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Research Article

Shooting continuous Runge–Kutta method for delay optimal control problems

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Abstract

In this paper, we present an efficient method to solve linear time-delay optimal control problems with a quadratic cost function. In this regard, first, by employing the Pontryagin maximum principle to time-delay systems, the original problem is converted into a sequence of two-point boundary value problems (TPBVPs) that have both advance and delay terms. Then, using the continuous Runge–Kutta (CRK) method, the resulting sequences are recursively solved by the shooting method to obtain an optimal control law. This obtained optimal control consists of a linear feedback term, which is obtained by solving a Riccati matrix differential equation, and a forward term, which is an infinite sum of adjoint vectors, that can be obtained by solving sequences of delay TPBVPs by the shooting CRK method. Finally, numerical results and their comparison with other available results illustrate the high accuracy and efficiency of our proposed method.

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1 Introduction

In recent years, optimization and control of systems with time delay have been considered in much research because the time delay in many processes cannot be ignored. To more accurately express the behavior of a natural phenomenon, we need a more complex system. Some of the applications of these issues are in the chemical, electronic, medicine, engineering, biological, economy, and so on [22, 19, 12, 7, 8, 41].

In general, two methods are provided to solve optimal control problems (OCPs). The first approach involves the use of necessary and (or) sufficient conditions of optimality by applying the Pontryagin minimum (maximum) principle or optimality principle. The minimum principle was presented in 1956 by the Russian mathematician Lev Pontryagin and his students, and its primary application was to maximize the terminal velocity of a rocket. This result was obtained using the classical ideas of variational calculus. The equations obtained from these conditions can be solved numerically. This approach yields indirect methods, which are known as analytical-based methods; see [39, 43, 17, 13].

In another approach, an OCP is considered an optimization problem. instead of using the optimality conditions, the dynamic constraints are transformed into an algebraic equations system by discretizing the time interval and parameterizing the variables of the problem. Therefore, the OCP becomes a nonlinear programming problem of dimension finite. The resulting nonlinear programming problem can then be solved using optimization techniques. This approach yields direct methods. We refer the reader to [11, 2, 18, 26, 8]. Since direct methods do not need to calculate the optimality conditions, they can be used for a wide range of OCPs. However, the lack of guarantee for the optimal solution and the high amount of memory resources and time for producing a close approximation is among the disadvantages of these methods.

In the case of time-delay OCPs, in 1963, Oğuztöreli [35] was one of the pioneers in the analytical-based approach (also, see [36]). For the first time, Kharatishvili [24] generalized the Pontryagin maximum principle for OCPs with a constant delay in the state variable. Then in [25], he gave similar results on OCPs with delay in the control variable. After that, in 1968, a maximum principle for OCPs with multiple constant delays in state and control was proved by Halanay [16]. In 1972, Ray and Soliman [42] also obtained similar results. Guinn [14] transformed the delayed OCP with constant delay in the state variable into a higher-dimensional undelayed OCP. Banks [3] derives a maximum principle for control systems with a time-dependent delay in the state variable.

The system resulted from the necessary conditions that Kharatishvili provided, which was a two-point boundary value problem involving both advance and delay terms. This type of problem does not have an exact solution, except in exceptional cases. Therefore, there are many attempts available in the literature to approximately solve this problem; for example, see [29, 44, 30, 31, 32, 20, 21, 6].

The following articles can be mentioned as the latest studies. For OCPs with time-invariant delayed systems, Mirhosseini-Alizamini, the second author, and Heydari [32] applied the variational iteration method and then obtained a suboptimal solution for the two-point boundary value problem (TPBVP). Moreover, Mirhosseini-Alizamini and the second author [31] investigated infinite horizon OCPs with time-variant delayed systems. Also, using a Hermite interpolation polynomial for delay terms and employing a second-order finite difference formula for the first-order derivatives, Jajarmi and Hajipour [21] converted the TPBVP obtained from the time-delay OCP into a system of linear algebraic equations and then solved it. Recently, using an algorithm based on the forward and backward difference approximation, Bouajaji et al. [6] solved the system obtained from the application of the Pontryagin maximum principle to a delayed OCP.

In this work, we investigate a family of time-delay OCPs with a quadratic cost functional that should be minimized subject to a linear time-delay system with constant delay in the state variable. Using the Pontryagin minimum principle for delayed systems from [24] and then applying continuous Runge–Kutta (CRK) methods, we convert a time-delay OCP into a sequence of linear TPBVPs and thereafter solve it recursively by the shooting method to obtain the optimal control law.

The rest of the paper is organized as follows: The CRK methods are presented in Section 2. After that, in Section 3, we introduce the Shooting CRK (SCRK) method and apply it to a delayed TPBVP. Then, in the continuation of this section, we present a basic algorithm for the proposed method. In the next section, we will use a generalization of this algorithm to solve a time-delay OCP. Section 4 describes the Pontryagin maximum principle for our delayed OCP and designs an algorithm based on the previous algorithm defined in Section 3 for solving the final system. In Section 5, we give several numerical examples to demonstrate the effectiveness and accuracy of the proposed technique. Finally, with the conclusion in Section 6, we end the article.

2 CRK methods

In this section, we describe the CRK methods. Consider $f(t, x(t)) \in C^0([t_0, t_f] \times \mathbb{R}^d, \mathbb{R}^d)$. The CRK methods were originally designed to treat the initial value problem for the following ordinary differential equation:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & t_0 \leq t \leq t_f, \\ x(t_0) = x_0. \end{cases} \tag{1}$$

Some of the implicit Runge–Kutta methods are equivalent to collocation methods; see [46]. Thus, they sequentially provide a continuous extension of the approximate solution without any additional evaluation of f. Indeed, the next question is whether there is such a continuous extension for each Runge–Kutta process that is given sequentially by the method itself?

Nørsett and Wanner partially answered this question by proving that a large number of Runge–Kutta methods are the same as the somewhat pertubed collocation method that is somewhat perturbed. After that, Zennaro [47] presented a continuous extension of the solution provided by a Runge–Kutta method, which includes the collocation solution if it is equivalent to collocation and behaves similarly in other cases.
Let $\Delta = \{t_0, \ldots, t_n, \ldots, t_N = t_f\}$ be an arbitrary mesh. Then for the numerical solution of the ordinary differential equation (1), an s-stage discrete Runge–Kutta method has the form

$$x_{n+1} = x_n + h_{n+1} \sum_{i=1}^{s} b_i k_i, \tag{2}$$

$$k_i = f(t_n^i, x_n + h_{n+1} \sum_{j=1}^{s} a_{ij} k_j), \quad i = 1, \ldots, s, \tag{3}$$

where $c_i = \sum_{j=1}^{s} a_{ij}, t_n^i = t_n + c_i h_{n+1}, i = 1, \ldots, s$, and $h_{n+1} = t_{n+1} - t_n$. In addition, the Runge–Kutta method (2) and (3) is denoted by (A, b). Let the solution have advanced to the point $t = t_n$. Zennaro [47] showed that for this s-stage Runge–Kutta method of order p, there is a CRK method of degree d, if there exist s polynomials $b_i(\theta),\ i = 1, \ldots, s$, of degree less than or equal to d, independent of f. This method reads as follows:

$$\eta(t_n + \theta h_{n+1}) = x_n + h_{n+1} \sum_{i=1}^{s} b_i(\theta) k_i, \quad 0 \leq \theta \leq 1, \tag{4}$$

$$k_i = f(t_n^i, x_n + \sum_{j=1}^{s} a_{ij} k_j), \quad i = 1, \ldots, s, \tag{5}$$

where

$$\eta(t_n) = x_n, \qquad \eta(t_n + h_{n+1}) = x_{n+1},$$

and x_n is an approximate solution obtained by applying the Runge–Kutta method for $x(t_n)$. This method, which is usually expressed as $(A, b(\theta))$, can also be related to the following CRK tableau:

$$\begin{array}{c|c} \mathrm{C} & \mathrm{A} \\ \hline & b^T(\theta) \end{array}.$$

In fact, $\{c_i, a_{ij}\}$'s are the same as the coefficients of the discrete Runge–Kutta method. Now, we recall the consistency of the discrete Runge–Kutta method from [5].

Definition 1. [5, Definition 5.1.3] Consider $p \geq 1$ the largest integer having the following property: For every mesh point and C^p-continuous right-hand-side function $f(t,x)$ in (1), the local solution $z_{n+1}(t)$ to the local problem

$$\begin{cases} z_{n+1}^{'}(t) = f(t, z_{n+1}(t)), \quad t_n \leq t \leq t_{n+1}, \\ z_{n+1}(t) = x_n^*, \end{cases} \tag{6}$$

satisfies

$$\|z_{n+1}(t_{n+1}) - x_{n+1}\| = O(h_{n+1}^{p+1})$$

uniformly with respect to x_n^* belonging to a bounded subset of $\mathbb{R}^d$ and respect to $n = 0, \ldots, N-1$, Then we say that the discrete Runge–Kutta method (A, b) is consistent with order p.

Similarly, with the above notations, we say that the continuous extension (4) is consistent with uniform order q if $q \geq 1$ is the largest integer having the following property:

$$\max_{t_n \leq t \leq t_{n+1}} \|z_{n+1}(t) - \eta(t)\| = O(h_{n+1}^{q+1}),$$

for every mesh point and C^q-continuous right-hand-side function $f(t,x)$ in (1).

According to Definition 1, the convergence results in discrete and CRK methods for ordinary differential equations have been proved in the following theorem; see [5].

Theorem 1. [5, Theorem 5.1.4] Suppose that the Runge–Kutta method (2) and (3) is consistent with order p and that $f(t,x)$ defined in (1) is a right-hand-side C^p-continuous function. Then, on any bounded interval $[t_0, t_f]$, the method has discrete global order (or, equivalently, is convergent of order) p. In other words,

$$\max_{1 \leq n \leq N} \|x(t_n) - x_n\| = O(h^p),$$

in which $h = \max_{1 \le n \le N} h_n$.

Moreover, let the continuous extension (4) have the uniform order q. Then the CRK method (4) and (5) has the uniform global order (or, equivalently, uniformly convergent of order) $q' = \min(q+1, p)$, which means that

$$\max_{t_0 \le t \le t_f} \|x(t) - \eta(t)\| = O(h^{q'}).$$

Then, Baker and Paul [1] generalized this idea for a CRK method to delay differential equations with a general delay differential equation of the form

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t - \tau(t))), & t > t_0, \\ x(t) = \phi(t), & t_0 - \tau(t_0) \le t \le t_0, \end{cases} \tag{7}$$

in which $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\tau(t) \ge 0$. Moreover, $\phi \in C^0[t_0 - \tau(t_0), t_0]$ denotes the initial information of the state variable x. For delay differential equations, Baker and Paul [1] modified (4) and (5) as follows :

$$\eta(t_n + \theta h_{n+1}) = x_n + h_{n+1} \sum_{i=1}^{s} b_i(\theta) k_i, \quad 0 \le \theta \le 1, \tag{8}$$

$$k_i = f(t_n^i, X_i, \eta(t_n^i - \tau(t_n^i))), \quad i = 1, \ldots, s, \tag{9}$$

$$X_i = x_n + h_{n+1} \sum_{j=1}^{s} a_{ij} k_j, \quad i = 1, \ldots, s. \tag{10}$$

When the delay is constant and $h_{n+1} \le \tau$, then $\eta(t_n^i - \tau)$ is known for any i $(0 \le c_i \le 1)$. In this case, $\eta(t_n^i - \tau)$ is available from the past, so this method is an explicit CRK method. The pair formed by (A, b) and $(A, b(\theta))$ is called the underlying CRK method.

Theorem 2. [5, Theorem 6.3.1] Assuming the delay differential equation (7), suppose that $f(t, x, y) \in [t_0, t_f] \times \mathbb{R}^n \times \mathbb{R}^n$ is a C^p-continuous function. Then the delay $\tau(t) \in [t_0, t_f] \times \mathbb{R}^n$ is a C^p-continuous function and $\phi(t)$ is the initial C^p-continuous function. In addition, let $\Delta = \{t_0, t_1, \ldots, t_n, \ldots, t_N = t_f\}$ be the mesh containing all points of discontinuity with the order less than or equal to p being in $[t_0, t_f]$. Also, assume that the underlying CRK method has the uniform and discrete orders q and p, respectively. Then for the delay differential equation, the CRK method (8), (9), and (10) has uniform global and discrete global orders $q' = \min(p, q+1)$. In other words,

$$\max_{1 \le n \le N} \|x(t) - \eta(t)\| = O(h^{q'}),$$

and

$$\max_{1\le n\le N}\|x(t_n)-x_n\| = O(h^{q'}),$$

where $h = \max_{1\le n\le N} h_n$.

3 Outline of SCRK method for a delay TPBVP

In the present section, we first state details of the proposed method on a TPBVP with only a time-delay term. Therefore, consider the following basic form of a first-order TPBVP with a time delay:

$$\begin{cases} \dot{x}(t) = f_1(t, x(t), y(t), x(t-\tau), y(t-\tau)), & t_0 \le t \le t_f, \\ \dot{y}(t) = f_2(t, x(t), y(t), x(t-\tau), y(t-\tau)), & t_0 \le t \le t_f, \\ x(t) = \phi(t), & t_0 - \tau \le t \le t_0, \\ y(t_f) = \beta. \end{cases} \tag{11}$$

For solving this problem, we need to use the solutions to a sequence of initial value problems that are made by substituting the initial guess $y(t_0) = z$ instead of the terminal condition $y(t_f) = \beta$ in (11).

To approximate a solution to the boundary value problem (11), we involve a parameter z, by choosing the parameters $z = z_k$ such that

$$\lim_{k\to\infty} y(t_f, z_k) = y(t_f) = \beta,$$

where $y(t)$ is the solution to the boundary value problem (11) and $y(t, z_k)$ denotes the solutions to the constructed initial value problem with initial conditions $x(t) = \phi(t),\ t_0 - \tau \le t \le t_0$ and $y(t_0) = z_k$.

This technique is called the Shooting method. For starting, we choose a parameter z_1 such that it determines the initial evaluation at which the object is fired from the point $(t_0, \phi(t_0))$ and along the curve indicated by the solution to the problem

$$\begin{cases} \dot{x}(t) = f_1(t, x(t), y(t), x(t-\tau), y(t-\tau)), & t_0 \le t \le t_f, \\ \dot{y}(t) = f_2(t, x(t), y(t), x(t-\tau), y(t-\tau)), & t_0 \le t \le t_f, \\ x(t) = \phi(t), & t_0 - \tau \le t \le t_0, \\ y(t_0) = z_1. \end{cases} \tag{12}$$

If $y(t_f, z_1)$ is not sufficiently close to β, then we correct the approximation by choosing elevations z_2, z_3, and so on, until $y(t_f, z_k)$ is sufficiently close to β.

For determining the parameters z_k, we must solve this problem:

$$y(t_f, z) - \beta = 0. \tag{13}$$

To solve this nonlinear equation, we use the secant method. For this method, we need to choose initial approximations z_1 and z_2 and then generate the remaining terms of the sequence by the following formula:

$$z_{k+1} = z_k - \frac{y(t_f, z_k) - \beta}{y(t_f, z_k) - y(t_f, z_{k-1})}(z_k - z_{k-1}), \quad k = 3, 4, \dots. \tag{14}$$

To obtain $y(t_f, z_1)$ in (12), we use the CRK method (8), (9), and (10) for a system of delay differential equations. For a given mesh $\Delta = \{t_0, \dots, t_n, \dots, t_N = t_f\}$, let $h = \frac{t_f - t_0}{N}$. In each underlying mesh interval $[t_n, t_{n+1}]$, CRK formulas for (12) are as follows:

$$\begin{cases} k_{1,i} = f_1(t_n^i, X_i, Y_i, \eta_x(t_n^i - \tau_1), \eta_y(t_n^i - \tau_2)), & i = 1, \dots, s, \\ k_{2,i} = f_2(t_n^i, X_i, Y_i, \eta_x(t_n^i - \tau_1), \eta_y(t_n^i - \tau_2)), & i = 1, \dots, s, \\ X_i = x_n + h \sum_{j=1}^s a_{ij} k_{1,j}, & i = 1, \dots, s, \\ Y_i = y_n + h \sum_{j=1}^s a_{ij} k_{2,j}, & i = 1, \dots, s, \\ \eta_x(t_n + \theta h) = x_n + h \sum_{i=1}^s b_i(\theta) k_{1,i}, & 0 \le \theta \le 1, \\ \eta_y(t_n + \theta h) = y_n + h \sum_{i=1}^s b_i(\theta) k_{2,i}, & 0 \le \theta \le 1, \end{cases} \tag{15}$$

Note that at the endpoint of the interval, the stop condition must be checked. In the following algorithm, we describe an SCRK method for a time-delay TPBVP.

Example 1. Consider the following second-order delay boundary value problem:

$$\begin{cases} x''(t) = -\frac{1}{16} \sin x(t) - (t+1)x(t-1) + t, & 0 \le t \le 2, \\ x(t) = t - \frac{1}{2}, & t \le 0, \\ x(2) = -\frac{1}{2}. \end{cases} \tag{16}$$

With the new condition $x'(0) = z$, instead of solving (16), we need to solve a sequence of initial value problems of the form

$$\begin{cases} x''(t) = -\frac{1}{16} \sin x(t) - (t+1)x(t-1) + t, & 0 \le t \le 2, \\ x(t) = t - \frac{1}{2}, & t \le 0, \\ x'(0) = z. \end{cases} \tag{17}$$

Now, we try to make the value of $y(2, z)$ as close to $\beta = -\frac{1}{2}$ as possible by adjusting the value of z. Before that, by assuming $y(t) = x'(t)$, we turn the delay second-order system (17) into a delay first-order system as follows:

$$\begin{cases} x'(t) = y(t), & 0 \le t \le 2, \\ y'(t) = -\frac{1}{16} \sin x(t) - (t+1)x(t-1) + t, & \\ x(t) = t - \frac{1}{2}, & t \le 0, \\ y(0) = z. \end{cases} \tag{18}$$

Algorithm 1 SCRK method for time-delay TPBVP

Step 1. Set N (the number of subintervals), $h = \frac{t_f - t_0}{N}$, $K = 1$, M (the maximum number of iterations), and s (the number of stages of the CRK method), and choose z_1, z_2, and tolerance error bound ϵ.

Step 2. While $L \leq M$, do
- Set $x_0 = \alpha$ and $y(t_0) = z_1$,

Step 3. For $k = 1, 2, \cdots$,
- solve (12), using the CRK method (15).
- Set $x_0 = \alpha$ and $y(t_0) = z_1$,

Step 4. Check the stop condition,
- If $|y_N - \beta| < \epsilon$, then the procedure is complete, and jump to Step 7,
- else, go to the next step.

Step 5. If $k = 1$, then set $y(t_0) = z_2$ and back to Step 3,
- else, go to the next step,

Step 6. Calculate the next approximation for z_{k+1} from (14), set $y(t_0) = z_{k+1}$, and back to Step 3.
- end for

Step 7. Stop the algorithm and output (t_n, x_n, y_n).
- end while

Step 8. Output (maximum number of iterations exceeded).
- Stop

We solve this problem by applying Algorithm 1. For this purpose, we use the explicit Runge–Kutta of discrete order $p = 4$ with the following coefficients:

$$\begin{array}{c|cccc} 0 & 0 & & & \\ \frac{1}{2} & \frac{1}{2} & 0 & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \\ \frac{1}{2} & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

Moreover, we set

$$b_1(\theta) = \frac{1}{2}\theta^2 + \frac{2}{3}\theta, \qquad b_3(\theta) = \frac{1}{3}\theta,$$
$$b_2(\theta) = \frac{1}{3}\theta, \qquad b_4(\theta) = \frac{1}{2}\theta^2 - \frac{1}{3}\theta.$$

Table 1 indicates a comparison between the approximate result of our SCRK method and the results obtained in [34].

Table 1: Approximation values of $x(t)$ in Example 1

	$x_n(1)$		$x_n(1.5)$		$x_n(2)$	
n	Ref. [34]	Proposed method	Ref. [34]	Proposed method	Ref. [34]	Proposed method
4	-1.854384	-1.983957	-1.719174	-1.884111	-0.499976	-0.499999
6	-2.018854	-2.032385	-1.896332	-1.922809	-0.499999	-0.500000
8	-2.066385	-2.052802	-1.946231	-1.939199	-0.499999	-0.500000
10	-2.078723	-2.063029	-1.959110	-1.947469	-0.499999	-0.500000
12	-2.081821	-2.068830	-1.962343	-1.952141	-0.500000	-0.500000

4 Design of SCRK method for an OCP with time delay in state variable

In this section, we first use the Pontryagin maximum principle to solve our delayed OCP. Then, for solving the final system, we describe an algorithm based on Algorithm 1. Through this section, by $PC^1([t_0,t_f],\mathbb{R}^n)$ we denote the class of continuous functions from $[t_0,t_f]$ into $\mathbb{R}^n$ whose first-order derivatives are piecewise continuous, and similarly, $PC([t_0,t_f],\mathbb{R}^n)$ denotes the class of piecewise continuous functions from $[t_0,t_f]$ into $\mathbb{R}^n$.

Consider the linear system with delay in the state variable

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t-\tau) + Bu(t), & t_0 \le t \le t_f, \\ x(t) = \phi(t), & t_0 - \tau \le t \le t_0, \end{cases} \tag{19}$$

where $u(t)$ in $PC([t_0,t_f],\mathbb{R}^n)$ and $x(t)$ in $PC^1([t_0-\tau,t_f],\mathbb{R}^n)$ are the control and state variables, respectively. In fact, the parameter $\tau > 0$ is nonnegative and indicates the time delay. Furthermore, the initial state function $\phi(t)$ is continuous in $C([t_0-\tau,t_0],\mathbb{R}^n)$, and finally, the matrices A, B, and A_1 are real constants with appropriate dimensions. For $t \in [t_0,t_f]$, our aim is to obtain, $u^*(t)$, the optimal control law minimizing the quadratic cost function

$$J = \frac{1}{2}\int_{t_0}^{t_f} (u^T(t)Ru(t) + x^T(t)Qx(t))dt + \frac{1}{2}x^T(t_f)Q_fx(t_f), \tag{20}$$

in which $R \in \mathbb{R}^{m\times n}$ is a positive definite matrix and Q and $Q_f \in \mathbb{R}^{n\times n}$ are positive semi-definite matrices.

For time-delay OCPs, it follows from [24] that the pontryagin maximum principle provides the necessary conditions of optimality for the problem (19) and (20) as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t-\tau) - BR^{-1}B^T\lambda(t), & t_0 \le t \le t_f, \\ \dot{\lambda}(t) = \begin{cases} -Qx(t) - A^T\lambda(t) - A_1^T\lambda(t+\tau), & t_0 \le t \le t_f - \tau, \\ -Qx(t) - A^T\lambda(t), & t_f - \tau < t \le t_f, \end{cases} \\ x(t) = \phi(t), \quad t_0 - \tau \le t \le t_0, \\ \lambda(t_f) = Q_f x(t_f). \end{cases} \tag{21}$$

The Hamiltonian function from which the above conditions are derived is

$$H(x,u,\lambda,t) = \lambda^T(t)[Ax(t) + Bu(t) + A_1x(t-\tau) + \frac{1}{2}x^T(t)Qx(t) + \frac{1}{2}u^T(t)Ru(t)], \tag{22}$$

where $\lambda(t) \in PC^1([t_0,t_f],\mathbb{R}^n)$ is called co-state vector. Moreover,

$$u^*(t) = -R^{-1}B^T\lambda(t), \tag{23}$$

for $t_0 \le t \le t_f$, is the optimal control law. We recall that the system (21) is a TPBVP with both time-advance and time-delay terms. Unfortunately, in general, this problem does not have any analytical solution. Therefore, providing an efficient method for solving this difficult problem numerically is very important.

At first, we produce a sequence of TPBVP as

$$\begin{cases} \dot{x}^{(k)}(t) = -S\lambda^{(k)}(t) + Ax^{(k)}(t) + A_1x^{(k)}(t-\tau), \quad t_0 \le t \le t_f, \\ \dot{\lambda}^{(k)}(t) = \begin{cases} -A^T\lambda^{(k)}(t) - Qx^{(k)}(t) - A_1^T\lambda^{(k-1)}(t+\tau), & t_0 \le t \le t_f - \tau, \\ -A^T\lambda^{(k)}(t) - Qx^{(k)}(t), & t_f - \tau < t \le t_f, \end{cases} \\ x^{(k)}(t) = \phi(t), \quad t_0 - \tau \le t \le t_0, \\ \lambda^{(k)}(t_f) = Q_f x^{(k)}(t_f), \\ x^{(0)}(t) \equiv 0, \quad \lambda^{(0)}(t) \equiv 0, \quad t_0 \le t \le t_f, \end{cases} \tag{24}$$

where $S = BR^{-1}B^T$ and $k = 1, 2, \ldots$. Therefore,

$$u^{(k)}(t) = -R^{-1}B^T\lambda^{(k)}(t) \tag{25}$$

is the sequence of controls. Now, we are ready to obtain a closed-loop optimal control. We can define the co-state vector by

$$\lambda^{(k)}(t) = g^{(k)}(t) + P(t)x^{(k)}(t), \tag{26}$$

in which $g^{(k)}(t) \in \mathbb{R}^n$ is the kth adjoint vector and $P(t) \in \mathbb{R}^{n\times n}$ is an unknown function matrix with positive-definite property [45, 44].
Consider the following extended sequence of the TPBVP (24):

$$\begin{cases} \dot{x}^{(k)}(t) = [A - SP(t)]x^{(k)}(t) - Sg^{(k)}(t) + A_1 x^{(k)}(t-\tau), & t_0 \le t \le t_f, \\ \dot{g}^{(k)}(t) = \begin{cases} -P(t)A_1 x^{(k)}(t-\tau) - [A - SP(t)]^T g^{(k)}(t) \\ -A_1^T P(t+\tau)x^{(k-1)}(t+\tau) - A_1^T g^{(k-1)}(t+\tau), & t_0 \le t \le t_f - \tau, \\ -P(t)A_1 x^{(k)}(t-\tau) - [A - SP(t)]^T g^{(k)}(t), & t_f - \tau < t \le t_f, \end{cases} \\ x^{(k)}(t) = \phi(t), \qquad t_0 - \tau \le t \le t_0, \\ g^{(k)}(t_f) = 0, \\ x^{(0)}(t) \equiv 0, \quad g^{(0)}(t) \equiv 0, \qquad t_0 \le t \le t_f. \end{cases} \tag{27}$$

We note that by substituting (26) into the first equation of (24), the kth optimal closed-loop system is constructed, which is the first equation of the system (27). Similarly, substituting (26) in the second equation of (24) and comparing the result with the derivative of (26), we obtain the second equation of the system (27). Also,

$$\begin{aligned} -\dot{P}(t) &= P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t) + Q, \\ P(t_f) &= Q_f, \end{aligned} \tag{28}$$

is a Riccati matrix differential equation.
Moreover, from (25) and (26), the sequence of controls is converted to

$$u^{(k)}(t) = -R^{-1}B^T(P(t)x^{(k)}(t) + g^{(k)}(t)), \quad k = 1, 2, \dots. \tag{29}$$

The system (27) is similar to (11), except that (27) has advance terms in addition to the delay terms. Now, we want to use Algorithm 1 to solve this advance-delay TPBVP. By using the SCRK method, we have the following CRK iteration formula of (27) in the mesh interval $[t_n, t_{n+1}]$:

$$\begin{aligned} \eta_x^{(k)}(t_n + \theta h) =& x_n^{(k)} + h\sum_{i=1}^{s} b_i(\theta)[\Psi(t_n^i)(x_n^{(k)} + h\sum_{j=1}^{s} a_{ij}k_{1,j}) \\ & - S(g_n^{(k)} + h\sum_{j=1}^{s} a_{ij}k_{2,j}) + A_1\eta_x^{(k)}(t_n^i - \tau)], \quad t_0 \le t \le t_f, \end{aligned} \tag{30}$$

$$\eta_g^{(k)}(t_n + \theta h) = \begin{cases} g_n^{(k)} + h\sum_{i=1}^{s} b_i(\theta)[-\Psi^T(t_n^i)(g_n^{(k)} + h\sum_{j=1}^{s} a_{ij}k_{2,j}) \\ -P(t_n^i)A_1\eta_x^{(k)}(t_n^i - \tau) - A_1^T P(t_n^i + \tau)x^{(k-1)}(t_n^i + \tau) \\ -A_1^T g^{(k-1)}(t_n^i + \tau)], \qquad t_0 \le t \le t_f - \tau, \\ g_n^{(k)} + h\sum_{i=1}^{s} b_i(\theta)[-\Psi^T(t_n^i)(g_n^{(k)} + h\sum_{j=1}^{s} a_{ij}k_{2,j}) \\ -P(t_n^i)A_1\eta_x^{(k)}(t_n^i - \tau)], \qquad t_f - \tau < t \le t_f, \end{cases} \tag{31}$$

where $t_n^i = t_n + c_i h$, $\Psi(t_n^i) = A - SP(t_n^i)$, and $0 \le \theta \le 1$. Also,

$$\begin{cases} x^{(k)}(t) = \phi(t), \quad t_0 - \tau \le t \le t_0, \\ g^{(k)}(t_f) = 0, \end{cases} \tag{32}$$

are the known initial and final conditions.

As already mentioned, for the constant delay, if $0 \le c_i \le 1$ and $h \le \tau$, then $\eta(t_n^i - \tau)$ is known for any i. Hence, $\eta_x(t_n^i - \tau)$ in (30) and (31) is known, and there is no so-called overlapping. On the other hand, $x^{(k-1)}(t_n^i + \tau)$ and $g^{(k-1)}(t_n^i + \tau)$ are obtained from the previous iteration by the assumptions $x^{(0)}(t) \equiv 0$ and $g^{(0)}(t) \equiv 0$.

Theorem 3. Consider TPBVP (27).

i) Assume that the right-hand-side functions corresponding to $\dot{x}^{(k)}(t)$ and $\dot{g}^{(k)}(t)$ and the initial function $\phi(t)$ are C^p-continuous in their domains (p is the discrete order of the underlying CRK method). Then the sequences $\{\eta_x^{(k)}(t)\}$ and $\{\eta_g^{(k)}(t)\}$ obtained from CRK formulas (30) and (31) with initial and boundary conditions (32), converge uniformly to the solution of TPBVP (27).

ii) Under the assumptions of part (i), the sequences $\{u^{(k)}(t)\}$ and $\{J^{(k)}\}$, which are defined as follows

$$u^{(k)}(t) = -R^{-1}B^T[P(t)\eta_x^{(k)}(t) + \eta_g^{(k)}(t)], \tag{33}$$

$$J^{(k)} = \frac{1}{2}(\eta_x^{(k)}(t_f))^T Q_f \eta_x^{(k)}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [(\eta_x^{(k)}(t))^T Q \eta_x^{(k)}(t) + (u^{(k)}(t))^T R u^{(k)}(t)]dt, \tag{34}$$

converge to optimal control $u^*(t)$ and the optimal value of objective function, J^*, respectively.

Proof. i) Consider the vector function F as follows:

$$F(t, x, g, u, v, w, z) = (\dot{x}(t), \dot{g}(t))^T, \quad t_0 \le t \le t_f,$$

and u, v, w, z denote delay and advance terms corresponding to the variables $x(t)$ and $g(t)$. Also, $\dot{x}(t)$ and $\dot{g}(t)$ are the functions defined in (27). Because F and ϕ are C^p-continuous functions and τ is a constant delay, according to Theorem 2, the sequences $\{\eta_x^{(k)}(t)\}$ and $\{\eta_g^{(k)}(t)\}$ from the CRK method are uniformly convergence to the exact solutions of (27).

ii) Suppose that $\{\eta_x^{(k)}(t)\}$ and $\{\eta_g^{(k)}(t)\}$ are solution sequences produced by the CRK method, which are convergence to $\hat{\eta}_x(t)$ and $\hat{\eta}_g(t)$ under the assumptions of part (i). We take the limit from the (33) as $k \to \infty$,

$$\hat{u}(t) := \lim_{k\to\infty} u^{(k)}(t) = -R^{-1}B^T[P(t)(\lim_{k\to\infty} \eta_x^{(k)}(t)) + \lim_{k\to\infty} \eta_g^{(k)}(t)]$$

$$= -R^{-1}B^T[P(t)\hat{\eta}_x(t) + \hat{\eta}_g(t)].$$

Since $\hat{\eta}_x(t)$ and $\hat{\eta}_g(t)$ are the exact solutions of necessary conditions (27), so $\hat{u}(t)$ is the optimal control $u^*(t)$.
Similarly, we take the limit from the (34) as follows:

$$\begin{aligned}
\hat{J} &:= \lim_{k\to\infty} J^{(k)} \\
&= \lim_{k\to\infty}((\frac{1}{2}(\eta_x^{(k)}(t_f))^T Q_f \eta_x^{(k)}(t_f)) \\
&\quad + \frac{1}{2}\lim_{k\to\infty}(\int_{t_0}^{t_f}[(\eta_x^{(k)}(t))^T Q \eta_x^{(k)}(t) + (u^{(k)}(t))^T R u^{(k)}(t)]dt) \\
&= \frac{1}{2}(\lim_{k\to\infty}(\eta_x^{(k)}(t_f))^T Q_f(\lim_{k\to\infty}\eta_x^{(k)}(t_f)) \\
&\quad + \frac{1}{2}\int_{t_0}^{t_f}[(\lim_{k\to\infty}(\eta_x^{(k)}(t))^T)Q(\lim_{k\to\infty}\eta_x^{(k)}(t)) \\
&\quad + (\lim_{k\to\infty}(u^{(k)}(t))^T)R(\lim_{k\to\infty}u^{(k)}(t))]dt \\
&= \frac{1}{2}\hat{\eta}_x^T(t_f)Q_f\hat{\eta}_x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}[\hat{\eta}_x^T(t)Q\hat{\eta}_x(t) + \hat{u}^T(t)R\hat{u}(t)]dt,
\end{aligned}$$

so, $\hat{J}$ is the optimal value of the performance index J.

□

According to Theorem 3, it can be concluded that for enough iterations of the CRK method, for example, N iterations, where N depends on a given error criterion, we can obtain a suboptimal control as follows:

$$u^{(N)}(t) = -R^{-1}B^T[P(t)\eta_x^{(N)}(t) + \eta_g^{(N)}(t)]. \tag{35}$$

In this case, the continuous suboptimal state function is as $\eta_x(t) \cong \eta_x^{(N)}(t)$. To calculate a more accurate state function, the suboptimal control function resulting from equation (35), can be placed in (19), and we then solve the obtained initial value problem. Finally, by placing this pair of suboptimal control and state in the objective function, we have

$$\begin{aligned}
J^{(N)} = \frac{1}{2}(\eta_x^{(N)}(t_f))^T Q_f \eta_x^{(N)}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}((\eta_x^{(N)}(t))^T Q \eta_x^{(N)}(t) \\
+ (u^{(N)}(t))^T R u^{(N)}(t))dt.
\end{aligned} \tag{36}$$

For given $\varepsilon > 0$, if the stop condition,

$$\left|\frac{J^{(N)} - J^{(N-1)}}{J^{(N)}}\right| < \varepsilon,$$

is satisfied, then the suboptimal control (35) will have the desired accuracy. Now, to implement the above method, we provide the following simple algorithm.

Algorithm 2 SCRK method for time-delay OCPs

Step 1. Solve $P(t)$ from (28).

Step 2. Put $k = 1$, $x^{(0)} \equiv 0$, and $g^{(0)} \equiv 0$. Then obtain a continuous approximation for $x^{(k)}(t)$ and $g^{(k)}(t)$ from the kth TPBVP (30), (31), and (32) with the shooting method (Algorithm 1).

Step 3. Let $N = k$ and obtain $u^{(N)}(t)$ from (35).

Step 4. Obtain $J^{(N)}$ from (36).

Step 5. If $\left|\frac{J^{(N)}-J^{(N-1)}}{J^{(N)}}\right| < \varepsilon$, then the procedure is complete, and go to the next step;

- else, let $k := k+1$, and back to Step 2.

Step 6. Stop the algorithm and consider the output $u^{(N)}(t)$ as the desired closed-loop suboptimal control law.

5 Numerical examples

Now, we are ready to present several examples for showing the efficiency of the SCRK method.

Example 2. Consider the delay system

$$\begin{cases} \dot{x} = x(t) + u(t) + x(t-1), & t \geq 0, \\ x(t) = 1, & -1 \leq t \leq 0, \end{cases} \tag{37}$$

to minimize this quadratic cost functional

$$J = \frac{3}{2}x^2(2) + \frac{1}{2}\int_0^2 u^2(t)dt. \tag{38}$$

It follows from [4] that the exact solution for $u(t)$ is

$$u^*(t) = \begin{cases} \delta(e^{2-t} + (1-t)e^{1-t}), & 0 \leq t \leq 1, \\ \delta e^{2-t}, & 1 \leq t \leq 2, \end{cases} \tag{39}$$

and that $J^* = 3.1017$, where $\delta = -0.3932$. According to (37) and (38), we have $Q = 0,\ R = 1,\ Q_f = 3,\ A = 1,\ B = 1$, and $A_1 = 1$. Hence, (28) can be rewritten as

$$\begin{cases} \dot{p}(t) + 2p(t) - p^2(t) = 0, \\ p(2) = 3, \end{cases} \tag{40}$$

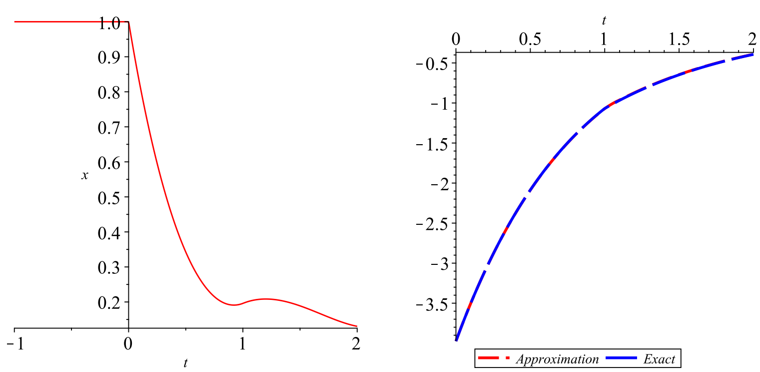
which has the unique solution

$$p(t) = \frac{6e^{4-2t}}{2 - 3(1 - e^{4-2t})}. \tag{41}$$

For the first time, Banks and Burns [4] proposed a numerical method to solve this problem based on averaging approximations. Then Pananisamy and Rao [38] solved it by using the Walsh functions. After that, Mirhosseini-Alizamini, the second author, and Heydari [32] used the variational iteration method. Furthermore, Jajarmi and Hajipour [20] employed a finite difference method for solving this problem. We apply our proposed method according to Algorithm 2 to this example. Comparison results of the optimal values of J obtained by our proposed technique and other mentioned methods are listed in Table 2. The curves depicted from the obtained approximations for the state and control variables of problems (37) and (38) are shown in Figure 6.

Table 2: Value of cost functional for various methods in Example 2

Method	J
Banks and Burns [4]	3.0833
Pananismay and Rao [38]	3.0879
Mirhosseini-Alizamini, Effati, and Heydari [32]	3.1091
Jajarmi and Hajipour [20]	3.101717
Proposed SCRK method	3.101667
Optimal cost J^*	3.1017



(a) State variable $x(t)$ (b) Control variable $u(t)$

Figure 1: Simulated curves of (a) state variable and (b) approximation and exact values of control variable for Example 2

Now, we give another example.

Example 3. Consider the time-delay system

$$\begin{cases} \dot{x} = u(t) - x(t-1), & 0 \leq t \leq 1, \\ x(t) = 1, & -1 \leq t \leq 0, \end{cases} \tag{42}$$

to minimize this quadratic cost functional

$$J = \int_0^1 [\frac{1}{2}x^2(t) + \frac{1}{2}u^2(t)]dt. \tag{43}$$

Now, our aim is to obtain the optimal control, $u(t)$, subject to (42) that minimizes (43). Moreover, the Riccati equation for this example is

$$\begin{cases} \dot{p}(t) - p^2(t) + 1 = 0, \\ p(1) = 0, \end{cases} \tag{44}$$

and has the unique solution

$$p(t) = -\tanh(t-1) \tag{45}$$

The exact solutions for $u(t)$ and $x(t)$ are, respectively, obtained as follows:

$$u^*(t) = 1 + \frac{1}{\cosh(1)}(\sinh(t-1) - \cosh(t)), \tag{46}$$

$$x^*(t) = \frac{1}{\cosh(1)}(\cosh(t-1) - \sinh(t)). \tag{47}$$

Moreover, it follows from [33] that the optimal value of cost functional is $J^* = 0.1480542786$. It can be shown that the approximate value of the cost functional calculated by the proposed SCRK method is equal to $J = 0.1480542988$. It is clear that the approximate value of J is very close to the optimal value. Also, we depict the simulation curves of the trajectory of $x(t)$, control variable $u(t)$, and their exact values in Figure 2.

For the first time, Eller, Aggarwal, and Banks [10] presented the next example and then studied by other authors in [23, 37, 9, 40].

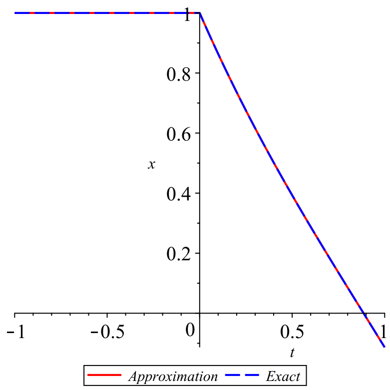
Example 4. Consider the linear time-varying delay system

$$\begin{cases} \dot{x} = x(t) + u(t) + x(t-1), & 0 \leq t \leq 2, \\ x(t) = 1, & -1 \leq t \leq 0, \end{cases} \tag{48}$$

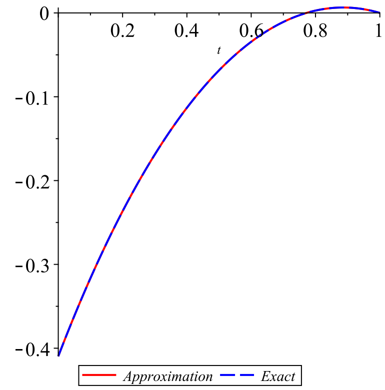
to minimize this quadratic functional

$$J = \int_0^2 [x^2(t) + u^2(t)]dt. \tag{49}$$

Therefore, the Riccati equation for this example is



(a) State variable $x(t)$



(b) Control variable $u(t)$

Figure 2: Approximation and exact values of state and control variables for Example 3

$$\begin{cases} \dot{p}(t) + 2p(t) - \frac{1}{2}p^2(t) + 2 = 0, \\ p(2) = 0, \end{cases} \tag{50}$$

and the unique solution for this Riccati equation is

$$p(t) = 2 - 2\sqrt{2}\tanh(\sqrt{2}t + \tanh^{-1}(\frac{\sqrt{2}}{2}) - 2\sqrt{2}). \tag{51}$$

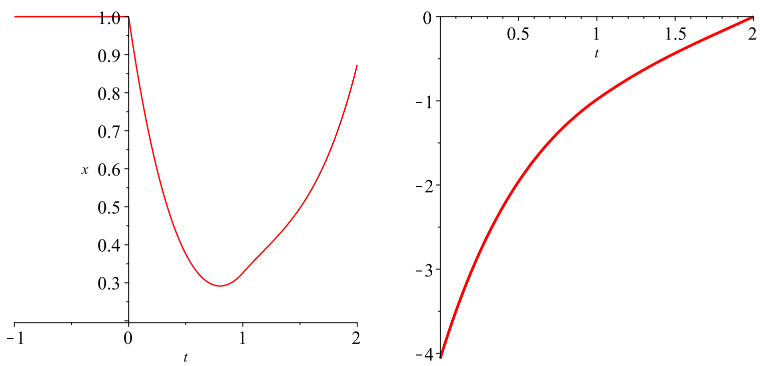
In Table 3, we compare the results of the suggested method with the reported results in [10, 23, 37, 9, 40, 21]. Figure 3 shows the approximate values of the state and control variables of the problem (48) and (49).

Table 3: Values of cost functional for various methods in Example 4

Method	J
Eller, Aggarwal, and Banks [10]	6.45
Dadebo and luus [9]	6.26775
Oh and Luus [37]	6.23711
Jamshidi and malek-Zavarei [23]	6.5
Santos and Sanchez-Diaz [40]	6.97
Jajarmi and Hajipour [21]	6.219615
Proposed SCRK method	6.200623

Example 5. In this example, we want to minimize the cost functional

$$J = 5x_1^2(2) + \frac{1}{2}\int_0^2 u^2(t)dt, \tag{52}$$



(a) State variable $x(t)$ (b) Control variable $u(t)$

Figure 3: Simulated curves of (a) state and (b) control variables for Example 4

with the following two-dimensional delay system:

$$\begin{cases} \dot{x}_1(t) = x_2(t), & 0 \leq t \leq 2, \\ \dot{x}_2(t) = -x_1(t) - x_2(t-1) + u(t), & 0 \leq t \leq 2, \\ x_1(0) = 10, \qquad x_2(0) = 0, & -1 \leq t \leq 0. \end{cases} \tag{53}$$

Now, our aim is to obtain the optimal control $u^*(t)$ subject to (53) that minimizes (52). It follows from [4] that this problem has the exact solution

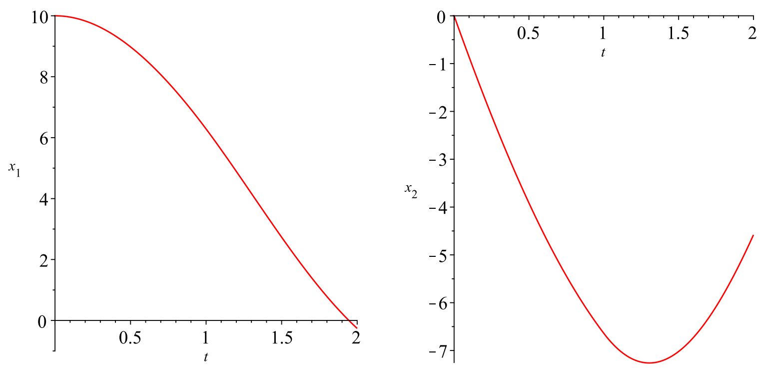
$$u^*(t) = \begin{cases} \delta \sin(2-t) + \frac{\delta}{2}(1-t)\sin(t-1), & 0 \leq t \leq 1, \\ \delta \sin(2-t), & 1 \leq t \leq 2, \end{cases} \tag{54}$$

in which the optimal cost is $J^* = 3.3991$ and $\delta = 2.5599$. In this two-dimensional example, we have $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $Q_f = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$, and $R = 1$.

Thus, instead of the Riccati equation, we have a system consisting of four equations and four variables. After applying the proposed method to this example, we obtained the minimum value of $J = 3.3993$. In Table 4, the comparison of the result obtained with our proposed method and the result based on the techniques presented in [4, 28, 27, 15, 32] is shown. Also, Figures 3 and 5 show the corresponding state trajectories of $x_1(t)$, $x_2(t)$ and control variable $u(t)$, respectively.

Table 4: Cost functional values of various methods for Example 5

Method	J
Banks and Burns [4]	3.2587
Lee [28]	3.4827
Khellat [27]	3.43254
Haddadi, Ordokhani, and Razzaghi [15]	3.21663
Mirhosseini-Alizamini, Effati, and Heydari [32]	3.3991
Proposed SCRK method	3.3993



(a) State variable $x_1(t)$ (b) State variable $x_2(t)$

Figure 4: Simulated curves of state variables for Example 5

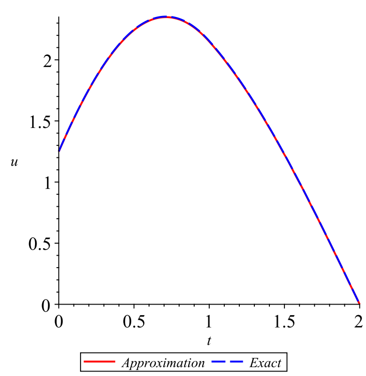


Figure 5: Control variable $u(t)$ for Example 5

6 Conclusion

We employed The CRK method to solve a class of time-delay OCPs with delay in the state variable and with quadratic cost functional in this paper. At first, by employing the Pontryagin maximum principle for time-delay systems, the delay OCP was converted to a sequence of TPBVPs that have both delays and advance terms. After that, by applying the CRK method together with the shooting method, we constructed two sequences in which the delay and advance terms are known. Then we showed that by establishing the continuity condition, these sequences converge to the exact solution of the problem. The numerical results were presented to illustrate the high accuracy and efficiency of our proposed approach. Further research can be done on the extension of the SCRK method for solving time-delay OCPs with time-dependent delays in the control and state variables.

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