) = y(\frac{T}{2}\bar{t} + \frac{T}{2}, \frac{b-a}{2}\bar{x} + \frac{b+a}{2}),$$

$$P(\bar{t},\bar{x}) = p(\frac{T}{2}\bar{t} + \frac{T}{2}, \frac{b-a}{2}\bar{x} + \frac{b+a}{2}),$$

$$Y_d(\bar{t},\bar{x}) = y_d(\frac{T}{2}\bar{t} + \frac{T}{2}, \frac{b-a}{2}\bar{x} + \frac{b+a}{2}),$$

$$Y_0(\bar{x}) = y_0(\frac{b-a}{2}\bar{x} + \frac{b+a}{2}),$$

$$\psi_1\left(Y, P, Y_{\bar{x}}, Y_{\bar{x}\bar{x}}\right) = \frac{T}{2}\left((\frac{2}{b-a})^2 \nu Y_{\bar{x}\bar{x}} - \frac{2}{b-a} Y Y_{\bar{x}} - \Phi(-\frac{1}{\alpha}P)\right),$$

$$\psi_2\left(Y, P, P_{\bar{x}}, P_{\bar{x}\bar{x}}\right) = \frac{T}{2}\left(-(\frac{2}{b-a})^2 \nu P_{\bar{x}\bar{x}} - \frac{2}{b-a} Y P_{\bar{x}} + Y_d - Y\right).$$

We assume that ψ_1 and ψ_2 have bounded and continuous derivatives with respect to their arguments. Hence, there exist constants M_1 and M_2 such that

$$|\psi_1\left(Y,P,Y_{\bar{x}},Y_{\bar{x}\bar{x}}\right)-\psi_1\left(\tilde{Y},\tilde{P},\tilde{Y}_{\bar{x}},\tilde{Y}_{\bar{x}\bar{x}}\right)|\le M_1\left(|Y-\tilde{Y}|+|P-\tilde{P}|\right),$$
$$|\psi_2\left(Y,P,P_{\bar{x}},P_{\bar{x}\bar{x}}\right)-\psi_2\left(\tilde{Y},\tilde{P},\tilde{P}_{\bar{x}},\tilde{P}_{\bar{x}\bar{x}}\right)|\le M_2\left(|Y-\tilde{Y}|+|P-\tilde{P}|\right). \tag{15}$$

Now, to approximate the optimal solution, we utilize the following polynomials interpolating:

$$\begin{cases} Y^N(\bar{t},\bar{x})=\sum_{i=0}^{N}\sum_{j=0}^{N}\bar{a}_{ij}^N L_i(\bar{t})L_j(\bar{x}),\\ P^N(\bar{t},\bar{x})=\sum_{i=0}^{N}\sum_{j=0}^{N}\bar{b}_{ij}^N L_i(\bar{t})L_j(\bar{x}),\end{cases} \tag{16}$$

where $\bar{a}_{ij}^N$ and $\bar{b}_{ij}^N$, for $i,j=0,1,\ldots,N$, are the unknown coefficients. By (12), we have

$$\begin{cases} Y^N(\bar{t}_i,\bar{x}_j)=\bar{a}_{ij}^N,\\ P^N(\bar{t}_i,\bar{x}_j)=\bar{b}_{ij}^N,\end{cases} \tag{17}$$

where

$$\bar{t}_i=-\cos(\frac{\pi i}{N}),\quad \bar{x}_j=-\cos(\frac{\pi j}{N}).$$

To express the derivatives $Y_{\bar{t}}^N(.,.)$, $Y_{\bar{x}}^N(.,.)$, $Y_{\bar{x}\bar{x}}^N(.,.)$, $P_{\bar{t}}^N(.,.)\,P_{\bar{x}}^N(.,.)$, and $P_{\bar{x}\bar{x}}^N(.,.)$, we can use the matrix multiplication $D=(D_{kj})$ and get

$$\begin{cases} Y_{\bar{t}}^N(\bar{t}_p,\bar{x}_k)=\sum_{i=0}^{N}\bar{a}_{ik}^N D_{pi},\\ Y_{\bar{x}}^N(\bar{t}_p,\bar{x}_k)=\sum_{j=0}^{N}\bar{a}_{pj}^N D_{kj},\\ Y_{\bar{x}\bar{x}}^N(\bar{t}_p,\bar{x}_k)=\sum_{j=0}^{N}\bar{a}_{pj}^N \hat{D}_{kj},\\ P_{\bar{t}}^N(\bar{t}_p,\bar{x}_k)=\sum_{i=0}^{N}\bar{b}_{ik}^N D_{pi},\\ P_{\bar{x}}^N(\bar{t}_p,\bar{x}_k)=\sum_{j=0}^{N}\bar{b}_{pj}^N D_{kj},\\ P_{\bar{x}\bar{x}}^N(\bar{t}_p,\bar{x}_k)=\sum_{j=0}^{N}\bar{b}_{pj}^N \hat{D}_{kj},\end{cases} \tag{18}$$

where

$$D_{kj}=L_j'(\bar{t}_k)=\begin{cases} \dfrac{\mu_k}{\mu_j}(-1)^{k+j}\dfrac{1}{\bar{t}_k-\bar{t}_j}, & j\ne k,\\ -\dfrac{\bar{t}_k}{2-2\bar{t}_k^2}, & 0\le j=k\le N-1,\\ -\dfrac{2N^2+1}{6}, & j=k=0,\\ \dfrac{2N^2+1}{6}, & j=k=N,\end{cases} \tag{19}$$

and $\hat{D}=D\cdot D=(\hat{D}_{kj})$ where $\hat{D}_{kj}=\sum_{l=0}^{N}D_{kl}D_{lj},\ k,j=0,1,\ldots,N$. In fact, multiplying by the matrix D transforms a vector of the state variables at the CGL points to the vector of approximate derivatives at these points.

Now, by relations (16), (17), and (18), conditions (7) can be written as the following discrete form:

$$\begin{cases} \sum_{i=0}^{N} \bar{a}_{ik}^{N} D_{pi} - \psi_1 \left(\bar{a}_{pk}^{N}, \bar{b}_{pk}^{N}, \sum_{j=0}^{N} \bar{a}_{pj}^{N} D_{kj}, \sum_{j=0}^{N} \bar{a}_{pj}^{N} \hat{D}_{kj} \right) = 0, \\ \sum_{i=0}^{N} \bar{b}_{ik}^{N} D_{pi} - \psi_2 \left(\bar{a}_{pk}^{N}, \bar{b}_{pk}^{N}, \sum_{j=0}^{N} \bar{b}_{pj}^{N} D_{kj}, \sum_{j=0}^{N} \bar{b}_{pj}^{N} \hat{D}_{kj} \right) = 0, \\ \bar{a}_{p0}^{N} = \bar{a}_{pN}^{N} = 0, \ \bar{b}_{p0}^{N} = \bar{b}_{pN}^{N} = 0, \ \bar{a}_{0k}^{N} = Y_0(\bar{x}_k), \ \bar{b}_{Nk}^{N} = 0, k, p = 0, 1, \ldots, N. \end{cases} \tag{20}$$

By solving the system of algebraic equations (20), we can obtain pointwise and continuous approximate optimal solutions as (16) and (17), respectively. Also, by (6), the approximate optimal control can be given as

$$U^N(\bar{t}, \bar{x}) = \frac{-1}{\alpha} \sum_{i=0}^{N} \sum_{j=0}^{N} \bar{b}_{ij}^{N} L_i(\bar{t}) L_j(\bar{x}), \quad (\bar{t}, \bar{x}) \in [-1, 1] \times [-1, 1].$$

Moreover, the optimal value of functional (1) can be approximated as follows:

$$J_N = \frac{T}{8} \sum_{k=0}^{N} \sum_{p=0}^{N} w_k w_p [\bar{a}_{pk}^{N} - z \left(\frac{T}{2}(\bar{t}_p + 1), \frac{b-a}{2}\bar{x}_k + \frac{b+a}{2} \right) + \alpha \bar{c}_{pk}^{N2}] \tag{21}$$

where $\bar{c}_{pk}^{N} = \frac{1}{\alpha} \bar{b}_{pk}^{N}$, and $w_s, s = 0, 1, \ldots, N$, are the quadrature weights of the numerical approximation (21). For even N, the weights are

$$\begin{cases} w_0 = w_N = \frac{1}{N^2 - 1} \\ w_s = w_{N-s} = \frac{4}{N} \sum_{j=0}^{\frac{N}{2}}{}'' \frac{1}{1-4j^2} \cos(\frac{2\pi j s}{N}), \quad s = 1, 2, \ldots, \frac{N}{2}, \end{cases} \tag{22}$$

and for odd N,

$$\begin{cases} w_0 = w_N = \frac{1}{N^2} \\ w_s = w_{N-s} = \frac{4}{N} \sum_{j=0}^{\frac{N-1}{2}}{}'' \frac{1}{1-4j^2} \cos(\frac{2\pi j s}{N}), \quad s = 1, 2, \ldots, \frac{N-1}{2}. \end{cases} \tag{23}$$

The double prime in the weights formula denotes the first and the last elements have to be halved.

Remark 1. Notice that the approximation solutions (16) can be written as the following Kronecker (tensor) product form:

$$Y(\bar{t}, \bar{x}) \approx Y^N(\bar{t}, \bar{x}) = \sum_{i=0}^{N} \sum_{j=0}^{N} \bar{a}_{ij}^{N} L_i(\bar{t}) L_j(\bar{x}) = L(\bar{t}, \bar{x}) T_Y$$

$$P(\bar{t}, \bar{x}) \approx P^N(\bar{t}, \bar{x}) = \sum_{i=0}^{N} \sum_{j=0}^{N} \bar{b}_{ij}^{N} L_i(\bar{t}) L_j(\bar{x}) = L(\bar{t}, \bar{x}) T_P,$$

where

$$\begin{aligned}L(\bar{t},\bar{x}) = [&L_0(\bar{t})L_0(\bar{x})\cdots L_0(\bar{t})L_N(\bar{x})L_1(\bar{t})L_0(\bar{x})\cdots L_1(\bar{t})L_N(\bar{x})\cdots \\ &L_N(\bar{t})L_0(\bar{x})\cdots L_N(\bar{t})L_N(\bar{x})] \\ &= [L_0(\bar{t})\ L_1(\bar{t})\cdots L_N(\bar{t})] \otimes [L_0(\bar{x})\ L_1(\bar{x})\cdots L_N(\bar{x})] \\ &= L(\bar{t}) \otimes L(\bar{x}),\end{aligned}$$

and the symbol " $\otimes$ " denotes the Kronecker product. Also

$$T_Y = [\bar{a}^N_{00}\cdots \bar{a}^N_{0N}\bar{a}^N_{10}\cdots \bar{a}^N_{1N}\cdots \bar{a}^N_{N0}\cdots \bar{a}^N_{NN}]',$$
$$T_P = [\bar{b}^N_{00}\cdots \bar{b}^N_{0N}\bar{b}^N_{10}\cdots \bar{b}^N_{1N}\cdots \bar{b}^N_{N0}\cdots \bar{b}^N_{NN}]',$$

where the symbol " $'$ " denotes the transpose. All the components of T_Y and T_P are unknown and our aim is to obtain them (A similar representation can be seen in [22, 32, 33]).

5 The convergence and error analysis

In this section, first we give the following definition and then analyze the convergence of the presented method.

Assume that $\bar{\Omega} = [-1, 1] \times [-1, 1]$. We apply $C^r(\bar{\Omega})$ for the space of functions with continuous derivatives of rth order.

Definition 3. [19] The function $W : \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties is called a modulus of continuity if

i) W is increasing,

ii) $\lim_{z\to 0} W(z) = 0$,

iii) $W(z_1 + z_2) \leq W(z_1) + W(z_2)$, for any z_1 and $z_2 \in \mathbb{R}^+$,

iv) there exists a constant c such that $cW(z) \geq z$ for all $0 < z \leq 2$.

Some important modulus of continuity can be defined as

$$W(z) = z^\alpha, \quad 0 < \alpha \leq 1. \tag{24}$$

Now, assume that B^2 is the unit circle in $\mathbb{R}^2$. We say that a continuous function $f(\cdot,\cdot)$ on $\bar{\Omega}$ admits $W(.)$ as a modulus of continuity, if the following value is finite

$$|f(\cdot,\cdot)|_W = \sup\{\frac{|f(\bar{t},\bar{x}) - f(\tilde{t},\tilde{x})|}{W\left(\|(\bar{t},\bar{x}) - (\tilde{t},\tilde{x})\|_\infty\right)} : (\bar{t},\bar{x}),\ (\tilde{t},\tilde{x}) \in \bar{\Omega}, (\bar{t},\bar{x}) \neq (\tilde{t},\tilde{x})\}\cdot \tag{25}$$

Suppose that space $C^1_W(B^2)$ includes all functions $f(\cdot,\cdot)$ on B^2 with continuous first order partial derivatives, and it equips with the following norm:

$$\|f(\cdot,\cdot)\|_{1,W} = \|f(\cdot,\cdot)\|_\infty + \|f_{\bar{t}}(\cdot,\cdot)\|_\infty + \|f_{\bar{x}}(\cdot,\cdot)\|_\infty + |f_{\bar{t}}(\cdot,\cdot)|_W + |f_{\bar{x}}(\cdot,\cdot)|_W. \tag{26}$$

Also, we define the space $C^1_W(\bar{\Omega})$ as follows:

$$\begin{aligned} C^1_W(\bar{\Omega}) = \; &\{f(\cdot,\cdot) \in C^1(\bar{\Omega}) : \text{ for all } (\tilde{t},\tilde{x}) \in \bar{\Omega}, \text{ there exists a map} \phi : B^2 \to \bar{\Omega}, \\ &s.t. \;\; (\tilde{t},\tilde{x}) \in \; Int(\phi(B^2)) \; and \; f \circ \phi(\cdot,\cdot) \in C^1_W(B^2)\}. \end{aligned} \tag{27}$$

It can be proved that if $\bar{\Omega} = \bigcup_{i=1}^{l} Int(\phi_i(B^2))$ for some $\phi_1,\ldots,\phi_l$, then $f(\cdot,\cdot) \in C^1_W(\bar{\Omega})$ if and only if $f \circ \phi_i(\cdot,\cdot) \in C_W(B^2)$ for each $i = 1,\ldots,l$. Moreover, $C^1_W(\bar{\Omega})$ with norm

$$\|f(\cdot,\cdot)\|_{1,W} = \sum_{i=1}^{l} \|f \circ \phi_i(\cdot,\cdot)\|_{1,W} \tag{28}$$

is a Banach space (for more details, see [19]). Now, we show the space of all polynomials of total degree at most $2N$ on $\bar{\Omega}$ by $Pol(N,N,\bar{\Omega})$, that is,

$$Pol(N,N,\bar{\Omega}) = \{\eta(\tilde{t},\tilde{x}) = \sum_{i=0}^{N}\sum_{j=0}^{N} \gamma_{ij}\tilde{t}^i\tilde{x}^j : (\tilde{t},\tilde{x}) \in \bar{\Omega}, \gamma_{ij} \in \mathbb{R}\}.$$

Theorem 2. *For any $f(\cdot,\cdot) \in C^1_W(\bar{\Omega})$, there is a polynomial $\eta(\cdot,\cdot) \in Pol(N,N,\bar{\Omega})$ such that*

$$\|f(\cdot,\cdot) - \eta(\cdot,\cdot)\|_\infty \le \frac{c_0 c_1}{2N} W(\frac{1}{2N}), \tag{29}$$

where $c_1 = \|f(\cdot,\cdot)\|_{1,W}$ and c_0 is a constant that depends on $W(\cdot)$, but independent of N.

Proof. The proof is a result of Theorem 2.1 in [19]. □

Now, to guarantee the existence of solution for the system of algebraic equations (20), we relax it as follows:

$$\begin{cases} |\sum_{i=0}^{N} \bar{a}^N_{ik} D_{pi} - \psi_1\left(\bar{a}^N_{pk}, \bar{b}^N_{pk}, \sum_{i=0}^{N} \bar{a}^N_{ik} D_{pi}, \sum_{j=0}^{N} \bar{a}^N_{pj} D_{kj}, \sum_{j=0}^{N} \bar{a}^N_{pj} \hat{D}_{kj}\right)| \\ \qquad \le \frac{\sqrt{N}}{2N-1} W(\frac{1}{2N-1}), \quad p,k = 1,2,\ldots,N-1 \\ |\sum_{i=0}^{N} \bar{b}^N_{ik} D_{pi} - \psi_2\left(\bar{a}^N_{pk}, \bar{b}^N_{pk}, \sum_{i=0}^{N} \bar{b}^N_{ik} D_{pi}, \sum_{j=0}^{N} \bar{b}^N_{pj} D_{kj}, \sum_{j=0}^{N} \bar{b}^N_{pj} \hat{D}_{kj}\right)| \\ \qquad \le \frac{\sqrt{N}}{2N-1} W(\frac{1}{2N-1}), \quad p,k = 1,2,\ldots,N-1 \\ |\bar{a}^N_{p0}| \le \frac{\sqrt{N}}{2N-1} W(\frac{1}{2N-1}), |\bar{a}^N_{pN}| \le \frac{\sqrt{N}}{2N-1} W(\frac{1}{2N-1}), \quad p = 0,1,\ldots,N, \\ |\bar{b}^N_{p0}| \le \frac{\sqrt{N}}{2N-1} W(\frac{1}{2N-1}), |\bar{b}^N_{pN}| \le \frac{\sqrt{N}}{2N-1} W(\frac{1}{2N-1}), \quad p = 0,1,\ldots,N, \\ |\bar{a}^N_{0k} - Y_0(\bar{x}_k)| \le \frac{\sqrt{N}}{2N-1} W(\frac{1}{2N-1}), |\bar{b}^N_{Nk}| \le \frac{\sqrt{N}}{2N-1} W(\frac{1}{2N-1}), \qquad k = 0,1,\ldots,N, \end{cases} \tag{30}$$

where N is sufficiently big and $W(\cdot)$ is a given modulus of continuity. Since $\lim_{N\to\infty} \frac{\sqrt{N}}{2N-1}W(\frac{1}{2N-1}) = 0$, any solution $(\bar{a},\bar{b})_N = ((\bar{a}^N_{pk},\bar{b}^N_{pk}); p,k = 0,1,\ldots,N)$ for system of algebraic inequalities (30) is a solution for system of algebraic equations (20) when N tends to infinity.

Now, we show that system (30) is feasible, that is, it has at least one solution $(\bar{a},\bar{b})_N$.

Theorem 3. *Let $(Y(\cdot,\cdot),P(\cdot,\cdot))$ be a solution for system (14), where $Y(\cdot,\cdot)$ and $P(\cdot,\cdot)$ are in $C^1_W(\bar{\Omega})$. Then there is a positive integer K such that, for any $N \geq K$, the system (30) has a solution as*

$$(\bar{a},\bar{b})_N = ((\bar{a}_{pk},\bar{b}_{pk}); p,k = 0,1,\ldots,N). \tag{31}$$

Moreover, $(\bar{a},\bar{b})_N$ satisfies

$$\begin{cases} |Y(\bar{t}_p,\bar{x}_k) - \bar{a}^N_{pk}| \leq \frac{L}{2N-1}W(\frac{1}{2N-1}),\ p,k = 0,1,\ldots,N, \\ |P(\bar{t}_p,\bar{x}_k) - \bar{b}^N_{pk}| \leq \frac{L}{2N-1}W(\frac{1}{2N-1}),\ p,k = 0,1,\ldots,N, \end{cases} \tag{32}$$

where L is a positive constant independent of N.

Proof. Assume that $\eta_1(\cdot,\cdot)$ and $\eta_2(\cdot,\cdot)$ in $Pol(N-1,N,\bar{\Omega})$ are the best polynomial approximations of $Y_{\bar{t}}(\cdot,\cdot)$ and $P_{\bar{t}}(\cdot,\cdot)$, respectively. By Theorem 2, we get

$$\begin{cases} \|Y_{\bar{t}}(\bar{t},\bar{x}) - \eta_1(\bar{t},\bar{x})\|_\infty \leq \frac{\gamma_1}{2N-1}W(\frac{1}{2N-1}), \quad (\bar{t},\bar{x}) \in \bar{\Omega}, \\ \|P_{\bar{t}}(\bar{t},\bar{x}) - \eta_2(\bar{t},\bar{x})\|_\infty \leq \frac{\gamma_2}{2N-1}W(\frac{1}{2N-1}), \quad (\bar{t},\bar{x}) \in \bar{\Omega}, \end{cases} \tag{33}$$

where γ_1 and γ_2 are two constants independent of N. We define

$$\begin{cases} \tilde{Y}(\bar{t},\bar{x}) = Y(-1,\bar{x}) + \int_{-1}^{\bar{t}} \eta_1(\tau,\bar{x})d\tau, \quad (\bar{t},\bar{x}) \in \bar{\Omega} \\ \tilde{P}(\bar{t},\bar{x}) = P(-1,\bar{x}) + \int_{-1}^{\bar{t}} \eta_2(\tau,\bar{x})d\tau, \quad (\bar{t},\bar{x}) \in \bar{\Omega}, \end{cases} \tag{34}$$

and

$$\bar{a}^N_{pk} = \tilde{Y}(\bar{t}_p,\bar{x}_k), \quad \bar{b}^N_{pk} = \tilde{P}(\bar{t}_p,\bar{x}_k), \qquad p,k = 0,1,\ldots,N. \tag{35}$$

We show that $(\bar{a},\bar{b})_N = ((\bar{a}^N_{pk},\bar{b}^N_{pk});\ p,k = 0,1,\ldots,N)$ satisfies system (30). By (33), (34), and (35), for $(\bar{t},\bar{x}) \in \bar{\Omega}$, we get

$$\begin{aligned} |Y(\bar{t},\bar{x}) - \tilde{Y}(\bar{t},\bar{x})| &= |\int_{-1}^{\tau} (Y_{\bar{t}}(\tau,\bar{x}) - \eta_1(\tau,\bar{x}))d\tau| \leq \int_{-1}^{\tau} |Y_{\bar{t}}(\tau,\bar{x}) - \eta_1(\tau,\bar{x})|\, d\tau \\ &\leq \frac{\gamma_1}{2N-1}W(\frac{1}{2N-1}) \int_{-1}^{\tau} d\tau \leq \frac{2\gamma_1}{2N-1}W(\frac{1}{2N-1}). \end{aligned} \tag{36}$$

Also, by a similar procedure, for $(\bar{t},\bar{x}) \in \bar{\Omega}$, we gain

$$|P(\bar{t},\bar{x}) - \tilde{P}(\bar{t},\bar{x})| \leq \frac{2\gamma_2}{2N-1} W(\frac{1}{2N-1}). \tag{37}$$

Now, by relation (34), the functions $\tilde{Y}(\cdot,\bar{x})$ and $\tilde{P}(\cdot,\bar{x})$, for any $\bar{x} \in [-1,1]$, are polynomials of total degree at most $2N$. Hence, their derivatives at CGL nodes $\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_N$ are exactly equal to the value of polynomial at the nodes multiplied by the differential matrix D, defined by (19). Thus, we have

$$\sum_{i=0}^{N} \bar{a}_{ik}^N D_{pi} = \tilde{Y}_{\bar{t}}(\bar{t}_p,\bar{x}_k), \quad \sum_{i=0}^{N} \bar{b}_{ik}^N D_{pi} = \tilde{P}_{\bar{t}}(\bar{t}_p,\bar{x}_k); \quad p,k = 0,1,\ldots,N. \tag{38}$$

Therefore, by relations (15) and (36), we get

$$\begin{aligned}
&\left| \sum_{i=0}^{N} a_{ik}^N D_{pi} - \psi_1 \left(\bar{a}_{pk}^N, \bar{b}_{pk}^N, \sum_{j=0}^{N} \bar{a}_{pj}^N D_{kj}, \sum_{j=0}^{N} \bar{a}_{pj}^N \hat{D}_{kj} \right) \right| \\
&\leq \left| \tilde{Y}_{\bar{t}}(\bar{t}_p,\bar{x}_k) - Y_{\bar{t}}(\bar{t}_p,\bar{x}_k) \right| \\
&\quad + \left| Y_{\bar{t}}(\bar{t}_p,\bar{x}_k) - \psi_1 \left(\bar{a}_{pk}^N, \bar{b}_{pk}^N, \sum_{j=0}^{N} \bar{a}_{pj}^N D_{kj}, \sum_{j=0}^{N} \bar{a}_{pj}^N \hat{D}_{kj} \right) \right| \\
&= |\eta_1(\bar{t}_p,\bar{x}_k) - Y_{\bar{t}}(\bar{t}_p,\bar{x}_k)| \\
&\quad + \Bigg| \psi_1 \left(Y(\bar{t}_p,\bar{x}_k), P(\bar{t}_p,\bar{x}_k), Y_{\bar{x}}(\bar{t}_p,\bar{x}_k), Y_{\bar{x}\bar{x}}(\bar{t}_p,\bar{x}_k) \right) \\
&\qquad - \psi_1 \left(\bar{a}_{pk}^N, \bar{b}_{pk}^N, \sum_{j=0}^{N} \bar{a}_{pj}^N D_{kj}, \sum_{j=0}^{N} \bar{a}_{pj}^N \hat{D}_{kj} \right) \Bigg| \\
&\leq |\eta_1(\bar{t}_p,\bar{x}_k) - Y_{\bar{t}}(\bar{t}_p,\bar{x}_k)| + M_1 \left(\left| Y(\bar{t}_p,\bar{x}_k) - \bar{a}_{pk}^N \right| + \left| P(\bar{t}_p,\bar{x}_k) - \bar{b}_{pk}^N \right| \right) \\
&\leq \frac{\gamma_1}{2N-1} W(\frac{1}{2N-1}) + M_1 \left(\frac{2\gamma_1}{2N-1} W(\frac{1}{2N-1}) + \frac{2\gamma_2}{2N-1} W(\frac{1}{2N-1}) \right) \\
&= \frac{\gamma_1(2M_1+1) + 2M_1\gamma_2}{2N-1} W(\frac{1}{2N-1}), \quad p,k = 1,\ldots,N-1,
\end{aligned} \tag{39}$$

where M_1 and M_2 are Lipschitz constants of ψ_1 and ψ_2, respectively. Moreover, by a similar process, we obtain

$$\begin{aligned}
&\left| \sum_{i=0}^{N} b_{ik}^N D_{pi} - \psi_2 \left(\bar{a}_{pk}^N, \bar{b}_{pk}^N, \sum_{j=0}^{N} \bar{b}_{pj}^N D_{kj}, \sum_{j=0}^{N} \bar{b}_{pj}^N \hat{D}_{kj} \right) \right| \\
&\leq \frac{\gamma_2(2M_2+1) + 2M_2\gamma_1}{2N-1} W(\frac{1}{2N-1}), \quad p,k = 1,\ldots,N-1.
\end{aligned} \tag{40}$$

Further, for boundary conditions, we get

$$\left|\tilde{Y}(-1,\bar{x}_k) - Y_0(\bar{x}_k)\right| \leq \left|\tilde{Y}(-1,\bar{x}_k) - Y(-1,\bar{x}_k)\right| + |Y(-1,\bar{x}_k) - Y_0(\bar{x}_k)|$$
$$\leq \frac{2\gamma_1}{2N-1}W(\frac{1}{2N-1}), \quad k = 0,1,\dots,N. \tag{41}$$

Also, for all $p, k = 0,1,\dots,N$, we have

$$\left|\tilde{P}(1,\bar{x}_k)\right| = \left|\tilde{P}(1,\bar{x}_k) - P(1,\bar{x}_k)\right| \leq \frac{2\gamma_2}{2N-1}W(\frac{1}{2N-1}), \tag{42}$$
$$\left|\tilde{Y}(\bar{t}_p,-1)\right| = \left|\tilde{Y}(\bar{t}_p,-1) - Y(\bar{t}_p,-1)\right| \leq \frac{2\gamma_1}{2N-1}W(\frac{1}{2N-1}), \tag{43}$$
$$\left|\tilde{Y}(\bar{t}_p,1)\right| = \left|\tilde{Y}(\bar{t}_p,1) - Y(\bar{t}_p,1)\right| \leq \frac{2\gamma_1}{2N-1}W(\frac{1}{2N-1}), \tag{44}$$
$$\left|\tilde{P}(\bar{t}_p,-1)\right| = \left|\tilde{P}(\bar{t}_p,-1) - P(\bar{t}_p,-1)\right| \leq \frac{2\gamma_2}{2N-1}W(\frac{1}{2N-1}), \tag{45}$$
$$\left|\tilde{P}(\bar{t}_p,1)\right| = \left|\tilde{P}(\bar{t}_p,1) - P(\bar{t}_p,1)\right| \leq \frac{2\gamma_2}{2N-1}W(\frac{1}{2N-1}). \tag{46}$$

Hence, if we select K such that

$$\max\{\gamma_1(2M_1+1) + 2M_1\gamma_2, \gamma_2(2M_2+1) + 2M_2\gamma_1, 2\gamma_1, 2\gamma_2\} \leq \sqrt{N},$$

for $N \geq K$, then by (39)–(46), pair $(\bar{a},\bar{b})_N$ satisfies system (30). □

Now, we show that the sequence of solutions of problem (30) and sequence of their interpolating polynomials converge to the solution of the problem (14).

Theorem 4. *Let* $\{(\bar{a},\bar{b})_N = ((\bar{a}^N_{pk}, \bar{b}^N_{pk});\ p,k = 0,1,\dots,N\}_{N=K}^{\infty}$ *be a sequence of solution of system* (30) *and let* $\{(Y^N(\cdot,\cdot), P^N(\cdot,\cdot))\}_{N=K}^{\infty}$ *be their interpolating polynomials sequence defined by* (16)*. Also, we assume that, for any* $\bar{x}$ *in* $[-1,-1]$*, the sequence* $\{(Y^N(-1,\bar{x}), P^N(-1,\bar{x}), Y^N_{\bar{t}}(\cdot,\cdot), P^N_{\bar{t}}(\cdot,\cdot))\}_{N=K}^{\infty}$ *has a subsequence* $\{(Y^{N_i}(-1,\bar{x}), P^{N_i}(-1,\bar{x}), Y^{N_i}_{\bar{t}}(\cdot,\cdot), P^{N_i}_{\bar{t}}(\cdot,\cdot))\}_{i=0}^{\infty}$ *that uniformly converges to* $(\phi_1^\infty(\bar{x}), \phi_2^\infty(\bar{x}), \lambda_1(\cdot,\cdot), \lambda_2(\cdot,\cdot))$ *where* $\lambda_1(\cdot,\cdot), \lambda_2(\cdot,\cdot) \in C^2(\bar{\Omega})$*,* $\phi_1^\infty(\cdot)$*,* $\phi_2^\infty(\cdot) \in C^2([-1,1])$ *and* $\lim_{i\to\infty} N_i = \infty$*. Then the pair*

$$(\tilde{Y}(\bar{t},\bar{x}), \tilde{P}(\bar{t},\bar{x})) = \left(\lim_{i\to\infty} Y^{N_i}(\bar{t},\bar{x}), \lim_{i\to\infty} P^{N_i}(\bar{t},\bar{x})\right), \tag{47}$$

for $(\bar{t},\bar{x}) \in \bar{\Omega}$*, is a solution of system* (14)*.*

Proof. By attention to the assumptions, we have

$$\begin{cases} \tilde{Y}(\bar{t},\bar{x}) = \phi_1^\infty(\bar{x}) + \int_{-1}^{\bar{t}} \lambda_1(\tau,\bar{x})d\tau, \\ \tilde{P}(\bar{t},\bar{x}) = \phi_2^\infty(\bar{x}) + \int_{-1}^{\bar{t}} \lambda_2(\tau,\bar{x})d\tau. \end{cases} \tag{48}$$

We first show that $(\tilde{Y}(\bar{t},\bar{x}),\tilde{P}(\bar{t},\bar{x}))$ for $\bar{t}\in[-1,1]$ and $\bar{x}=x_k$, $k=0,1,\ldots,N$ satisfy system (14). Assume that $(\tilde{Y}(\cdot,\bar{x}_k),\tilde{P}(\cdot,\bar{x}_k))$ for some $k=1,\ldots,N-1$ does not satisfy the first equation of (14). Then, there is a τ in $(-1,1)$ such that

$$\tilde{Y}_{\bar{t}}(\tau,\bar{x}_k)-\psi_1\left(\tilde{Y}(\tau,\bar{x}_k),\tilde{P}(\tau,\bar{x}_k),\tilde{Y}_{\bar{x}}(\tau,\bar{x}_k),\tilde{Y}_{\bar{x}\bar{x}}(\tau,\bar{x}_k)\right)\neq 0.$$

Since CGL nodes $\{\bar{t}_p\}_{p=0}^{N}$ when $N\to\infty$ are dense in $[-1,1]$, there is a sequence $\{\bar{t}_{l_{N_i}}\}_{i=1}^{\infty}$ such that $0<l_{N_i}<N_i$ and $\lim_{i\to\infty}\bar{t}_{l_{N_i}}=\tau$. Thus

$$\lim_{i\to\infty}\left(\tilde{Y}_{\bar{t}}(\bar{t}_{l_{N_i}},\bar{x}_k)-\psi_1\left(\tilde{Y}(\bar{t}_{l_{N_i}},\bar{x}_k),\tilde{P}(\bar{t}_{l_{N_i}},\bar{x}_k),\tilde{Y}_{\bar{x}}(\bar{t}_{l_{N_i}},\bar{x}_k),\tilde{Y}_{\bar{x}\bar{x}}(\bar{t}_{l_{N_i}},\bar{x}_k)\right)\right)$$
$$=\tilde{Y}_{\bar{t}}(\tau,\bar{x}_k)-\psi_1\left(\tilde{Y}(\tau,\bar{x}_k),\tilde{P}(\tau,\bar{x}_k),\tilde{Y}_{\bar{x}}(\tau,\bar{x}_k),\tilde{Y}_{\bar{x}\bar{x}}(\tau,\bar{x}_k)\right)\neq 0. \quad (49)$$

On the other hand, since $\lim_{i\to\infty}\frac{\sqrt{N_i}}{2N_i-1}W(\frac{1}{2N_i-1})=0$, by (30), we obtain

$$\lim_{i\to\infty}\left(\tilde{Y}_{\bar{t}}(\bar{t}_{l_{N_i}},\bar{x}_k)-\psi_1\left(\tilde{Y}(\bar{t}_{l_{N_i}},\bar{x}_k),\tilde{P}(\bar{t}_{l_{N_i}},\bar{x}_k),\tilde{Y}_{\bar{x}}(\bar{t}_{l_{N_i}},\bar{x}_k),\tilde{Y}_{\bar{x}\bar{x}}(\bar{t}_{l_{N_i}},\bar{x}_k)\right)\right)=0,$$

which is a contradiction to (49). Thus $(\tilde{Y}(\bar{t},\bar{x}),\tilde{P}(\bar{t},\bar{x}))$ (for all $\bar{t}\in[-1,1]$ and $\bar{x}=\bar{x}_k$, $k=1,\ldots,N-1$) satisfies the first equation of (14). By a similar procedure, we can show that it satisfies the second equation. Also, it can be easily proved that $(\tilde{Y}(\cdot,\bar{x}_k),\tilde{P}(\cdot,\bar{x}_k))$, for $k=0,1,\ldots,N$, satisfies the boundary conditions. For example we show that $\tilde{Y}(-1,\bar{x}_k)=Y_0(\bar{x}_k)$ for $k=0,1,\ldots,N$. We have

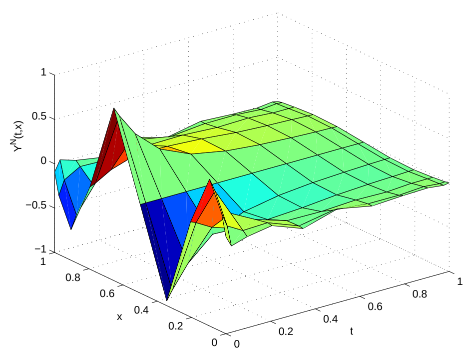
$$\begin{aligned}0\leq|\tilde{Y}(-1,\bar{x}_k)-Y_0(\bar{x}_k)|&=|\lim_{i\to\infty}Y^{N_i}(-1,\bar{x}_k)-Y_0(\bar{x}_k)|\\&=\lim_{i\to\infty}|Y^{N_i}(-1,\bar{x}_k)-Y_0(\bar{x}_k)|=\lim_{i\to\infty}|\bar{a}_{0k}^{N_i}-Y_0(\bar{x}_k)|\\&\leq\lim_{i\to\infty}\frac{\sqrt{N_i}}{2N_i-1}W(\frac{1}{2N_i-1})=0.\end{aligned}$$

Hence $\tilde{Y}(-1,\bar{x}_k)=Y_0(\bar{x}_k)$ for all $k=0,1,\ldots,N$. Now, we know that nodes $\{\bar{x}_k\}_{k=0}^{N}$ when $N\to\infty$, are dense in $[-1,1]$. Therefore the pair $(\tilde{Y}(\cdot,\cdot),\tilde{P}(\cdot,\cdot))$ defined by (47) is a solution for (14) on $\bar{\Omega}=[-1,1]\times[-1,1]$. $\square$

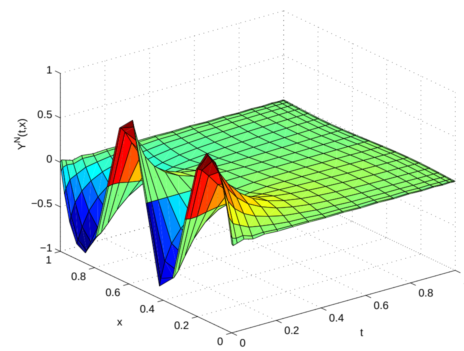
6 Numerical examples

In the following examples, we use the Levenberg–Marquardt method (a quasi-Newton method) for FSOLVE command in MATLAB software to solve algebraic system (20).

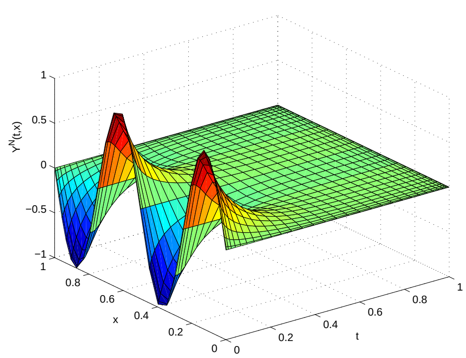
Example 1. Consider problems (1)–(4), where $T = 1$, $\alpha = 1$, $\nu = 0.01$ and 0.05, $y_0 = \sin(4\pi x)$, $\Phi(u) = u$, and $z(t,x) = 0$. The approximate optimal value of objective function computed by the CPS and LPS [20] methods for $\nu = 0.01$ and 0.05 and $N = 10,\ 20,\ 30$ and 40 are shown in Table 1. We observe that our numerical results are better than the results of the LPS method. In Figures 1 and 2, we show the obtained approximate optimal state and control for $\nu = 0.05$ and different values of N.



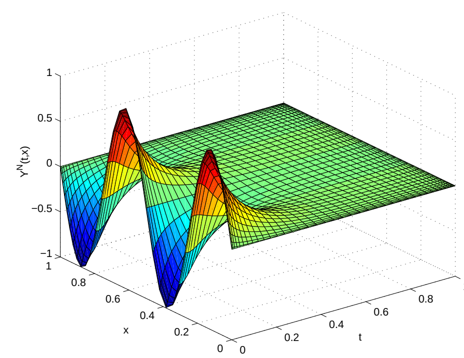
(a) The approximate optimal state for $N = 10$ and $\nu = 0.05$



(b) The approximate optimal state for $N = 20$ and $\nu = 0.05$

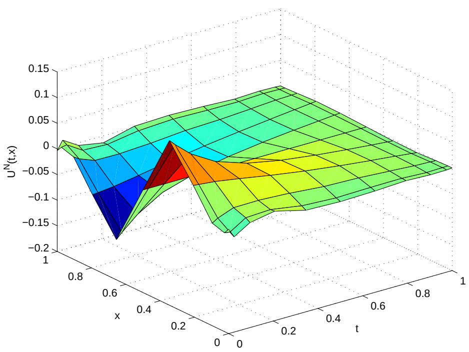


(c) The approximate optimal state for $N = 30$ and $\nu = 0.05$

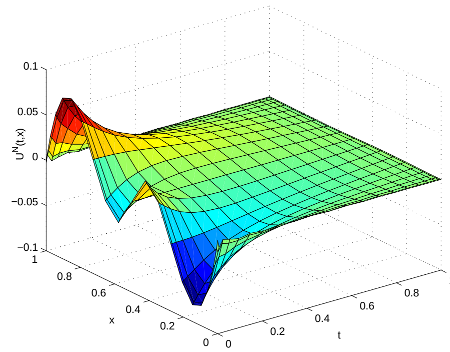


(d) The approximate optimal state for $N = 40$ and $\nu = 0.05$

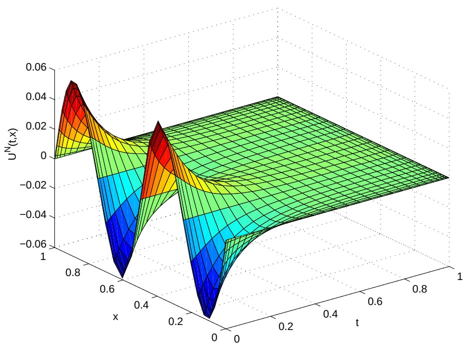
Figure 1: The approximate optimal state for Example 1



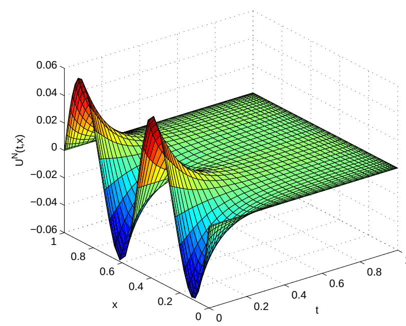
(a) The approximate optimal control for $N = 10$ and $\nu = 0.05$



(b) The approximate optimal control for $N = 20$ and $\nu = 0.05$



(c) The approximate optimal control for $N = 30$ and $\nu = 0.05$



(d) The approximate optimal control for $N = 40$ and $\nu = 0.05$

Figure 2: The approximate optimal control for Example 1

Table 1: Comparison of objective function values for Example 1

	$\nu = 0.01$	$\nu = 0.01$	$\nu = 0.05$	$\nu = 0.05$
N	Presented method	LPS method [20]	Presented method	LPS method [20]
10	0.033014881174862	0.0828638100277	0.016911085761598	0.01590006952876
20	0.040440356851832	0.0620867108909	0.013730446517693	0.01519309308076
30	0.029728232559725	0.0466282421253	0.015082505388501	0.01519227846689
40	0.029005091596013	0.0463124455511	0.015073940981453	0.01519176630695

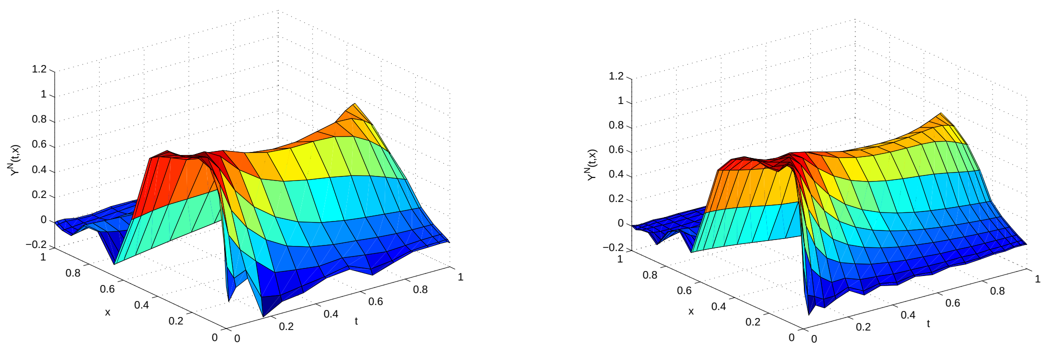
Example 2. Consider optimal control problem (1)–(4). Let $T = 1$, $\alpha = 0.05$, $\nu = 0.01$, and $z(t,x) = y_0(x)$ where

$$y_0(x) = \begin{cases} 1, & x \in \ (0, \frac{1}{2}], \\ 0 & otherwise. \end{cases}$$

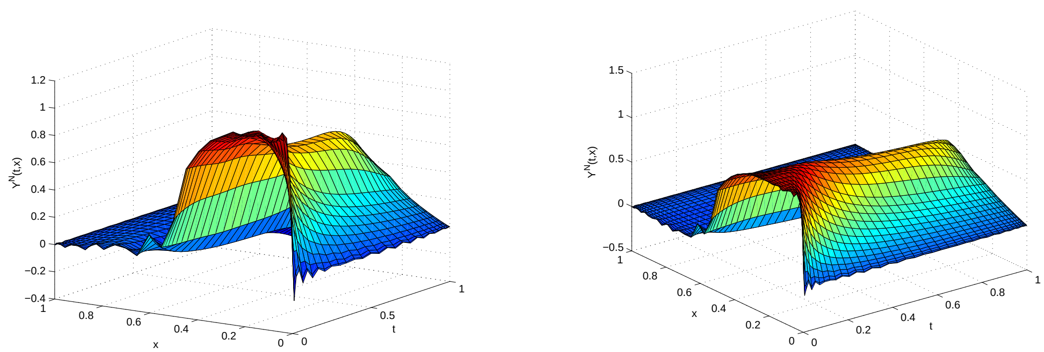
is the initial condition and

$$\Phi(u) = \begin{cases} u, & u \ on \ (0,T) \times (\frac{1}{4}, \frac{3}{4}), \\ 0, & otherwise. \end{cases}$$

Table 2 shows the approximate optimal values of objective function for different values of N by the CPS method. In Figures 3 and 4, the optimal state and optimal control are presented, respectively.

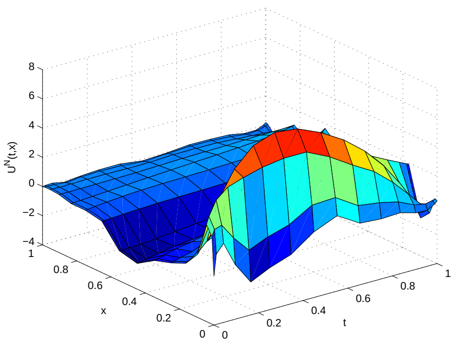


(a) The approximate optimal state for $N = 15$ (b) The approximate optimal state for $N = 20$

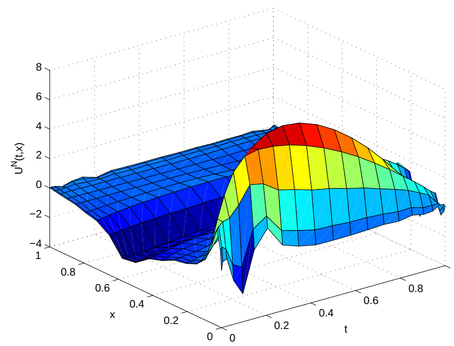


(c) The approximate optimal state for $N = 30$ (d) The approximate optimal state for $N = 40$

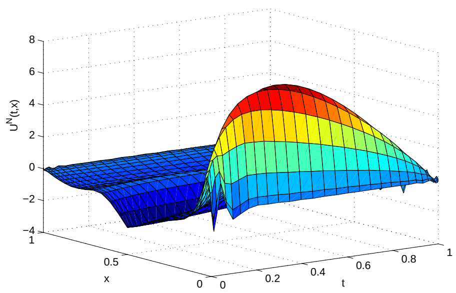
Figure 3: The approximate optimal state for Example 2



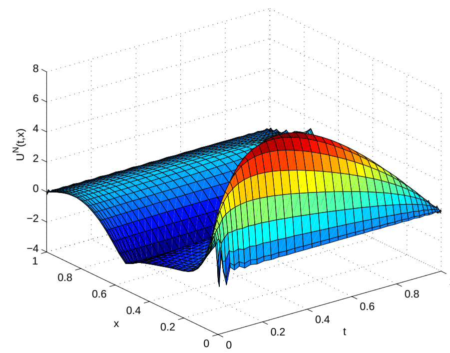
(a) The approximate optimal control for $N = 15$



(b) The approximate optimal control for $N = 20$



(c) The approximate optimal control for $N = 30$



(d) The approximate optimal control for $N = 40$

Figure 4: The approximate optimal control for Example 2

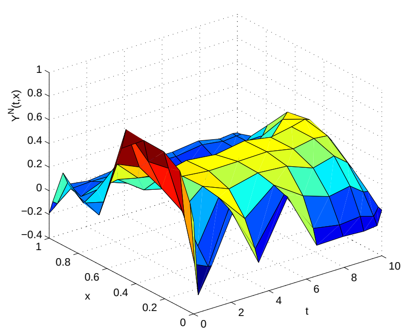
Table 2: Approximate values of objective functional for Example 2

N	J_N
20	0.106416588965259
25	0.085739365971895
30	0.083129631634715

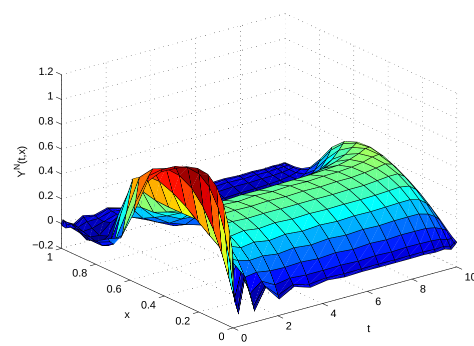
Example 3. Consider optimal control problem (1)–(4). Assume that $T = 10$, $\alpha = 1$, $\Phi(u) = u$, and $z(t, x) = y_0(x)$ where y_0 is defined in Example 2. We apply the CPS method to this problem. In Table 3, the values of cost functional using CPS method for different values of N are listed. In Figures 5 and 6, the approximate optimal state and optimal control are illustrated, respectively.

Table 3: Approximate values of objective functional for Example 3

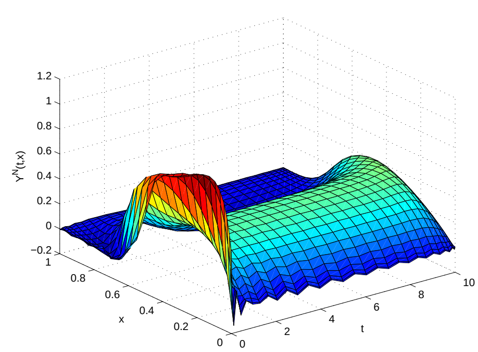
N	J_N
10	0.795502231705757
20	0.732649373291908
30	0.605003194783743



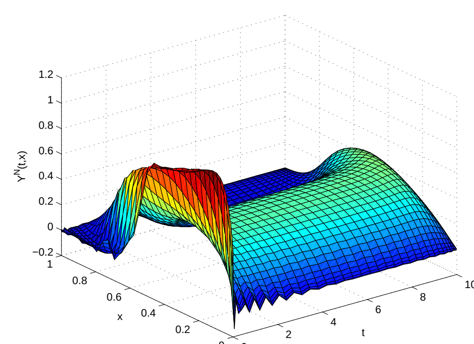
(a) The approximate optimal state for $N = 10$



(b) The approximate optimal state for $N = 20$

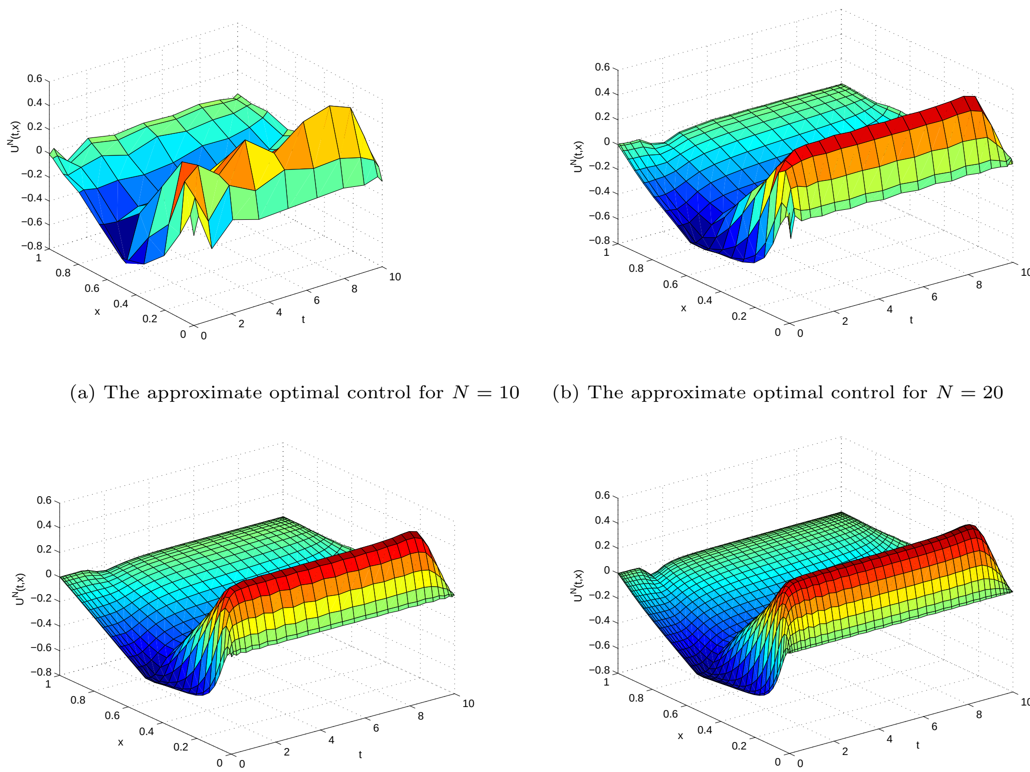


(c) The approximate optimal state for $N = 30$



(d) The approximate optimal state for $N = 40$

Figure 5: The approximate optimal state for Example 3



(a) The approximate optimal control for $N = 10$ (b) The approximate optimal control for $N = 20$

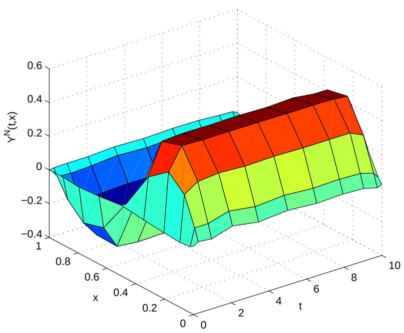
(c) The approximate optimal control for $N = 30$ (d) The approximate optimal control for $N = 40$

Figure 6: The approximate optimal control for Example 3

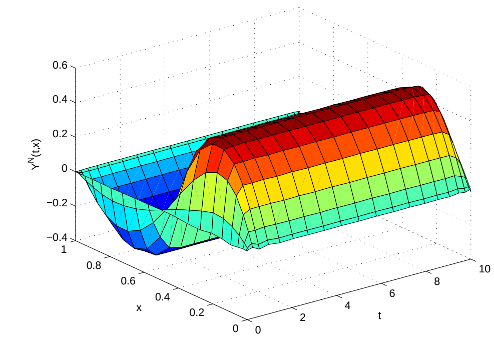
Example 4. Consider the optimal control of Burgers' equation (1)–(4) with $T = 10$, $\alpha = 0.1$, $\nu = 0.01$, $\Phi(u) = u$, and the desired state $z(t,x) = y_0(x)$ where $y_0 = \exp(-x)\sin(2\pi x)$, $\quad x \in [0,1]$. The numerical results are displayed in Table 4 for different values of N by using our method. In Figures 7 and 8, we show the approximate optimal state and control, respectively.

Table 4: Approximate values of objective functional for Example 4

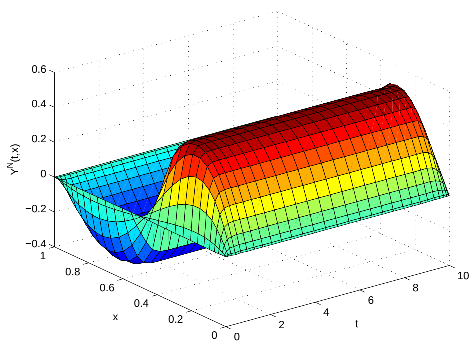
N	J_N
20	0.121081851037199
30	0.116980721067958
40	0.115852005217238



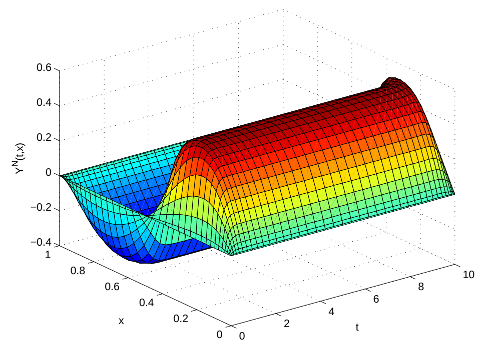
(a) The approximate optimal state for $N = 10$



(b) The approximate optimal state for $N = 20$

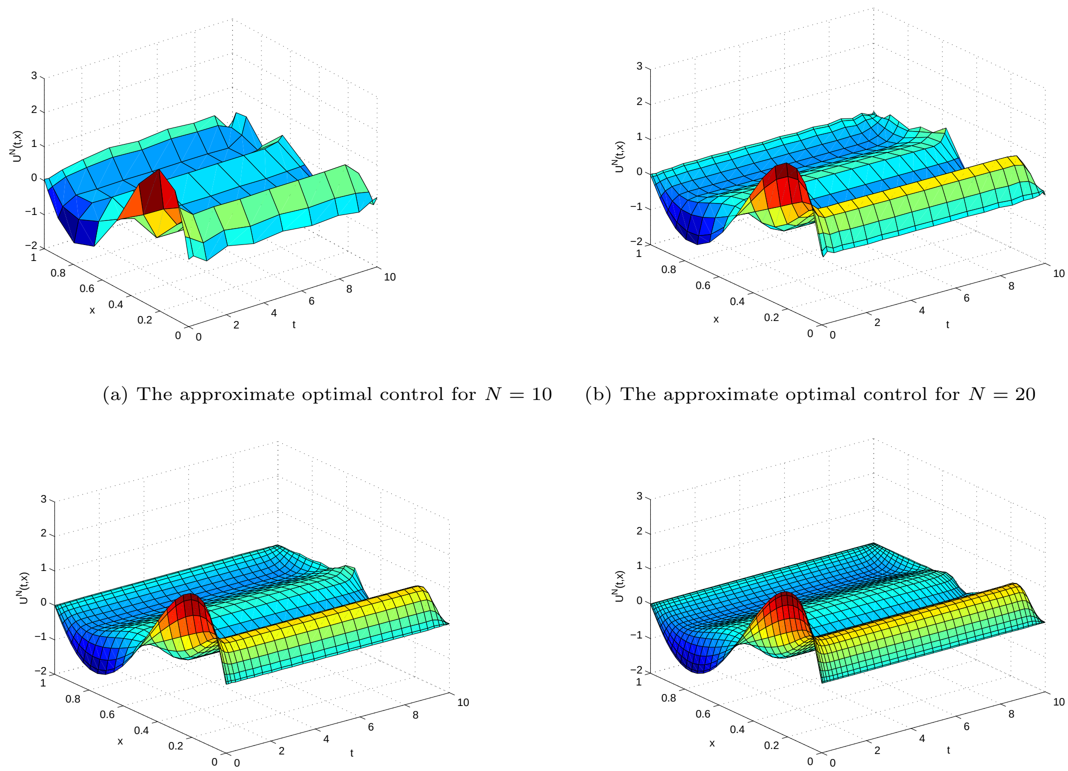


(c) The approximate optimal state for $N = 30$



(d) The approximate optimal state for $N = 40$

Figure 7: The approximate optimal state for Example 4



(a) The approximate optimal control for $N = 10$ (b) The approximate optimal control for $N = 20$

(c) The approximate optimal control for $N = 30$ (d) The approximate optimal control for $N = 40$

Figure 8: The approximate optimal control for Example 4

7 Conclusions

In this paper, we proposed an efficient Chebyshev pseudo spectral method to solve the optimal control problem governed by Burgers' equation. By applying this method, we discretized the optimality conditions and obtained a system of algebraic equations. We achieved a good approximate optimal solution with good accuracy. We analyzed the feasibility and convergence of the presented method.

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