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A matrix method for system of integro-differential equations by using generalized Laguerre polynomials

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Abstract

The purpose of this research is to present a matrix method for solving system of linear Fredholm integro-differential equations(FIDEs) of the second kind on unbounded domain with degenerate kernels in terms of generalized Laguerre polynomials(GLPs). The method is based on the approximation of the truncated generalized Laguerre series. Then the system of (FIDEs) along with initial conditions are transformed into the matrix equations, which corresponds to a system of linear algebraic equations with the unknown generalized Laguerre coefficients. Combining these matrix equations and then solving the system yields the generalized Laguerre coefficients of the solution function. In addition, several numerical examples are given to demonstrate the validity, efficiency and applicability of the technique.

Keywords: Systems of linear Fredholm integro-differential equations; Unbounded domain; Generalized Laguerre polynomials; Operational matrix of integration.

1 Introduction

The main object of this paper is to approximate the solution system of Fredholm integro-differential equations of the second kind on a semi-infinite domain of the following form:

$$U'(x) = F(x) + \rho \int_0^\infty w(t)K(x,t)U(t)dt, \quad x \in \mathbb{R}_+, \tag{1}$$

along with initial condition $U(0) = A$, where $\rho \in R$, and

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$$U(x) = [u_1(x), u_2(x), ..., u_m(x)]^T,$$
$$F(x) = [f_1(x), f_2(x), ..., f_m(x)]^T,$$
$$K(x,t) = [k_{ij}], \quad i,j = 1,2,...,m,$$
$$A = [a_1, a_2, ..., a_m]^T. \tag{2}$$

In system (1), $w(t) = t^{\alpha}e^{-t}(\alpha > -1)$ and $K(x,t)$ a function of two variables x and t, is called the kernel that might have singularity in the region $D = \{(x,t) : 0 \leq x, t < \infty\}$ and $F(x)$ is continuous function and A is fixed constant vector, and $U(x)$ is the unknown vector function of the solution that will be determined. The considered equation arises in a number of important problem of elasticity theory, neutron transport, particle scattering and the theory of mixed-type equations [11,13,17]. System of linear Fredholm integro-differential equations of the second kind on unbounded domain can not be analytically solved easily. Therefore, it is required to obtain the approximate solutions. It's the reason of great interest for solving these equations. But numerical methods includes Quadrature, Petrov-Galerkin, Nystrom and Galerkin methods with Laguerre polynomial as a bases function for solving infinite boundary integral and integro-differential equations are used ago that their analysis may be found in [1, 7, 9, 10, 12, 16]. On the other hand, there are several numerical techniques for solving fractional differential equations (FDEs) on the half line using generalized Laguerre polynomials [2–6]. However, method of solution for equation (1) is too rear in the literature. In the present work, we are going to use the operational matrix of generalized Laguerre polynomials to find the approximate solutions for the system of FIDEs on the half-line. Next sections of this paper are organized as follows: In Section 2, we describe some necessary definitions and give some relevant properties of the GPLs which is required for our subsequent development. Section 3, is devoted to the approximation of the function $f(x)$ and also the kernel function $k(x,t)$ by using GPLs basis. Also the upper bound of the approximation error is presented. In Section 4, we obtain the operational matrix of integration by GPLs. In Section 5, we implemented the matrix method on the system of linear Fredholm integral-differential equations on unbounded domain and convert them to a linear algebraic system of equations. In Section 6, presented numerical examples that shows the efficiency and accuracy of the proposed method. Also a tall conclusion is given in Section 7.

2 Preliminaries and basic definitions

In this part, for the reader's convenience, we give some basic definitions and properties of the generalized Laguerre polynomials, which are used further in this article.

Let $\mathbb{R}_+ := \Lambda = [0, \infty)$ and $w^{(\alpha)}(x) = x^\alpha e^{-x}$ be a weight function on Λ in the usual sense. We define the following:

$$L^2_{w^{(\alpha)}}(\Lambda) = \{v : v \ \ \textit{is measurable on} \ \ \Lambda \ \ \textit{and} \ \ \|v\|_{w^{(\alpha)}} < \infty\}, \tag{3}$$

equipped with the following inner product and norm:

$$(u, v)_{w^{(\alpha)}} = \int_\Lambda u(x)v(x)w^{(\alpha)}(x)dx, \quad \|v\|_{w^{(\alpha)}} = (v, v)^{\frac{1}{2}}_{w^{(\alpha)}}. \tag{4}$$

Next, suppose $L_n^{(\alpha)}(x)$ be the generalized Laguerre polynomials of degree n, defined by the following:

$$L_n^{(\alpha)}(x) = \frac{1}{n!}x^{-\alpha}e^x\partial_x^n\left(e^{-x}x^{n+\alpha}\right), \qquad n = 0, 1, ... \tag{5}$$

$L_n^{(\alpha)}(x)$(generalized Laguerre polynomials) are the nth eigenfunction of the Sturm-Liouville problem:

$$x^{-\alpha}e^x\left(x^{\alpha+1}e^{-x}\left(L_n^{(\alpha)}(x)\right)'\right)' + \lambda_n L_n^{(\alpha)}(x) = 0, \quad x \in \Lambda, \tag{6}$$

with the eigenvalues $\lambda_n = n$ $[8, 14]$.
Generalized Laguerre polynomials are orthogonal in $L^2_{w^{(\alpha)}}(\Lambda)$ Hilbert space with the weight function $w^{(\alpha)}(x) = x^\alpha e^{-x}$ satisfy in the following relation

$$\int_0^\infty x^\alpha e^{-x}L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx = \gamma_n^\alpha\delta_{n,m}, \qquad \forall n, m \geq 0, \tag{7}$$

where $\delta_{n,m}$ is the Kronecher delta function and $\gamma_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}$. The explicit form of these polynomials is in the form

$$L_n^{(\alpha)}(x) = \sum_{i=0}^{n} E_i^\alpha x^i, \tag{8}$$

where

$$E_i^\alpha = \frac{\binom{n+\alpha}{n-i}(-1)^i}{i!}. \tag{9}$$

These polynomials are satisfied in the following recurrence formula

$$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = 1 + \alpha - x,$$

$$L_{n+1}^{(\alpha)}(x) = \frac{1}{n+1}\left[(2n + \alpha + 1 - x)L_n^{(\alpha)}(x) - (n + \alpha)L_{n-1}^{(\alpha)}(x)\right], \ n = 1, 2, ... \tag{10}$$

The case $\alpha = 0$ leads to the classical Laguerre polynomials, which are used most frequently in practice and will simply be denoted by $L_n(x)$. An important property of the Laguerre polynomials is the following derivative relation [10]:

$$\left(L_n^{(\alpha)}(x)\right)' = \sum_{i=0}^{n-1} L_i^{(\alpha)}(x). \tag{11}$$

Further, $\left(L_i^{(\alpha)}(x)\right)^{(k)}$ are orthogonal with respect to the weight function $w^{(\alpha+k)}(x)$. i.e.

$$\int_0^\infty (L_i^{(\alpha)})^{(k)}(x)(L_j^{(\alpha)})^{(k)}(x)w^{(\alpha+k)}(x)dx = \gamma_{n-k}^{\alpha+k}\delta_{i,j}, \qquad \forall i,j \geq 0, \tag{12}$$

where $\gamma_{n-k}^{\alpha+k}$ is defined in (7).

3 Approximation of functions by using GLPs

An arbitrary function $f(x) \in L^2_{w^{(\alpha)}}(\Lambda)$ may be expanded into generalized Laguerre polynomials as:

$$f(x) = \sum_{i=0}^{\infty} f_i^{(\alpha)} L_i^{(\alpha)}(x), \tag{13}$$

where the generalized Laguerre coefficients $f_i^{(\alpha)}$ are given by

$$f_i^{(\alpha)} = \int_0^\infty \frac{L_i^{(\alpha)}(x)}{\binom{i+\alpha}{i}} \cdot \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} \cdot f(x)dx, \qquad i = 0, 1, ... \tag{14}$$

The series converges in the associated Hilbert space $L^2_{w^{(\alpha)}}(\Lambda)$, iff

$$\|f\|^2_{L^2} := \int_0^\infty \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} |f(x)|^2 dx = \sum_{i=0}^{\infty} \binom{i+\alpha}{i} |f_i^{(\alpha)}|^2 < \infty. \tag{15}$$

In practice, only the first $(n+1)$ terms of generalized Laguerre polynomials are considered. Then we have

$$f(x) \simeq \sum_{i=0}^{n} f_i^{(\alpha)} L_i^{(\alpha)}(x) = F^T L_x, \tag{16}$$

where the generalized Laguerre coefficient vector F and generalized Laguerre vector L_x are given by as follows:

$$F = [f_0^{(\alpha)},\ f_1^{(\alpha)}, \ldots,\ f_n^{(\alpha)}]^T,\ L_x = [L_0^{(\alpha)}(x),\ L_1^{(\alpha)}(x), \ldots,\ L_n^{(\alpha)}(x)]^T. \quad (17)$$

Now in the following lemma we present an upper bound to estimate the error.

Lemma 1. *Suppose that the function $f : \Lambda \to \mathbb{R}$ is $n+1$ times continuously differentiable (i.e. $f \in C^{n+1}(\Lambda)$), and $Y = Span\{L_0^{(\alpha)}(x),\ L_1^{(\alpha)}(x), \ldots,\ L_n^{(\alpha)}(x)\}$. If $F^T L_x$ be the best approximation f out of Y then mean error bound is presented as follows:*

$$\|f - F^T L_x\|_{L^2_{w^{(\alpha)}}(\Lambda)} \leq \frac{N\sqrt{(2n+\alpha+2)!}}{(n+1)!}, \quad (18)$$

where $N = \max_{x \in \Lambda} |f^{(n+1)}(x)|$.

Proof. We know that the power basis $\{1, x, ..., x^n\}$ forms a basis for the space of all polynomials of degree less than or equal to n. Therefore, we define $y_1(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0)$. From Taylor expansion we have

$$|f(x) - y_1(x)| \leq |f^{(n+1)}(\eta_x)\frac{x^{n+1}}{(n+1)!}|, \quad (19)$$

where $\eta_x \in (0, \infty)$. Since $F^T L_x$ is the best approximation f out of Y, $y_1 \in Y$ and using (19) we have

$$\|f - F^T L_x\|^2_{L^2_{w^{(\alpha)}}(\Lambda)} \leq \|f - y_1\|^2_{L^2_{w^{(\alpha)}}(\Lambda)} \leq \frac{N^2(2n+\alpha+2)!}{(n+1)!^2}. \quad (20)$$

Then, by taking square roots we have the above bound. □

This Lemma shows that the error vanishes as $n \to \infty$.
We can also approximate the function of two variables, $k(x,t) \in L^2_{w^{(\alpha)}}(\Lambda^2)$ as follows:

$$k(x,t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{n} L_i^{(\alpha)}(x)k_{ij}^{(\alpha)}L_j^{(\alpha)}(t) = L_x^T K L_t. \quad (21)$$

Here the entries of matrix $K = [k_{ij}^{(\alpha)}]_{(n+1)\times(n+1)}$ will be obtained by

$$k_{ij}^{(\alpha)} = \frac{(L_i^{(\alpha)}(x), (k(x,t), L_j^{(\alpha)}(t)))}{(L_i^{(\alpha)}(x), L_i^{(\alpha)}(x))(L_j^{(\alpha)}(t), L_j^{(\alpha)}(t))}, \quad for \quad i,j = 0,1,...,n, \quad (22)$$

so that, $(.,.)$ denotes the inner product.

4 Operational matrix of integration, development and applications

The main objective of this part is to obtain the operational matrix of the integration by GPLs.

Theorem 1. *Suppose L_x be the generalized Laguerre vector defined in (17) then,*

$$\int_0^x L_t dt \simeq PL_x, \tag{23}$$

where P is the $(n+1)\times(n+1)$ operational matrix for integration as follows:

$$P = \begin{bmatrix} \Omega(0,0,\alpha) & \Omega(0,1,\alpha) & \Omega(0,2,\alpha) & \cdots & \Omega(0,n,\alpha) \\ \Omega(1,0,\alpha) & \Omega(1,1,\alpha) & \Omega(1,2,\alpha) & \cdots & \Omega(1,n,\alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega(i,0,\alpha) & \Omega(i,1,\alpha) & \Omega(i,2,\alpha) & \cdots & \Omega(i,n,\alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega(n,0,\alpha) & \Omega(n,1,\alpha) & \Omega(n,2,\alpha) & \cdots & \Omega(n,n,\alpha) \end{bmatrix}, \tag{24}$$

where

$$\Omega(i,j,\alpha) = \sum_{k=0}^{i}\sum_{r=0}^{j} \frac{(-1)^{k+r} j!\Gamma(i+\alpha+1)\Gamma(k+\alpha+r+2)}{(i-k)!(j-r)!(k+1)!r!\Gamma(k+\alpha+1)\Gamma(r+\alpha+1)}. \tag{25}$$

Proof. (See [4]). □

Also, we can see the extent of this theorem for solving fractional differential equations. For example, see [2, 5, 6].

5 Implementation of the matrix method

In this section, we solve the system of linear Fredholm integro-differential equations of the second kind on unbounded domain (1). To this end, we consider the *ith* equation of (1) as follows:

$$u_i'(x) = f_i(x) + \rho \int_0^\infty t^\alpha e^{-t} \sum_{j=1}^{m} k_{ij}(x,t) u_j(t) dt, u_i(0) = a_i, \ i = 1, ..., m, \tag{26}$$

where $f_i \in L^2_{w(\alpha)}(\Lambda)$, $k_{ij} \in L^2_{w(\alpha)}(\Lambda^2)$, and $u_i'(x)$ represents the first order derivative of $u_i(x)$ with respect to x, a_i are constants that give the initial conditions and u_i is an unknown function. In order to approximate the solution of equation (26), we approximate functions $f_i(x)$, $u_i(x)$ and $k_{ij}(x,t)$ with respect to generalized Laguerre polynomials as mentioned in the previous section as follows:

$$f_i(x) \simeq F_i^T L_x, \quad u_i'(x) \simeq C_i'^T L_x, \quad u_i(0) \simeq C_{i0}^T L_x, \quad k_{ij}(x,t) \simeq L_x^T K_{ij} L_t, \quad (27)$$

where F_i, C_i' for $i = 1, ..., m$ are known $(n+1) \times 1$ vectors and K_{ij} for $i, j = 1, 2, ..., m$ are known $(n+1) \times (n+1)$ matrices. Then, for $i = 1, ..., m$, we have:

$$u_i(x) = \int_0^x u_i'(t)dt + u_i(0) \simeq \int_0^x C_i'^T L_t dt + C_{i0}^T L_x \simeq (C_i'^T P + C_{i0}^T)L_x, \quad (28)$$

where P is a $(n+1) \times (n+1)$ operational matrix of integration given in (23). By substituting the approximations (27) and (28) into equation (26), we get the following:

$$\begin{aligned} L_x^T C_i' &= L_x^T F_i + \rho \int_0^\infty t^\alpha e^{-t} \sum_{j=1}^m L_x^T K_{ij} L_t L_t^T (p^T C_j' + C_{j0}) dt \\ &= L_x^T F_i + \rho L_x^T \sum_{j=1}^m K_{ij} \{ \int_0^\infty t^\alpha e^{-t} L_t L_t^T dt \} (p^T C_j' + C_{j0}) \\ &= L_x^T F_i + \rho L_x^T \sum_{j=1}^m K_{ij} Q (p^T C_j' + C_{j0}). \end{aligned} \quad (29)$$

Then we have following system of linear equations:

$$C_i' = F_i + \rho \sum_{j=1}^m K_{ij} Q (P^T C_j' + C_{j0}), \qquad i = 1, ..., m, \quad (30)$$

where

$$Q = \int_0^\infty t^\alpha e^{-t} L_t L_t^T dt = [q_{ij}^{(\alpha)}], \qquad i, j = 0, 1, ..., n, \quad (31)$$

and Q is a $(n+1) \times (n+1)$ matrix with elements

$$q_{ij}^{(\alpha)} = \int_0^\infty t^\alpha e^{-t} L_i^{(\alpha)}(t) L_j^{(\alpha)}(t) dt, \qquad i, j = 0, 1, ..., n. \quad (32)$$

By solving the linear system of algebraic equations (30), we can achieve the vector C_i' for $i = 1, ..., m$, then we will have

$$C_i^T = C_i'^T P + C_{i0}^T \Longrightarrow u_i(x) \simeq C_i^T L_x, \quad i = 1, ..., m. \quad (33)$$

That are the approximate solution for our system of (1).

6 Numerical Examples

In this section, we give several illustrative examples for demonstrate the efficiency of our proposed method to approximate the solutions system of Fredholm integro-differential equations of the second kind along with initial condition on a semi-infinite domain. For each example, we find the approximate solutions using different degree of generalized Laguerre polynomials. The results obtained by the present method reveal that the proposed method is very effective and convenient for system (1) on the half line. In all examples, the package of Matlab (2013) has been used to solve the test problems considered in this paper.

Example 1. For the first example, consider the following of Fredholm integral-differential equation on unbounded domain (constructed):

$$u'(x) = -\frac{247131410303000045}{36028797018963968}x^2 - \frac{389031992318478309 19}{144115188075855872} + \int_0^\infty t^{\frac{1}{2}} e^{-t}(x^2+t^2)u(t)dt, \quad u(0)=1. \tag{34}$$

Exact solution of this problem is $u(x) = x^3 - 2x + 1$. If we apply the technique described in the section 5, with $\alpha = \frac{1}{2}$ and $n = 3$, then the approximate solution can be expanded as follows:

$$u(x) \simeq \sum_{i=0}^{3} c_i^{(\alpha)} \; L_i^{(\alpha)}(x) = C^T L_x, \tag{35}$$

where

$$C = [c_0^{(\alpha)},\; c_1^{(\alpha)},\; c_2^{(\alpha)},\; c_3^{(\alpha)}]^T. \tag{36}$$

Hence, from Eqs. (16), (21), (23), and (31), we find the matrices

$$F = \begin{bmatrix} -125363/424 \\ 17011/496 \\ -6434/469 \\ 0 \end{bmatrix}, \; K = \begin{bmatrix} 15/2 & -5 & 2 & 0 \\ -5 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; P = \begin{bmatrix} 3/2 & -1 & 0 & 0 \\ 3/8 & 1 & -1 & 0 \\ 5/16 & 0 & 1 & -1 \\ 35/128 & 0 & 0 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 148/167 & 0 & 0 & 0 \\ 0 & 222/167 & 0 & 0 \\ 0 & 0 & 555/334 & 0 \\ 0 & 0 & 0 & 2053/1059 \end{bmatrix}.$$

Next, we substitute these matrices into equation (30) and then simplify to obtain

$$\begin{bmatrix} c_0^{(\alpha)'} \\ c_1^{(\alpha)'} \\ c_2^{(\alpha)'} \\ c_3^{(\alpha)'} \end{bmatrix} = \begin{bmatrix} -119/5475 & -19/7262 & -574/2251 & -865/5409 \\ 161/2349 & 347/1578 & 93/632 & -247/3501 \\ -181/6602 & 552/1769 & 1487/1580 & 193/6839 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -13873/48 \\ 4211/141 \\ -6427/538 \\ 0 \end{bmatrix}. \tag{37}$$

By solving the linear system (37), we have the following:

$$c_0^{(\alpha)'} = \frac{37}{4},\ c_1^{(\alpha)'} = -15,\ c_2^{(\alpha)'} = 6,\ c_3^{(\alpha)'} = 0. \tag{38}$$

By substituting the obtained coefficients in (33) the solution of (34) becomes

$$u(x) \simeq \frac{89}{8} L_0^{(\alpha)}(x) - \frac{97}{4} L_1^{(\alpha)}(x) + 21 L_2^{(\alpha)}(x) - 6 L_3^{(\alpha)}(x), \tag{39}$$

or briefly

$$u(x) \simeq x^3 - 2x + 1, \tag{40}$$

which is the exact solution. Also, if we choose $n \geq 4$, we get the same approximate solution as obtained in equation (40). Numerical results will not be presented since the exact solution is obtained.

Example 2. As the second example, we consider the following system of linear Fredholm integro-differential equations on unbounded domain (constructed):

$$u_1'(x) = f_1(x) + \int_0^\infty t^{\frac{1}{2}} e^{-t}(2x + t^2)(u_1(t) + u_2(t))dt,$$

$$u_2'(x) = f_2(x) + \int_0^\infty t^{\frac{1}{2}} e^{-t}(t - x^2)(u_1(t) - u_2(t))dt, \tag{41}$$

where $f_1(x) = 3x^2 - \frac{87307746120759955}{2251799813685248} x - \frac{66317884995750748881}{18014398509481984}$ and

$f_2(x) = \frac{98782478468059837}{9007199254740992} x^2 + 2x - \frac{853121404951425865}{18014398509481984}$.

Subject to the initial conditions $u_1(0) = 1$ and $u_2(0) = 1$. The exact solutions of this problem are $u_1(x) = x^3 + 2x + 1$ and $u_2(x) = x^2 + 1$. If we apply the technique described in this paper and solve equation (41) with $\alpha = \frac{1}{2}$ and $n = 3$. For this system we get:

$$u_1(x) = \frac{137}{8} L_0^{(\alpha)}(x) - \frac{113}{4} L_1^{(\alpha)}(x) + 21 L_2^{(\alpha)}(x) - 6 L_3^{(\alpha)}(x) = x^3 + 2x + 1,$$

$$u_2(x) = \frac{19}{4} L_0^{(\alpha)}(x) - 5 L_1^{(\alpha)}(x) + 2 L_2^{(\alpha)}(x) + (0) L_3^{(\alpha)}(x) = x^2 + 1, \tag{42}$$

which is the exact solution. Also, if we choose $n \geq 4$, we get the same approximate solution as obtained in equation (42). Numerical results will not be presented since the exact solution is obtained.

Example 3. As the third example, consider the following system of linear Fredholm integro-differential equations on unbounded domain (constructed):

$$u_1'(x) = f_1(x) + \int_0^\infty te^{-t-x}(sin(t-x)u_1(t) + tu_2(t))dt,$$

$$u_2'(x) = f_2(x) + \int_0^\infty te^{-t}(xtu_1(t) - e^{-x}u_2(t))dt, \qquad (43)$$

with $f_1(x) = 1 - \frac{1}{4}(1 + 2sinx + 2cosx)e^{-x}$ and $f_2(x) = -6x - \frac{5}{4}e^{-x}$ and with the exact solutions $u_1(x) = x$, $u_2(x) = e^{-x}$ and boundary conditions $u_1(0) = 0$ and $u_2(0) = 1$. We apply the generalized Laguerre series approach and solve equation (43). Table 1 shows the absolute values of error $|e| = |u_2(x) - \overline{u_2}(x)|$, where $u_2(x)$ is the exact solution of equation (43) and $\overline{u_2}(x)$ is the approximate of $u_2(x)$ for $n = 20$, and $n = 30$ with $\alpha = 1$ using the described method in equally divided interval $[0, 1]$. Note that absolute

Table 1: Absolute errors for Example 3

i	x_i	$n = 20$	$n = 30$
0	0.0	$5.4836e-006$	$7.6834e-009$
1	0.1	$1.2376e-006$	$3.6294e-010$
2	0.2	$4.7027e-007$	$1.1165e-009$
3	0.3	$8.2669e-007$	$5.9280e-010$
4	0.4	$5.7470e-007$	$1.8341e-010$
5	0.5	$1.4911e-007$	$5.9768e-010$
6	0.6	$2.2241e-007$	$5.8849e-010$
7	0.7	$4.4523e-007$	$3.0976e-010$
8	0.8	$5.0477e-007$	$4.8167e-011$
9	0.9	$4.2952e-007$	$3.3937e-010$
10	1.0	$2.6633e-007$	$4.8646e-010$

errors for $u_1(x)$ is zero.
Corollary: If the exact solution of the system (1) be a polynomial, then the proposed method will obtain the real solution.

Example 4. Our last example is following of linear Fredholm integro-differential equation on a semi infinite interval (constructed):

$$u'(x) = e^{-x} - \frac{7}{4}\sqrt{x} + \int_0^\infty t^{\frac{1}{2}}e^{-t}\sqrt{xt}u(t)dt, \quad u(0) = 1. \qquad (44)$$

With the exact solution $u(x) = 2 - e^{-x}$. In Table 2, the numerical results of the presented method at some selected nods for $n = 10$, and $n = 12$ are displayed.

Table 2: Absolute errors for Example 4

i	x_i	$n = 10$	$n = 12$
0	0.0	$1.3000e-003$	$3.6116e-004$
1	0.1	$4.4620e-004$	$9.3426e-005$
2	0.2	$7.1845e-005$	$4.5500e-005$
3	0.3	$3.2705e-004$	$9.9666e-005$
4	0.4	$4.0407e-004$	$1.0182e-004$
5	0.5	$3.6812e-004$	$7.5666e-005$
6	0.6	$2.6834e-004$	$3.7765e-005$
7	0.7	$1.4076e-004$	$8.8208e-007$
8	0.8	$1.0787e-005$	$3.3527e-005$
9	0.9	$1.0468e-004$	$5.6602e-005$
10	1.0	$1.9540e-004$	$6.8843e-005$

7 Conclusion

Obtaining the analytic solutions for system of linear Fredholm integro-differential equations of the second kind, along with initial conditions on unbounded domain are usually difficult. In many cases, it is required to approximate solutions. For this reason, a new matrix approach which is based on the generalized Laguerre operational matrix of integration is proposed. The solution procedure is very simple by means of generalized Laguerre polynomials expansion and only in a few terms lead to high accurate solutions. The main goal of the presented technique was deriving an approximation to the solution system of linear Fredholm integro-differential equations on unbounded domain. To illustrate the method and its efficiency, four examples were provided. In the first and second examples, we obtained the exact solution. Another considerable advantage of the method is that the nth-order approximation gives the exact solution when the solution is polynomial of degree equal to or less than n. If the solution is not polynomial, generalized Laguerre series approximation converges to the exact solution as n increases.

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يك روش ماتريسی براي سيستمی از معادلات انتگرال-ديفرانسيل با استفاده از چندجمله ایهای لاگر تعميم يافته

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دریافت مقاله ۲۴ اسفند ۱۳۹۳، دریافت مقاله اصلاح شده ۱۶ دی ۱۳۹۴، پذیرش مقاله ۵ اسفند ۱۳۹۴

چکیده : هدف از این مقاله، ارائه يك روش ماتريسی براي حل سيستمی از معادلات انتگرال-ديفرانسيل فردهلم خطي از نوع دوم روي دامنه بي‌كران با هسته های جدایی پذیر با جملاتی از چندجمله ای های لاگر تعمیم یافته می باشد. این روش مبتنی بر تقریب بوسیله ی سری لاگر تعمیم یافته است. سیستم معادلات انتگرال-دیفرانسیل همراه با شرایط اولیه تبدیل به معادلاتی ماتریسی می شود که متناظر با سیستمی از معادلات جبری خطی با مجهولاتی از ضرایب لاگر تعمیم یافته است. با ترکیب معادلات ماتریسی و سپس حل سیستم مذکور می توان به ضرایب لاگر تعمیم یافته دست یافت و لذا تابع جواب بدست می آید. به علاوه در این مقاله، چندین مثال عددی براي نشان دادن درستی و کارایی روش ارائه گردیده است.

کلمات کلیدی : سیستم معادلات انتگرال-دیفرانسیل فردهلم خطی؛ دامنه ی بی کران؛ چندجمله ای های لاگر تعمیم یافته؛ ماتریس عملیاتی انتگرال گیری.