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Research Article

Solving biobjective network flow problem associated with minimum cost-time loading†

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Abstract

We apply a primal-dual simplex algorithm for solving the biobjective minimum cost-time network flow problem such that the total shipping cost and the total shipping fixed time are considered as the first and second objective functions, respectively. To convert the proposed model into a single-objective parametric one, the weighted sum scalarization technique is commonly used. This problem is a mixed-integer programming, which the decision variables are directly dependent together. Generally, the previous works have considered the linear biobjective problem with the traditional network flow constraints, while in this paper, corresponding to each flow variable, a binary variable is defined. These zero-one variables are utilized to describe a fixed shipping time for positive flows. The proposed method is successful in finding all supported efficient solutions of a real numerical example.

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Keywords: Biobjective network flow; Minimum cost-time; Primal-dual algorithm; Fixed time.

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1 Introduction

In multi-objective network flow optimization, several objective functions, which generally are time, cost, risk, environmental concerns, and so on, have to be optimized such that the flow conservation and capacity constraints are satisfied. In actual problems, since the objectives are frequently conflicting, finding a feasible solution optimizing all objectives simultaneously is difficult or in most cases is impossible. Therefore, the aim of multi-objective optimization is to compute nondominated points rather than optimal values. A perfect review of the multi-objective cost flow problem is done in [14, 25].

The minimum cost-time network flow problem (MCTNFP) is a well-known and valuable topic in the scientific researches, which can be classified in a subset of the fixed charge network flow problems (FCNFPs). The first objective function considers the total shipping cost of flow, and the second objective function measures the total fixed shipping time of flow. Corresponding to any flow variables, there is a zero-one decision variable that is utilized to compute the total fixed shipping time on each directed arc. The weighted sum scalarization technique is used to convert the reformulated model into a parametric single objective optimization. Eusébio, Figueria, and Ehrgott [11] presented a modified primal-dual simplex algorithm based on the work done in [8] to find nondominated extreme points of the linear biobjective network flow problems. The main idea in both algorithms is that the scalarization parameter $\lambda \in (0, 1)$ is broken into subintervals such that each nondominated extreme point is associated with one and only one set in the partition. According to these papers, the primal-dual approach is considered for computing all efficient solutions of MCTNFP (see also [22, 4]).

FCNFP is a classical NP-hard combinatorial problem, (see [13]). This problem consists of two costs corresponding to each incident arc in the network: A fixed cost for the use of the arc and a variable cost proportional to the number of units sending along the arc (see [7]). The end of the FCNFP is to choose some arcs and assign feasible flows to them in order to transfer commodities from an origin to a destination at a minimal total cost. Some important traditional problems such as transportation problem, facility location problem, and network design problem can be formulated as an FCNFP. Many exact and heuristic techniques have been applied for solving the FCNFP. Analytic procedures commonly utilize branch and bound method to search an exact solution of the FCNFP (see [21]). Heuristic and evolutionary techniques have been developed to find a near-optimal solution to the under regarded problem (see [19, 20]).

Raith and Ehrgott [24] presented an algorithm to compute a complete set of efficient solutions for the biobjective integer minimum cost flow (BOIMCF) problem based on a two-phase method (see [23]). Using the mean of weighted sum technique, Abo-Sinna and Rizk-Allah [2] introduced a new gradient-based neural network approach for solving biobjective optimization, which is stable in the sense of Lyapunov. According to Benson's scalarization [6] and min-max methods, Mohammadi, PourKarimi, and Pedram [17] studied the scientific workflow scheduling problem in a multi-cloud environment with cost and makespan minimization objectives. Eusébio and Figueria [10] solved a sequence of ϵ-constraint problem to find all integer nondominated extreme points of small or medium size of BOIMCF problem by a branch-and-bound method. Ehrgott, Shao, and Schöbel [9] implemented a method for approximating the nondominated set of a multi-objective nonlinear convex programming problem. Abareshi and Zaferanieh [1] introduced a bilevel model to solve the capacitated p-median facility location problem with the most likely allocation solution. Applying Lagrangian dual theory, the proposed bilevel problem is reduced to a new one-level nonlinear mixed-integer problem whose solution is obtained by comparing two mixed-integer linear problems.

In this study, we introduce a strategy for formulating and solving biobjective minimum cost-time network flow problems (BOMCTNFPs), which in addition to the flow variables, some corresponding zero-one decision variables are used in their construction. Keshavarz and Toloo [16] considered the weighted sum scalarization approach to convert their proposed model to a corresponding parametric mixed-integer minimum cost flow problem with single objective function and finally reported the supported efficient solutions for only two values of the scalarization parameter $\lambda = 0.05,\ 0.5$. The main difference of our work with [16] is that, by utilizing the primal-dual technique, all of the nondominated extreme points and the supported efficient solutions of the proposed problem are computed.

The idea of the mathematical formulation of BOMCTNFP is taken from Hochbaum and Segev [15]. They offered an augmented formulation of the fixed charge problem and proved that an optimal solution of the augmented problem is an optimal solution of the fixed charge problem. The numerical procedure that they have been implemented is based on Lagrangian relaxation, which behaves well empirically.

2 Mathematical formulation of the BOMCTNFP

Ehrgott and Puerto [8] suggested the primal-dual algorithm for solving a general form of the multi-objective programming problem. This algorithm solves the problem by constructing a partition of the set $(0, 1)$ such that each subinterval of this partition is attributed to only one efficient solution. A modified primal-dual approach for solving linear biobjective minimum cost network flow problem was developed in [11], which uses reduced cost information to avoid redundancy. Here, we apply a similar idea to find nondominated points for solving a mixed BOMCTNFP.

Suppose that $G = (\mathcal{V}, \mathcal{A})$ is a directed and connected network in which $\mathcal{V}$ is a finite set of nods with $|\mathcal{V}| = m$, and $\mathcal{A}$ is a collection of directed pairs of elements of $\mathcal{V}$ called arcs, with $|\mathcal{A}| = n$. Corresponding to any arc $(i, j) \in \mathcal{A}$, a triple label as (c_{ij}, t_{ij}, u_{ij}) is assigned, where the first, second, and third components are shipping cost, shipping fixed time, and capacity, respectively. Associated with every node $i \in \mathcal{V}$, there is a value b_i, which shows the supply (if $b_i > 0$) or demand (if $b_i < 0$). If $b_i = 0$ for some $i \in \mathcal{V}$, then the node i is called a transshipment node. The interested reader can consult [5, 3].

Consider the following BOMCTNFP with mixed continuous-binary variables:

$$
\begin{aligned}
\min cx &= \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}, \\
\min ty &= \sum_{(i,j)\in\mathcal{A}} t_{ij}y_{ij}, \\
\text{s.t.} \quad & \sum_{(k,j)\in\mathcal{A}} x_{kj} - \sum_{(i,k)\in\mathcal{A}} x_{ik} = b_k && \text{for all } k \in \mathcal{V}, \\
& 0 \le x_{ij} \le u_{ij} && \text{for all } (i,j) \in \mathcal{A}, \\
& y_{ij} = \begin{cases} 1 & x_{ij} > 0, \\ 0 & x_{ij} = 0, \end{cases} \\
& y_{ij} \in \{0, 1\} && \text{for all } (i,j) \in \mathcal{A},
\end{aligned}
\tag{1}
$$

where the binary variable y_{ij} is an auxiliary variable to consider the fixed time t_{ij} for the positive flow x_{ij} on arc (i, j). If we assume that the values of the flow variables are real, then the model (1) is a biobjective mixed-integer problem, and if the value of the flow variables is integer, then this model is a biobjective integer problem. We refer to the first case with minimum cost-time continuous flow (MCTCF) problem, and to the second one with minimum cost-time integer flow (MCTIF) problem.

The efficient solution is an important concept in the problem of multi-objective optimization. In fact, an efficient solution to a multi-objective problem is the response, which cannot improve some of the objectives without worsening other objectives. Assume that X is a set of feasible solutions to problem (1), providing the constraint $y_{ij} \in \{0, 1\}$, $(i, j) \in \mathcal{A}$ is replaced with the continuous constraint $y_{ij} \in [0, 1]$, $(i, j) \in \mathcal{A}$. The set X is called the decision space. We denote the feasible set in the objective space as $Z(x, y) = \{(cx, ty)|(x, y) \in X\}$.

Definition 1. [12] Let $(x, y), (x', y') \in X$. If $cx \leqslant cx', ty \leqslant ty'$ and $(cx, ty) \neq (cx', ty')$, then it is called (x, y) dominates (x', y') in the decision space, and equivalently (cx, ty) dominates (cx', ty') in the objective space. This notion is denoted by $Z(x, y) \preceq Z(x', y')$.

Definition 2. [18] The feasible solution $(x, y) \in X$ is an efficient solution, if there is no other feasible solution like $(x', y') \in X$ such that

$Z(x', y') \prec Z(x, y)$. If (x, y) is an efficient solution in the decision space X, then its image in the objective space is non-dominated, that is, $Z(x, y)$ is named a nondominated point. The set of all efficient solution is denoted by X_{E}, and the set of all nondominated points is denoted by Z_{E}.

Definition 3. [12] The efficient solution $(x^*, y^*) \in X_E$ is a supported efficient solution, whenever it is an optimal solution to the following weighted objective problem:

$$\min\{\lambda_1 cx + \lambda_2 ty : \ (x, y) \in X\}, \tag{2}$$

where $(\lambda_1, \lambda_2) \in \Lambda_0$ and $\Lambda_0 = \{(\lambda_1, \lambda_2) : \ \lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_1 + \lambda_2 = 1\}$.

Remark 1. [8] If (x^*, y^*) is a supported efficient solution, then $Z(x^*, y^*)$ is also called a supported nondominated point. We use X_{sE} and Z_{sN} to denote sets of supported efficient solutions and supported nondominated points, respectively. The efficient solution (x^*, y^*) is a nonsupported efficient solution, if there exist some nonpositive values of λ_1 or λ_2 such that (x^*, y^*) is an optimal solution of problem (2).

3 An augmented formulation of the BOMCTNFP

In this section, we look for the supported and nonsupported efficient solutions that are the main challenges in solving mixed BOMCTNFPs. In addition, we show the relation between x_{ij} and y_{ij}, which is an additional challenge. We solve this problem with an alternative linear constraint, which is expressed in many articles in the field of fixed-charge problem (see [15, 7, 20]). First, we reformulate model (1) by replacing the third constraint with $x_{i,j} \leq u_{i,j} y_{i,j}$, as follows:

$$\begin{aligned}
\min cx &= \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij}, \\
\min ty &= \sum_{(i,j)\in\mathcal{A}} t_{ij} y_{ij}, \\
\text{s.t.} \quad & \sum_{(k,j)\in\mathcal{A}^=} x_{kj} - \sum_{(i,k)\in\mathcal{A}} x_{ik} = b_k && \text{for all } k \in \mathcal{V}, \\
& x_{ij} \leq u_{ij} y_{ij} && \text{for all } (i,j) \in \mathcal{A}, \\
& x_{ij} \geq 0, \ y_{ij} \in \{0, 1\} && \text{for all } (i,j) \in \mathcal{A}.
\end{aligned} \tag{3}$$

Hochbaum and Segev [15] proved a theorem that states that an efficient solution to (3) is an efficient solution to (1). On the other hand, a fundamental theorem in multi-objective linear problem assures us that any efficient solution can be characterized as an optimal solution of the weighted sum single-objective problem (2). A proof for this fact can be found in [26]. Therefore, problem (3) can be reformulated as the following parametric problem, which provides a set of supported efficient solutions of the above problem:

$$\begin{aligned} &\min \lambda cx + (1-\lambda)ty = \lambda \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} + (1-\lambda)\sum_{(i,j)\in\mathcal{A}} t_{ij}y_{ij}, \\ \text{s.t} \quad & \sum_{(k,j)\in\mathcal{A}} x_{kj} - \sum_{(i,k)\in\mathcal{A}} x_{ik} = b_k \qquad \text{for all } k\in\mathcal{V}, \\ & x_{ij} \le u_{ij}y_{ij} \qquad \text{for all } (i,j)\in\mathcal{A}, \\ & x_{ij}\ge 0,\ y_{ij}\in\{0,1\} \qquad \text{for all } (i,j)\in\mathcal{A}, \end{aligned} \tag{4}$$

for all $\lambda \in (0,1)$.

4 Primal-dual algorithm for solving mixed BOMCTNFP

In the following, we use a primal-dual algorithm for the weighted sum single-objective FCNFP (4) and evaluate the efficiency of the algorithm. Since the primal-dual algorithm is based on the duality theory of problem (4), the relaxation relation $y_{ij} \in [0,1]$ is used instead of $y_{ij} \in \{0,1\}$. Hence, the dual problem of (4) is cast as follows:

$$\begin{aligned} \max \quad & \sum_{k\in\mathcal{V}} b_k\pi_k - \sum_{(i,j)\in\mathcal{A}} \delta_{ij}, \\ \text{s.t.} \quad & \pi_i - \pi_j - \mu_{ij} \le c_{ij}(\lambda) \qquad \text{for all } (i,j)\in\mathcal{A}, \\ & \mu_{ij}u_{ij} - \delta_{ij} \le t_{ij}(\lambda) \qquad \text{for all } (i,j)\in\mathcal{A}, \\ & \mu_{ij}\ge 0,\ \delta_{ij}\ge 0 \qquad \text{for all } (i,j)\in\mathcal{A}, \end{aligned} \tag{5}$$

where $c_{ij}(\lambda) = \lambda c_{ij}$, $t_{ij}(\lambda) = (1-\lambda)t_{ij}$, for some $\lambda \in (0,1)$, $\pi = (\pi_1, \dots, \pi_m)$ is the vector of dual variables corresponding to conservation constraint $\sum_{(k,j)\in\mathcal{A}} x_{kj} - \sum_{(i,k)\in\mathcal{A}} x_{ik} = b_k$, $\mu = (\mu_{i,j})_{(i,j)\in\mathcal{A}}$ is the vector of dual variables corresponding to the constraint $x_{ij} \le u_{ij}y_{ij}$, and $\delta = (\delta_{i,j})_{(i,j)\in\mathcal{A}}$ is the the vector of dual variables associated with the constraint $y_{ij} \le 1$, that is,

$$\begin{aligned} \sum_{(k,j)\in\mathcal{A}} x_{kj} - \sum_{(i,k)\in\mathcal{A}} x_{ik} = b_k : &\qquad \pi_k \quad \text{for all } k\in\mathcal{V}, \\ x_{ij} \le u_{ij}y_{ij} : &\qquad \mu_{i,j} \quad \text{for all } (i,j)\in\mathcal{A}, \\ y_{ij} \le 1 : &\qquad \delta_{i,j} \quad \text{for all } (i,j)\in\mathcal{A}. \end{aligned} \tag{6}$$

Since the right side of the constraints in (5) depends on λ, the feasible solutions of this problem inevitably depends on λ. Suppose that (x^*_{ij}, y^*_{ij}) and $(\pi(\lambda)^*, \mu(\lambda)^*, \delta(\lambda)^*)$ are the feasible solutions of the primal problem (4) and the dual problem (5), respectively. By using the complementary slackness conditions, (x^*, y^*) and $(\pi^*, \mu^*, \delta^*)(\lambda)$ are optimized for their corresponding problems if and only if

$$\begin{aligned}
&i) && \Big(- \pi^*_i + \pi^*_j + \mu^*_{ij} + c_{ij}(\lambda)\Big) x^*_{ij} = 0 && \text{for all } i, j \in \mathcal{V} \text{ \& for all } (i,j) \in \mathcal{A},\\
&ii) && \Big(- \mu^*_{ij} u_{ij} + \delta^*_{ij} - t_{ij}(\lambda)\Big) y^*_{ij} = 0 && \text{for all } (i,j) \in \mathcal{A},\\
&iii) && \Big(- x^*_{ij} + u_{ij} y^*_{ij}\Big) \mu^*_{ij} = 0 && \text{for all } (i,j) \in \mathcal{A},\\
&iv) && \Big(1 - y^*_{ij}\Big) \delta^*_{ij} = 0 && \text{for all } (i,j) \in \mathcal{A}.
\end{aligned}$$

If $0 < x^*_{ij} < u_{ij}$, then $\mu^*_{ij} = 0$ and $-\pi^*_i + \pi^*_j + c_{ij}(\lambda) = 0$. Also, if $c_{ij}(\lambda) - \pi^*_i + \pi^*_j > 0$, then $x^*_{ij} = 0$ and $y^*_{i,j} = 0$. If $c_{ij}(\lambda) - \pi^*_i + \pi^*_j < 0$, then consequently $\mu^*_{i,j} > 0$ and $x^*_{ij} = y^*_{ij} u_{ij}$. Two cases may be occurred:

a) If $t_{ij}(\lambda) \geq \mu^*_{ij} u_{ij}$, then $\delta^*_{ij} = 0$; hence $y^*_{ij} = 0$ and $x^*_{ij} = 0$.

b) If $t_{ij}(\lambda) < \mu^*_{ij} u_{ij}$, then $\delta^*_{ij} > 0$; hence $y^*_{ij} = 1$ and $x^*_{ij} = u_{ij}$.

Let $(\pi, \mu, \delta)(\lambda)$ be a feasible solution to (5) and let Λ_q, $q = 1, 2, \ldots, r$ be a partition of the interval $\Lambda_0 = (0, 1)$. Also, let $\mathcal{A}^=_q \subset \mathcal{A}$ be a set of arcs (i, j) such that for all $\lambda \in \Lambda_q$, it satisfies $\pi_i(\lambda) - \pi_j(\lambda) = c_{ij}(\lambda)$, that is,

$$\mathcal{A}^=_q = \{(i,j) \in \mathcal{A} : \pi_i(\lambda) - \pi_j(\lambda) = c_{ij}(\lambda),\ \lambda \in \Lambda_q\}.$$

For the arcs not in the set $\mathcal{A}^=_x$, assuming $\lambda \in \Lambda_q$, if $\pi_i - \pi_j < c_{ij}(\lambda)$, then $x_{ij} = 0$ and $y_{ij} = 0$, and if $\pi_i - \pi_j > c_{ij}(\lambda)$, then $x_{ij} = u_{ij}$ and $y_{ij} = 1$ or $x_{ij} = 0$ and $y_{ij} = 0$. We define sets L_q and U_q for $q = 1, 2, \ldots, r$, as follows:

$$L_q = \{(i,j) \in \mathcal{A} : \pi_i - \pi_j < c_{ij}(\lambda),\ \lambda \in \Lambda_q\},$$

$$U_q = \{(i,j) \in \mathcal{A} : \pi_i - \pi_j > c_{ij}(\lambda),\ \lambda \in \Lambda_q\}.$$

For each interval Λ_q, we now define the restricted primal problem as follows. This problem tries to find a feasible solution for the problem of the parametric single-objective minimum cost-time network flow (4) using the the arcs in the set $\mathcal{A}^=_q$, only. Consider the following restricted primal (RP($\mathcal{A}^=_q$))) problem:

$$(\mathrm{RP}(\mathcal{A}_q^{=}))\ \min z = \sum_{k\in\mathcal{V}\backslash\{1\}} s_k,$$

$$\begin{aligned}
\text{s.t.} \sum_{(1,j)\in\mathcal{A}_q^{=}} x_{1j} - \sum_{(i,1)\in\mathcal{A}_q^{=}} x_{i1} - s_1 + \sum_{k=2}^{m}(-1)^{t_k} s_k = b_1^{\prime}, \qquad (7)\\
\sum_{(k,j)\in\mathcal{A}_q^{=}} x_{kj} - \sum_{(i,k)\in\mathcal{A}_q^{=}} x_{ik} - (-1)^{t_k} s_k = b_k^{\prime} \qquad \forall k\in\mathcal{V}\backslash\{1\},\\
x_{ij} \le u_{ij} y_{ij} \qquad \forall (i,j)\in\mathcal{A}_q^{=},\\
x_{ij} \ge 0,\ y_{ij}\in\{0,1\} \qquad \forall (i,j)\in\mathcal{A}_q^{=},\\
s_k \ge 0 \qquad \forall k\in\mathcal{V},
\end{aligned}$$

where

$$t_k = \begin{cases} 1 & b_k^{\prime} \ge 0,\\ 0 & b_k^{\prime} < 0,\end{cases}$$

in which for each $k \in \mathcal{V}$, $b_k^{\prime}$ is defined as

$$b_k^{\prime} = b_k + \sum_{(i,k)\in U_q} x_{ik} - \sum_{(k,i)\in U_q} x_{ki}.$$

Problem (7) has a feasible solution as follows:

$$x_{ij} = 0, \quad y_{ij} = 0, \quad s_1 = 0, \quad s_k = |b_k^{\prime}| \quad (k\in\mathcal{V}\backslash\{1\}).$$

Note that $\mathrm{RP}(\mathcal{A}_q^{=})$ does not depend on λ. Assume that $(\hat{x}, \hat{y}, \hat{s})$ is the optimal solution of problem (7). If the optimal value of the problem $\mathrm{RP}(\mathcal{A}_q^{=})$, $\hat{z}$, is zero, then the optimal solutions for problem (4) are as $x_{ij}^* = \hat{x}_{ij}$ and $y_{ij}^* = \hat{y}_{ij}$ for all $(i,j) \in \mathcal{A}_q^{=}$, and $x_{ij}^* = 0$ and $y_{ij}^* = 0$ or $x_{ij}^* = u_{ij}$ and $y_{ij}^* = 1$ for remaining arcs in $\mathcal{A}$. Now if $\hat{z} > 0$, then the given solution is not a feasible solution to the primal problem (4). At this time, either a new feasible dual solution should be found that improves the objective function (7) or it can be concluded that the primal problem is infeasible. Since $(\hat{x}, \hat{y}, \hat{s})$ is an optimal solution to problem (7) with the arcs in $\mathcal{A}_q^{=}$, so a new arc in $\mathcal{A}\backslash\mathcal{A}_q^{=}$ is needed to reduce the value of the objective function in problem (7). We now consider the dual of the $\mathrm{RP}(\mathcal{A}_q^{=})$ as follows:

$$
\begin{aligned}
(\mathrm{DRP}(\mathcal{A}_q^{=}))\ \max\ & \sum_{k\in\mathcal{V}} b'_k \pi_k - \sum_{(i,j)\in\mathcal{A}_q^{=}} \delta_{ij}, \\
\text{s.t.}\quad & \pi_i - \pi_j - \mu_{ij} \le 0 && \text{for all } (i,j)\in\mathcal{A}_q^{=}, \\
& \mu_{ij}u_{ij} - \delta_{ij} \le 0 && \text{for all } (i,j)\in\mathcal{A}_q^{=}, \\
& \pi_1 \ge 0 \\
& (-1)^{t_k}\pi_1 - (-1)^{t_k}\pi_k \le 1 && \text{for all } k\in\mathcal{V}\backslash\{1\}, \\
& \mu_{ij}\ge 0,\ \delta_{ij}\ge 0 && \text{for all } (i,j)\in\mathcal{A}_q^{=}.
\end{aligned} \tag{8}
$$

Suppose that $(\hat{\pi},\hat{\mu},\hat{\delta})$ is an optimal solution for DRP($\mathcal{A}_q^{=}$). According to the complementary conditions, we obtain

$$
\begin{aligned}
&\hat{i}) && \Big(\hat{\pi}_i - \hat{\pi}_j - \hat{\mu}_{ij}\Big)\hat{x}_{ij} = 0 && \text{for all } i,j\in\mathcal{V} \ \&\ \text{for all } (i,j)\in\mathcal{A}_q^{=}, \\
&\hat{ii}) && \Big(-\hat{\mu}_{ij}u_{ij} - \hat{\delta}_{ij}\Big)\hat{y}_{ij} = 0 && \text{for all } (i,j)\in\mathcal{A}_q^{=}, \\
&\hat{iii}) && \Big(-\hat{x}_{ij} + u_{ij}\hat{y}_{ij}\Big)\hat{\mu}_{ij} = 0 && \text{for all } (i,j)\in\mathcal{A}_q^{=}, \\
&\hat{iv}) && \Big(1-\hat{y}_{ij}\Big)\hat{\delta}_{ij} = 0 && \text{for all } (i,j)\in\mathcal{A}_q^{=}.
\end{aligned}
$$

If $0<\hat{x}_{ij}<u_{ij}$, then $\hat{\mu}_{ij}=0$ and $\hat{\pi}_i-\hat{\pi}_j=0$. In addition, if $\hat{\pi}_i-\hat{\pi}_j<0$, then $\hat{x}_{ij}=0$ and $\hat{y}_{ij}=0$. If $\hat{\pi}_i+\hat{\pi}_j>0$, then $\hat{\mu}_{ij}>0$ and $\hat{x}_{ij}=\hat{y}_{ij}u_{ij}$. From the dual constraint $\hat{\mu}_{ij}u_{ij}-\hat{\delta}_{ij}\le 0$, it can be deduced that $\hat{\delta}_{ij}>0$, therefore $\hat{y}_{ij}=1$ and so $\hat{x}_{ij}=0$.

We now define the following sets:

$$
\mathcal{A}_q^{==} = \{(i,j)\in\mathcal{A}_q^{=} : \hat{\pi}_i-\hat{\pi}_j=0\},
$$

$$
L_q^{=} = \{(i,j)\in\mathcal{A}_q^{=} : \hat{\pi}_i-\hat{\pi}_j<0\} = \{(i,j)\in\mathcal{A}_q^{=}\backslash\mathcal{A}_q^{==} : \hat{x}_{ij}=0, \hat{y}_{ij}=0\},
$$

$$
U_q^{=} = \{(i,j)\in\mathcal{A}_q^{=} : \hat{\pi}_i-\hat{\pi}_j>0\} = \{(i,j)\in\mathcal{A}_q^{=}\backslash\mathcal{A}_q^{==} : \hat{x}_{ij}=u_{ij}, \hat{y}_{ij}=1\}.
$$

Now Suppose that $(\acute{\pi},\acute{\mu},\acute{\delta})$ is a new solution to the dual problem (5), which is defined as follows:

$$
\begin{aligned}
\acute{\pi}(\lambda) &= \pi + \theta(\lambda)\hat{\pi}, \\
\acute{\mu}(\lambda) &= \mu + \theta(\lambda)\hat{\mu}, \\
\acute{\delta}(\lambda) &= \delta + \theta(\lambda)\hat{\delta},
\end{aligned} \tag{9}
$$

where (π,μ,δ) is a dual feasible solution for the original dual problem (5), $(\hat{\pi},\hat{\mu},\hat{\delta})$ is a dual feasible solution for DRP($\mathcal{A}_q^{=}$), $q=1,\ldots,r$, and $\theta(\lambda)>0$. For $(i,j)\in\mathcal{A}_q^{=}$, the new solution $(\acute{\pi},\acute{\mu},\acute{\delta})$ is a dual feasible solution for (5), because

$$\begin{aligned}\acute{\pi}_i - \acute{\pi}_j - \acute{\mu}_{ij} - c_{ij} &= \Big(\pi_i + \theta(\lambda)\hat{\pi}_i\Big) - \Big(\pi_j + \theta(\lambda)\hat{\pi}_j\Big) - \Big(\mu_{ij} + \theta(\lambda)\hat{\mu}_{ij}\Big) - c_{ij}(\lambda)\\ &= \Big(\pi_i - \pi_j - \mu_{ij} - c_{ij}(\lambda)\Big) + \theta(\lambda)\Big(\pi_i - \pi_j - \mu_{ij}\Big)\\ &\le 0.\end{aligned}$$

For $(i,j) \notin \mathcal{A}_q^=$, the dual feasibility depends on the parameter $\theta(\lambda)$. The new arc $(i,j) \in \mathcal{A}\backslash\mathcal{A}_q^=$, which will be added to the current set $\mathcal{A}_q^=$, must reduce the value of objective function $\mathrm{RP}(\mathcal{A}_q^=)$. To this end, any arc $(i,j) \in \mathcal{A}\backslash\mathcal{A}_q^=$ satisfying either $(i,j) \in L_q$ (i.e., $\pi_i - \pi_j < c_{ij}(\lambda)$) and $\hat{\pi}_i - \hat{\pi}_j > 0$ or $(i,j) \in U_q$ (i.e., $\pi_i - \pi_j > c_{ij}(\lambda)$) and $\hat{\pi}_i - \hat{\pi}_j < 0$ can be included to $\mathcal{A}_q^=$ provided that the dual feasibility for problem (5) is valid. Thus for keeping this condition, $\theta(\lambda)$ is chosen as

$$\begin{aligned}\theta(\lambda) = \min\Big\{\frac{c_{ij}(\lambda) - \pi_i + \pi_j + \mu_{ij}}{\hat{\pi}_i - \hat{\pi}_j - \hat{\mu}_{ij}} : &\Big((i,j) \in L_q,\ \hat{\pi}_i - \hat{\pi}_j > 0\Big),\\ &\Big((i,j) \in U_q,\ \hat{\pi}_i - \hat{\pi}_j < 0\Big)\Big\}.\end{aligned} \tag{10}$$

This process continues until $\hat{z} = 0$. At this time, we will have an optimal solution for the primal problem (4). If the condition $\hat{z} = 0$ is not satisfied, then an arc such as $(i,j) \in L_q$ such that $\hat{\pi}_i - \hat{\pi}_j > 0$ or $(i,j) \in U_q$ so that $\hat{\pi}_i - \hat{\pi}_j < 0$ cannot be found. Therefore, the starting problem (4) is infeasible.

The primal-dual algorithm for solving the BOMCTNFP is summarized in Algorithm 1.

5 A biobjective network flow example

Consider the following cities network. where 9 cities of Iran with 17 arcs are shown in Figure 1 (each city is a node). For each arc (i,j), there is a triple (c_{ij}, t_{ij}, u_{ij}) specifying the cost (million rials), time (hours) and capacity (million tons), respectively. Also, for each city i, there is a scalar b_i that indicates the value of supply or demand (million tons). As can be seen from Figure 1, three cities produce a special commodity, that is, Mashhad, Isfahan, and Ahwaz, and the rest cities are consumers. We trade 13.5 million tons of all network goods, and supply all demands with minimum cost at the minimum time. For simplicity, each city of the network is labeled with the numbers 1 to 9, as Table 1.

Algorithm 1 Primal-dual biobjective minimum cost-time network flow algorithm

Set $\Lambda_0 := (0,1)$, and **let** $(\pi^0, \mu^0, \delta^0)(\lambda) = (0,0,0)$, $\lambda \in \Lambda_0$ be an initial feasible solution of (5);
Set $\mathcal{C} := \{l_0\}$, where $l_0 := \Big(\Lambda_0, \mathcal{A}_0^{=} := \emptyset, L_0 := \mathcal{A}, U_0 := \emptyset, (\pi^0, \mu^0, \delta^0)(\lambda)\Big)$;
While $\mathcal{C} \neq \emptyset$ **do**
    Select $l_q \in \mathcal{C}$ and **solve** RP$(\mathcal{A}_q^{=})$ with l_q;
        (a) **If** the optimal value is 0, then any optimal solution is an efficient solution to (3). **Set** $\mathcal{C} := \mathcal{C} \backslash \{l_q\}$.
        (b) **Else** solve DRP$(\mathcal{A}_q^{=})$. Let $(\hat{\pi}, \hat{\mu}, \hat{\delta})(\lambda)$, $\lambda \in \Lambda_q$ be an optimal solution for DRP$(\mathcal{A}_q^{=})$;
            i. **If** there is no arc (i,j) such that $\pi_i - \pi_j < c_{ij}(\lambda)$ and $\hat{\pi}_i - \hat{\pi}_j > 0$ or $\pi_i - \pi_j > c_{ij}(\lambda)$ and $\hat{\pi}_i - \hat{\pi}_j < 0$, then (3) is infeasible and **STOP**.
            ii. **Else** set $\Lambda_{q,q'}$, $q' = 1 \ldots, r'$ of Λ_q employing (10) and compute the new feasible solution $(\pi^{qq'}, \mu^{qq'}, \delta^{qq'})(\lambda)$, $q' = 1, \ldots, r'$ according to (10).
            Put $\mathcal{C} := \mathcal{C} \backslash \{l_q\} \bigcup \{l_{qq'}\}$.
End while

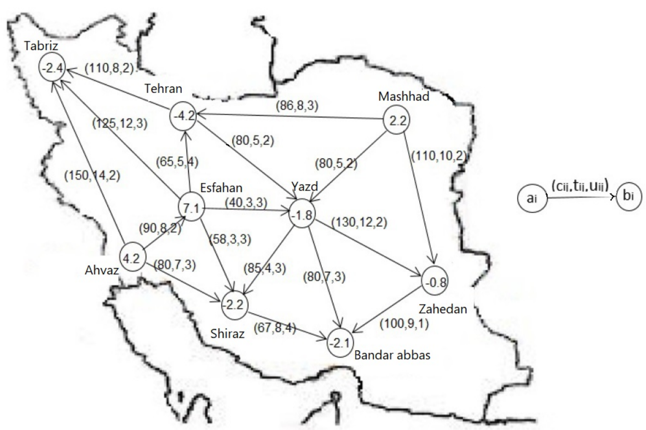

Figure 1: Graph of 9 cities of Iran with 17 arcs

Keshavarz and Toloo [16] computed the supported efficient solutions and supported nondominated points for only two values of the scalarization parameter $\lambda = 0.05,\ 0.5$, which is reported in Table 2. Utilizing the primal-dual technique proposed in this paper, we have obtained all the supported efficient solutions and nondominated points of the problem that are introduced in Ta-

Table 1: Labelling the cities of the network

City	Tabriz	Tehran	Esfahan	Ahvaz	Yazd	Shiraz	Mashhad	Zahedan	Bandar abbas
label	1	2	3	4	5	6	7	8	9

Table 2: Efficient solutions and nondominated points proposed in [16]

λ	Supported efficient solution	Supported nondominated point
0.05	$x^*_{32} = 3.2, x^*_{36} = 2.1, x^*_{35} = 1.8$ $x^*_{72} = 1.4, x^*_{78} = 0.8, x^*_{21} = 0.4$ $x^*_{46} = 2.2, x^*_{41} = 2.0, x^*_{69} = 2.1$	$(cx, ty) = (1270.9, 66)$
0.50	$x^*_{32} = 2.8, x^*_{36} = 0.1, x^*_{31} = 1.2$ $x^*_{35} = 3.0, x^*_{72} = 1.4, x^*_{78} = 0.8$ $x^*_{46} = 3.0, x^*_{41} = 1.2, x^*_{69} = 0.9$ $x^*_{59} = 1.2$	$(cx, ty) = (1242.5, 77)$

ble 3. As can be seen, some new supported efficient solutions are computed that such solutions were not investigated earlier in [16]. These new efficient solutions supply a decision maker with more alternatives, who prefers less to more in each objective.

Table 3: Efficient solutions and nondominated points computed by our method for the network of Figure 1

Supported efficient solutions		Supported nondominated points
$x_{i,j}$	$y_{i,j}$	
$x^*_{32} = 3.2, x^*_{36} = 2.1, x^*_{35} = 1.8$ $x^*_{72} = 1.4, x^*_{78} = 0.8, x^*_{21} = 0.4$ $x^*_{46} = 2.2, x^*_{41} = 2.0, x^*_{69} = 2.1$	$y^*_{32} = 1, y^*_{36} = 1, y^*_{35} = 1$ $y^*_{72} = 1, y^*_{78} = 1, y^*_{21} = 1$ $y^*_{46} = 1, y^*_{41} = 1, y^*_{69} = 1$	$(cx, ty) = (1270.9, 66)$
$x^*_{25} = x^*_{31} = x^*_{43} = x^*_{56} = 0$ $x^*_{58} = x^*_{59} = x^*_{75} = x^*_{89} = 0$	$y^*_{25} = y^*_{31} = y^*_{43} = y^*_{56} = 0$ $y^*_{58} = y^*_{59} = y^*_{75} = y^*_{89} = 0$	
$x^*_{32} = 2.8, x^*_{36} = 0.1, x^*_{31} = 1.2$ $x^*_{35} = 3.0, x^*_{72} = 1.4, x^*_{78} = 0.8$ $x^*_{46} = 3.0, x^*_{41} = 1.2, x^*_{69} = 0.9$ $x^*_{59} = 1.2$	$y^*_{32} = 1, y^*_{36} = 1, y^*_{31} = 1$ $y^*_{35} = 1, y^*_{72} = 1, y^*_{78} = 1$ $y^*_{46} = 1, y^*_{41} = 1, y^*_{69} = 1$ $y^*_{59} = 1$	$(cx, ty) = (1242.5, 77)$
$x^*_{21} = x^*_{25} = x^*_{43} = x^*_{56} = 0$ $x^*_{58} = x^*_{75} = x^*_{89} = 0$	$y^*_{21} = y^*_{25} = y^*_{43} = y^*_{56} = 0$ $y^*_{58} = y^*_{75} = y^*_{89} = 0$	
$x^*_{31} = 0.4, x^*_{32} = 2.8, x^*_{35} = 2.9$ $x^*_{36} = 3.0, x^*_{41} = 2.0, x^*_{43} = 2.0$ $x^*_{46} = 0.2, x^*_{59} = 1.1, x^*_{69} = 1.0$ $x^*_{72} = 1.4, x^*_{78} = 0.8$	$y^*_{31} = 1, y^*_{32} = 1, y^*_{35} = 1$ $y^*_{36} = 1, y^*_{41} = 1, y^*_{43} = 1$ $y^*_{46} = 1, y^*_{59} = 1, y^*_{69} = 1$ $y^*_{72} = 1, y^*_{78} = 1$	$(cx, ty) = (1381.4, 85)$
$x^*_{21} = x^*_{25} = x^*_{56} = 0$ $x^*_{58} = x^*_{75} = x^*_{89} = 0$	$y^*_{21} = y^*_{25} = y^*_{56} = 0$ $y^*_{58} = y^*_{75} = y^*_{89} = 0$	
$x^*_{31} = 0.4, x^*_{32} = 2.8, x^*_{35} = 1.8$ $x^*_{36} = 2.1, x^*_{41} = 2.0, x^*_{46} = 2.2$ $x^*_{69} = 2.1, x^*_{72} = 1.4, x^*_{78} = 0.8$	$y^*_{31} = 1, y^*_{32} = 1, y^*_{35} = 1$ $y^*_{36} = 1, y^*_{41} = 1, y^*_{46} = 1$ $y^*_{69} = 1, y^*_{72} = 1, y^*_{78} = 1$	$(cx, ty) = (1250.9, 70)$
$x^*_{21} = x^*_{25} = x^*_{43} = x^*_{56} = 0$ $x^*_{58} = x^*_{59} = x^*_{75} = x^*_{89} = 0$	$y^*_{21} = y^*_{25} = y^*_{43} = y^*_{56} = 0$ $y^*_{58} = y^*_{59} = y^*_{75} = y^*_{89} = 0$	
$x^*_{31} = 0.4, x^*_{32} = 2.8, x^*_{35} = 3.0$ $x^*_{36} = 0.9, x^*_{41} = 2.0, x^*_{46} = 2.2$ $x^*_{59} = 1.2, x^*_{69} = 0.9, x^*_{72} = 1.4$	$y^*_{31} = 1, y^*_{32} = 1, y^*_{35} = 1$ $y^*_{36} = 1, y^*_{41} = 1, y^*_{46} = 1$ $y^*_{59} = 1, y^*_{69} = 1, y^*_{72} = 1$	$(cx, ty) = (1156.9, 67)$
$x^*_{21} = x^*_{25} = x^*_{43} = x^*_{56} = 0$ $x^*_{58} = x^*_{75} = x^*_{78} = x^*_{89} = 0$	$y^*_{21} = y^*_{25} = y^*_{43} = y^*_{56} = 0$ $y^*_{58} = y^*_{75} = y^*_{78} = y^*_{89} = 0$	
$x^*_{31} = 1.2, x^*_{32} = 2.8, x^*_{35} = 1.8$ $x^*_{36} = 1.3, x^*_{41} = 1.2, x^*_{46} = 3.0$ $x^*_{69} = 2.1, x^*_{72} = 1.4, x^*_{78} = 0.8$	$y^*_{31} = 1, y^*_{32} = 1, y^*_{35} = 1$ $y^*_{36} = 1, y^*_{41} = 1, y^*_{46} = 1$ $y^*_{69} = 1, y^*_{72} = 1, y^*_{78} = 1$	$(cx, ty) = (1248.5, 70)$
$x^*_{21} = x^*_{25} = x^*_{43} = x^*_{56} = 0$ $x^*_{58} = x^*_{59} = x^*_{75} = x^*_{89} = 0$	$y^*_{21} = y^*_{25} = y^*_{43} = y^*_{56} = 0$ $y^*_{58} = y^*_{59} = y^*_{75} = y^*_{89} = 0$	

6 Computational experiments

The primal-dual approach for solving five instances of the BOMCTNFP has been implemented using MATLAB R2017a software, on a laptop equipped with Intel Core i5-8250U 1.80GHz processor and 8 GB of RAM, to find all their efficient extreme points. Table 4 shows that when the number of nodes and arcs grows as compared to the number of supported efficient solutions, the iterations and computational cost of the algorithm extremely increase.

Table 4: Numerical results by Algorithm 1 for several instances of BOMCTNFP (nses stands for the number of supported efficient solutions)

	The parameters of each instance									
Instances	Nodes	Arcs	Sources	Sinks	max $c_{i,j}$	max $t_{i,j}$	$u_{i,j}$	nses	Iterations	CPU time (sec.)
1	4	6	2	2	4	2	$1-2$	1	2	0.02
2	5	8	3	2	7	3	$1-3$	2	5	0.98
3	8	14	3	5	26	3	$1-3$	3	43	1.24
4	9	16	5	4	34	4	$1-4$	3	137	2.78
5	14	29	10	4	62	14	$1-5$	6	198	4.53

7 Conclusion

The purpose of this paper was to solve the mixed biobjective problem of minimum cost-time network flow. To this end, we first formulated the mixed biobjective problem and proposed the primal-dual algorithm for solving it. In this article, an example with 9 city nodes and 17 arcs was solved. We found 6 supported efficient solutions in the decision space. In comparison with the method presented in [16], we could find some new supported efficient solutions to provide a decision maker with more alternatives.

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