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Hopf bifurcation analysis of a delayed five-neuron BAM neural network with two neurons in the X-layer

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Abstract

In this paper, a bidirectional associative memory (BAM) neural network, which consists of two neurons in the X-layer and three neurons in the Y-layer, with two time delays will be studied. We conclude that under some assumptions, Hopf bifurcation occurs when the sum of two delays passes through a critical value. A numerical example is presented to support our theoretical results.

Keywords: Neural network; Hopf bifurcation; Characteristic equation; Time delay.

1 Introduction

The attention of many scientists (eg., mathematicians, physicists, computer scientists, engineers and so on) have been attracted toward the dynamical characteristics of artificial neural networks since Hopfield constructed a simplified neural network (NN) model [1]. As time delays always occur in the signal transmission, Marcus and Westervelt proposed an NN model with delay [2]. Many dynamical behaviours such as periodic phenomenon, bifurcation and chaos have been discussed on these systems (e.g. [3, 4, 5, 6, 7, 8, 9, 2, 10]).

The bidirectional associative memory (BAM) networks were first introduced by Kasko (e.g. [11, 12]). The properties of periodic solutions are significant in many applications. It is well known that BAM NNs are able to store multiple patterns, but most of NNs have only one storage pattern or memory pattern. BAM NNs have practical applications in storing paired patterns or memories and have the ability of searching the desired patterns through both forward and backward directions.

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The delayed BAM neural network is described by the following system:

$$\begin{cases} \dot{x_i}(t) = -\mu_i x_i(t) + \sum_{j=1}^{m} c_{ji} f_i(y_j(t-\tau_{ji})) + I_i \quad (i=1,2,\ldots,n) \\ \dot{y_j}(t) = -\upsilon_j y_j(t) + \sum_{i=1}^{n} d_{ij} g_j(x_i(t-\sigma_{ij})) + J_j \ (j=1,2,\ldots,m) \end{cases} \tag{1}$$

where c_{ji} and d_{ij} are the connection weights through the neurons in two layers: the X-layer and the Y-layer. The stability of internal neuron processes on the X-layer and Y-layer are described by μ_i and υ_j, respectively. On the X-layer, the neurons whose states are denoted by $x_i(t)$ receive the input I_i and the inputs outputted by those neurons in the Y-layer via activation function f_i, while the similar process happens on the Y-layer. Also, τ_{ji} and σ_{ij} correspond to the finite time delays of neural processing and delivery of signals. For further details, we refer to [12, 11].

Since a great number of periodic solutions indicate multiple memory patterns, the study of Hopf bifurcation is very important for the design and application of BAM NNs. In fact, various local periodic solutions can arise from the different equilibrium points of BAM NNs by applying Hopf bifurcation technique. But the exhaustive analysis of the dynamics of such a large system is complicated, so some authors have studied the dynamical behaviours of simplified systems. For example, the simplified three-neuron, four-neuron, five-neuron and six-neuron BAM NNs with multiple delays have been studied in [13, 14, 6, 15, 8, 9, 16, 17, 10, 18, 19, 20]. It should be noted that in the above papers, the systems which have been considered, just consist of one neuron in the X-layer and other neurons in the Y-layer. This way of choosing the systems simplifies the analysis. Also, [20] studied the stability and local Hopf bifurcation of a five-neuron ring neural network with delays and self connection. However, there are many other forms of BAM NNs that have not been studied.

Motivated by the above, in this paper, we consider the following five-neuron BAM neural network. We should point out that in [21], the following system has been studied through center manifold theory, but here, we study this system according to the distribution of roots. In [14], a more simplified form of (2) with some assumptions has been considered, but they have stated some results of synchronization.

$$\begin{cases} \dot{x_1}(t) = -\mu_1 x_1(t) + c_{11} f_1(y_1(t-\tau_2)) + c_{21} f_1(y_2(t-\tau_2)) \\ \qquad\qquad +c_{31} f_1(y_3(t-\tau_2)) \\ \dot{x_2}(t) = -\mu_2 x_2(t) + c_{12} f_2(y_1(t-\tau_2)) + c_{22} f_2(y_2(t-\tau_2)) \\ \qquad\qquad +c_{32} f_2(y_3(t-\tau_2)) \\ \dot{y_1}(t) = -\upsilon_1 y_1(t) + d_{11} g_1(x_1(t-\tau_1)) + d_{21} g_1(x_2(t-\tau_1)) \\ \dot{y_2}(t) = -\upsilon_2 y_2(t) + d_{12} g_2(x_1(t-\tau_1)) + d_{22} g_2(x_2(t-\tau_1)) \\ \dot{y_3}(t) = -\upsilon_3 y_3(t) + d_{13} g_3(x_1(t-\tau_1)) + d_{23} g_3(x_2(t-\tau_1)) \end{cases} \tag{2}$$

where $\mu_i > 0 (i = 1, 2)$, $\upsilon_j > 0 (j = 1, 2, 3)$, $c_{j1}, c_{j2} (j = 1, 2, 3)$ and $d_{i1}, d_{i2}, d_{i3} (i = 1, 2)$ are real constants. The time delay from the X-layer to another Y-layer is τ_1, while the time delay from the Y-layer back to the X-layer is τ_2, and there are two neurons in the X-layer and three neurons in the Y-layer. First, we take the sum of the delays $\tau = \tau_1 + \tau_2$ as parameter and we will show that the zero solution loses its stability and Hopf bifurcation occurs when τ passes through a critical value.

This paper is organized in four sections. In section 2, we will analyze the stability and Hopf bifurcation. To illustrate the results, numerical simulation is presented in section 3. Finally, in section 4, some main conclusions are stated.

2 Stability analysis and Hopf bifurcation

First, we need to explain some transformations stated in [21]. Letting $u_1(t) = x_1(t-\tau_1)$, $u_2(t) = x_2(t-\tau_1)$, $u_3(t) = y_1(t)$, $u_4(t) = y_2(t)$, $u_5(t) = y_3(t)$ and $\tau = \tau_1 + \tau_2$, system (2) can be rewritten as the following equivalent system:

$$\begin{cases} \dot{u_1}(t) = -\mu_1 u_1(t) + c_{11} f_1(u_3(t-\tau)) + c_{21} f_1(u_4(t-\tau)) \\ \qquad\quad +c_{31} f_1(u_5(t-\tau)) \\ \dot{u_2}(t) = -\mu_2 u_2(t) + c_{12} f_2(u_3(t-\tau)) + c_{22} f_2(u_4(t-\tau)) \\ \qquad\quad +c_{32} f_2(u_5(t-\tau)) \\ \dot{u_3}(t) = -\upsilon_1 u_3(t) + d_{11} g_1(u_1(t)) + d_{21} g_1(u_2(t)) \\ \dot{u_4}(t) = -\upsilon_2 u_4(t) + d_{12} g_2(u_1(t)) + d_{22} g_2(u_2(t)) \\ \dot{u_5}(t) = -\upsilon_3 u_5(t) + d_{13} g_3(u_1(t)) + d_{23} g_3(u_2(t)) \end{cases} \tag{3}$$

To establish the main results for system (3), it is necessary to make the following assumption:

$$\text{(H1) } f_i, g_j \in C^1,\ f_i(0) = g_j(0) = 0,\ (i = 1,2;\ j = 1,2,3).$$

Note that the above assumption is necessary for linearization. It is easily seen that the origin $(0,0,0,0,0)$ is an equilibrium point of (3). Under the hypothesis (H1), the linearization of (3) at $(0,0,0,0,0)$ is

$$\begin{cases} \dot{u_1}(t) = -\mu_1 u_1(t) + \alpha_{31} u_3(t-\tau) + \alpha_{41} u_4(t-\tau) + \alpha_{51} u_5(t-\tau) \\ \dot{u_2}(t) = -\mu_2 u_2(t) + \alpha_{32} u_3(t-\tau) + \alpha_{42} u_4(t-\tau) + \alpha_{52} u_5(t-\tau) \\ \dot{u_3}(t) = -\upsilon_1 u_3(t) + \alpha_{13} u_1(t) + \alpha_{23} u_2(t) \\ \dot{u_4}(t) = -\upsilon_2 u_4(t) + \alpha_{14} u_1(t) + \alpha_{24} u_2(t) \\ \dot{u_5}(t) = -\upsilon_3 u_5(t) + \alpha_{15} u_1(t) + \alpha_{25} u_2(t) \end{cases} \tag{4}$$

where $\alpha_{mi} = c_{ki} f_i'(0)$, $\alpha_{im} = d_{ik} g_k'(0)$ for $m = 3,4,5,\ k = m-2,\ i = 1,2.$ Then the associated characteristic equation of (4) is

$$det\begin{pmatrix} \lambda+\mu_1 & 0 & -\alpha_{31}e^{-\lambda\tau} & -\alpha_{41}e^{-\lambda\tau} & -\alpha_{51}e^{-\lambda\tau} \\ 0 & \lambda+\mu_2 & -\alpha_{32}e^{-\lambda\tau} & -\alpha_{42}e^{-\lambda\tau} & -\alpha_{52}e^{-\lambda\tau} \\ -\alpha_{13} & -\alpha_{23} & \lambda+\upsilon_1 & 0 & 0 \\ -\alpha_{14} & -\alpha_{24} & 0 & \lambda+\upsilon_2 & 0 \\ -\alpha_{15} & -\alpha_{25} & 0 & 0 & \lambda+\upsilon_3 \end{pmatrix} = 0,$$

i.e.,

$$\lambda^5 + a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e + (a_1\lambda^3 + b_1\lambda^2 + c_1\lambda + d_1)e^{-\lambda\tau} + (a_2\lambda + b_2)e^{-2\lambda\tau} = 0, \quad (5)$$

where
$a = \upsilon_1 + \upsilon_2 + \upsilon_3 + \mu_2 + \mu_1$,
$b = \upsilon_1\upsilon_2 + \mu_2\upsilon_1 + \mu_2\upsilon_2 + \upsilon_3\mu_1 + \upsilon_1\upsilon_3 + \upsilon_2\upsilon_3 + \upsilon_3\mu_2 + \upsilon_1\mu_1 + \upsilon_2\mu_1 + \mu_1\mu_2$,
$c = \mu_2\upsilon_1\upsilon_2 + \upsilon_1\upsilon_3\mu_1 + \upsilon_2\upsilon_3\mu_1 + \mu_2\upsilon_3\mu_1 + \upsilon_1\upsilon_2\upsilon_3 + \upsilon_1\upsilon_3\mu_2 + \upsilon_2\upsilon_3\mu_2 + \upsilon_1\upsilon_2\mu_1 + \mu_2\upsilon_1\mu_1 + \mu_2\upsilon_2\mu_1$,
$d = \mu_2\upsilon_1\upsilon_2\upsilon_3 + \mu_2\upsilon_1\upsilon_2\mu_1 + \upsilon_1\upsilon_2\upsilon_3\mu_1 + \mu_2\upsilon_1\upsilon_3\mu_1 + \mu_2\upsilon_2\upsilon_3\mu_1$,
$e = \upsilon_3\mu_1\mu_2\upsilon_1\upsilon_2$,
$a_1 = -\alpha_{52}\alpha_{25} - \alpha_{24}\alpha_{42} - \alpha_{32}\alpha_{23} - \alpha_{31}\alpha_{13} - \alpha_{41}\alpha_{14} - \alpha_{51}\alpha_{15}$,
$b_1 = -\alpha_{52}\alpha_{25}(\upsilon_1 + \upsilon_2 + \mu_1) - \alpha_{24}\alpha_{42}(\upsilon_1 + \upsilon_3 + \mu_1) - \alpha_{32}\alpha_{23}(\upsilon_3 + \upsilon_2 + \mu_1) - \alpha_{31}\alpha_{13}(\upsilon_3 + \upsilon_2 + \mu_2) - \alpha_{41}\alpha_{14}(\upsilon_1 + \upsilon_3 + \mu_2) - \alpha_{51}\alpha_{15}(\upsilon_1 + \upsilon_2 + \mu_2)$,
$c_1 = -\alpha_{52}\alpha_{25}(\upsilon_1\upsilon_2 + \mu_1\upsilon_2 + \mu_1\upsilon_1) - \alpha_{24}\alpha_{42}(\upsilon_1\upsilon_3 + \mu_1\upsilon_3 + \mu_1\upsilon_1) - \alpha_{32}\alpha_{23}(\upsilon_3\upsilon_2 + \mu_1\upsilon_2 + \mu_1\upsilon_3) - \alpha_{31}\alpha_{13}(\upsilon_3\upsilon_2 + \mu_2\upsilon_3 + \mu_2\upsilon_2) - \alpha_{41}\alpha_{14}(\upsilon_1\upsilon_3 + \mu_2\upsilon_3 + \mu_2\upsilon_1) - \alpha_{51}\alpha_{15}(\upsilon_1\upsilon_2 + \mu_2\upsilon_1 + \mu_2\upsilon_2)$,
$d_1 = -\alpha_{52}\alpha_{25}\mu_1\upsilon_1\upsilon_2 - \alpha_{24}\alpha_{42}\mu_1\upsilon_1\upsilon_3 - \alpha_{32}\alpha_{23}\mu_1\upsilon_2\upsilon_3 - \alpha_{31}\alpha_{13}\mu_2\upsilon_2\upsilon_3 - \alpha_{41}\alpha_{14}\mu_2\upsilon_3\upsilon_1 - \alpha_{51}\alpha_{15}\mu_2\upsilon_1\upsilon_2$,

$$\begin{aligned} a_2 = {} & \alpha_{31}\alpha_{13}\alpha_{52}\alpha_{25} + \alpha_{31}\alpha_{13}\alpha_{24}\alpha_{42} - \alpha_{31}\alpha_{14}\alpha_{42}\alpha_{23} - \\ & \alpha_{31}\alpha_{15}\alpha_{52}\alpha_{23} - \alpha_{41}\alpha_{13}\alpha_{32}\alpha_{24} + \alpha_{41}\alpha_{14}\alpha_{52}\alpha_{25} + \\ & \alpha_{41}\alpha_{14}\alpha_{23}\alpha_{32} - \alpha_{41}\alpha_{15}\alpha_{52}\alpha_{42} - \alpha_{51}\alpha_{13}\alpha_{32}\alpha_{25} - \\ & \alpha_{14}\alpha_{42}\alpha_{25}\alpha_{51} + \alpha_{51}\alpha_{15}\alpha_{42}\alpha_{24} + \alpha_{51}\alpha_{15}\alpha_{23}\alpha_{32}, \end{aligned}$$

$$\begin{aligned} b_2 = {} & \alpha_{31}\alpha_{13}\alpha_{52}\alpha_{25}\upsilon_2 + \alpha_{31}\alpha_{13}\alpha_{24}\alpha_{42}\upsilon_3 - \alpha_{31}\alpha_{14}\alpha_{42}\alpha_{23}\upsilon_3 \\ & - \alpha_{31}\alpha_{15}\alpha_{52}\alpha_{23}\upsilon_2 - \alpha_{41}\alpha_{13}\alpha_{32}\alpha_{24}\upsilon_3 + \alpha_{41}\alpha_{14}\alpha_{52}\alpha_{25}\upsilon_1 \\ & + \alpha_{41}\alpha_{14}\alpha_{23}\alpha_{32}\upsilon_3 - \alpha_{41}\alpha_{15}\alpha_{52}\alpha_{42}\upsilon_1 - \alpha_{51}\alpha_{13}\alpha_{32}\alpha_{25}\upsilon_2 \\ & - \alpha_{14}\alpha_{42}\alpha_{25}\alpha_{51}\upsilon_1 + \alpha_{51}\alpha_{15}\alpha_{42}\alpha_{24}\upsilon_1 + \alpha_{51}\alpha_{15}\alpha_{23}\alpha_{32}\upsilon_2. \end{aligned}$$

To study the distribution of the roots of (5), we make the following assumption: (If we assume that $a_2 = b_2 = 0$ instead of (H2), the results in this case can be obtained from [19] analogously.)

$$\text{(H2) } a_1 = b_1 = c_1 = d_1 = 0.$$

Then Eq. (5) reduces to

$$\lambda^5 + a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e + (a_2\lambda + b_2)e^{-2\lambda\tau} = 0. \tag{6}$$

Obviously, $i\omega(\omega > 0)$ is a root of Eq. (6) if and only if ω satisfies (the real and imaginary parts have been separated)

$$\begin{cases} -b_2 cos(2\omega\tau) - a_2\omega sin(2\omega\tau) = a\omega^4 - c\omega^2 + e, \\ -a_2\omega cos(2\omega\tau) + b_2 sin(2\omega\tau) = \omega^5 - b\omega^3 + d\omega. \end{cases} \tag{7}$$

Taking square on the both sides of the equations of (7) and summing them up, we obtain

$$\omega^{10} + (a^2 - 2b)\omega^8 + (b^2 + 2d - 2ac)\omega^6 + (c^2 + 2ae - 2bd)\omega^4 + (d^2 - 2ce - a_2^2)\omega^2 + e^2 - b_2^2 = 0. \tag{8}$$

Let $z = \omega^2$ and for convenience, denote

$$p = a^2 - 2b, q = b^2 + 2d - 2ac, r = c^2 + 2ae - 2bd, v = e^2 - b_2^2, s = d^2 - 2ce - a_2^2.$$

Then Eq. (8) becomes

$$z^5 + pz^4 + qz^3 + rz^2 + sz + v = 0. \tag{9}$$

Suppose that

$$h(z) = z^5 + pz^4 + qz^3 + rz^2 + sz + v.$$

The fact that Eq. (9) has positive roots is a necessary condition for the existence of pure imaginary roots of (6). The following four lemmas, which have been proved in [20], are going to be used to establish the distribution of positive real roots of Eq. (9). We should mention that the coefficients of Eq. (9) are different from those in Lemmas 2.1-2.4 in [20], but they have not changed the results of the lemmas. Hence, we can prove the following four lemmas analogously. So, we do not state the proofs.

Lemma 1. *If $v < 0$, then Eq.* (9) *has at least one positive root.*

Now, to study the distribution of positive roots of (9) when $v \geq 0$, consider the following equation that comes from $h'(z) = 0$:

$$5z^4 + 4pz^3 + 3qz^2 + 2rz + s = 0. \tag{10}$$

Substituting $z = y - \frac{p}{5}$ in Eq. (10), we have

$$y^4 + p_1 y^2 + q_1 y + r_1 = 0, \tag{11}$$

where $p_1 = -\frac{6}{25}p^2 + \frac{3}{5}q$, $q_1 = \frac{8}{125}p^3 + \frac{6}{25}pq + \frac{2}{5}r$, $r_1 = -\frac{3}{625}p^4 + \frac{3}{125}p^2 q - \frac{2}{25}pr + \frac{1}{5}s$. If $q_1 = 0$, then it is very easy to obtain the four roots of Eq. (11)

as follows:

$$y_1 = \sqrt{\frac{-p_1 + \sqrt{\Delta_0}}{2}},\ y_2 = -\sqrt{\frac{-p_1 + \sqrt{\Delta_0}}{2}},$$
$$y_3 = \sqrt{\frac{-p_1 - \sqrt{\Delta_0}}{2}},\ y_4 = -\sqrt{\frac{-p_1 - \sqrt{\Delta_0}}{2}}.$$

where $\Delta_0 = p_1^2 - 4r_1$.

Lemma 2. *Assume that $v \geq 0$ and $q_1 = 0$.*
(I) If $\Delta_0 < 0$, then Eq. (9) has no positive real roots.
(II) If $\Delta_0 \geq 0$, $p_1 \geq 0$ and $r_1 > 0$, then Eq. (9) has no positive real roots.
(III) If (I) and (II) are not satisfied, then Eq. (9) has positive real roots if and only if there exists at least one $z^* \in \{z_1, z_2, z_3, z_4\}$ such that $z^* > 0$ and $h(z^*) \leq 0$, where $z_i = y_i - \frac{p}{5}\,(i = 1, 2, 3, 4)$.

Denote $p_2 = -\frac{1}{3}p_1^2 - 4r_1$, $q_2 = -\frac{2}{27}p_1^3 + \frac{8}{3}p_1 r_1 - q_1^2$, $\Delta_1 = \frac{1}{27}p_2^3 + \frac{1}{4}q_2^2$,

$$s_* = \sqrt[3]{-\frac{q_2}{2} + \sqrt{\Delta_1}} + \sqrt[3]{-\frac{q_2}{2} - \sqrt{\Delta_1}} + \frac{1}{3}p_1,\ \Delta_2 = -s_* - p_1 + \frac{2q_1}{\sqrt{s_* - p_1}}$$

and $\Delta_3 = -s_* - p_1 - \frac{2q_1}{\sqrt{s_* - p_1}}$.

Lemma 3. *Suppose that $v \geq 0$, $q_1 \neq 0$ and $s_* > p_1$.*
(I) If $\Delta_2 < 0$ and $\Delta_3 < 0$, then Eq. (9) has no positive real roots.
(II) If (I) is not satisfied, then Eq. (9) has positive real roots if and only if there exists at least one $z^* \in \{z_1, z_2, z_3, z_4\}$ such that $z^* > 0$ and $h(z^*) \leq 0$, where $y_1 = \frac{-\sqrt{s_* - p_1} + \sqrt{\Delta_2}}{2}$, $y_2 = \frac{-\sqrt{s_* - p_1} - \sqrt{\Delta_2}}{2}$, $y_3 = \frac{\sqrt{s_* - p_1} + \sqrt{\Delta_3}}{2}$, $y_4 = \frac{\sqrt{s_* - p_1} - \sqrt{\Delta_3}}{2}$ and $z_i = y_i - \frac{p}{5}\,(i = 1, 2, 3, 4)$.

Lemma 4. *Assume that $v \geq 0$, $q_1 \neq 0$ and $s_* < p_1$, then Eq. (9) has positive real roots if and only if $\frac{q_1^2}{4(p_1 - s_*)^2} + \frac{s_*}{2} = 0$, $\bar{z} > 0$ and $h(\bar{z}) \leq 0$, where $\bar{z} = \frac{q_1}{2(p_1 - s_*)} - \frac{1}{5}p$.*

Suppose that Eq. (9) has positive roots and without loss of generality, we assume that it has five positive roots, denoted by z_k^*, $k = 1, 2, 3, 4, 5$. Then Eq. (8) has five positive roots $\omega_k = \sqrt{z_k^*}$, $k = 1, 2, 3, 4, 5$.

By Eq. (7), we have:

$$cos(2\omega_k \tau) = \frac{a_2\omega_k^6 + (ab_2 - a_2 b)\omega_k^4 + (da_2 - cb_2)\omega_k^2 + eb_2}{-b_2^2 - a_2^2\omega_k^2},$$

$$sin(2\omega_k \tau) = \frac{(b_2 - aa_2)\omega_k^5 + (ca_2 - bb_2)\omega_k^3 + (db_2 - ea_2)\omega_k}{a_2^2\omega_k^2 + b_2^2}.$$

Thus, we get the corresponding $\tau_k^{(j)} > 0$ such that the characteristic equation (6) has purely imaginary roots.

$$\tau_k^{(j)} = \frac{1}{2\omega_k}[cos^{-1}(-\frac{a_2\omega_k^6 + (ab_2 - a_2b)\omega_k^4 + (da_2 - cb_2)\omega_k^2 + eb_2}{b_2^2 + a_2^2\omega_k^2}) + 2j\pi], \tag{12}$$

where $k = 1, 2, 3, 4, 5$ and $j = 0, 1, 2, \ldots$, then $\pm i\omega_k$ is a pair of purely imaginary roots of Eq. (6) with $\tau = \tau_k^{(j)}$. Clearly, the sequence $\{\tau_k^{(j)}\}_{j=0}^{+\infty}$ is increasing, and $lim_{j\to+\infty}\tau_k^{(j)} = +\infty$, $k = 1, 2, 3, 4, 5$.

Therefore, we can define

$$\tau_0 = \tau_{k_0}^{(0)} = min_{k\in\{1,\ldots,5\}}\{\tau_k^{(0)}\},\ \omega_0 = \omega_{k_0},\ z_0 = z_{k_0}^*. \tag{13}$$

For convenience, we make the following hypotheses:

(H3) $a > 0$, $ab - c > 0$, $c(ab - c) + a(e + b_2 - a(d + a_2)) > 0$, $e + b_2 > 0$
$(ab - c)[c(d + a_2) - b(e + b_2)] - [a(d + a_2) - e - b_2]^2 > 0$.

We also need the following result from Ruan and Wei [22].

Lemma 5. *Consider the exponential polynomial*

$$\begin{aligned} P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \ldots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &+ [p_1^{(1)}\lambda^{n-1} + \ldots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} + \ldots \\ &+ [p_1^{(m)}\lambda^{n-1} + \ldots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m}, \end{aligned}$$

where $\tau_i \geq 0\,(i = 1, 2, \ldots, m)$ *and* $p_j^{(i)}\,(i = 0, 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ *are constants. As* $(\tau_1, \tau_2, \ldots, \tau_m)$ *vary, the sum of the orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *on the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

Proof. See [22]. □

Using Lemmas 1-5, we can easily obtain the following results on the distribution of roots of the Eq. (6).

Lemma 6. *Assume that (H3) holds.*
(I) If one of the following conditions holds:
(a) $v < 0$;
(b) $v \geq 0$, $q_1 = 0$, $\Delta_0 \geq 0$ *and* $p_1 < 0$ *or* $r_1 \leq 0$ *and there exists* $z^* \in \{z_1, z_2, z_3, z_4\}$ *such that* $z^* > 0$ *and* $h(z^*) \leq 0$;
(c) $v \geq 0$, $q_1 \neq 0$, $s_* > p_1$, $\Delta_2 \geq 0$ *or* $\Delta_3 \geq 0$ *and there exists* $z^* \in \{z_1, z_2, z_3, z_4\}$ *such that* $z^* > 0$ *and* $h(z^*) \leq 0$;
(d) $v \geq 0$, $q_1 \neq 0$, $s_* < p_1$, $\frac{q_1^2}{4(p_1 - s_*)^2} + \frac{1}{2}s_* = 0$, $\bar{z} > 0$ *and* $h(\bar{z}) \leq 0$,
then all roots of Eq. (6) *have negative real parts when* $\tau \in [0, \tau_0)$.

(II) If none of the conditions (a)-(d) of (I) is satisfied, then all roots of Eq. (6) *have negative real parts for all* $\tau \geq 0$.

Proof. When $\tau = 0$, Eq. (6) becomes

$$\lambda^5 + a\lambda^4 + b\lambda^3 + c\lambda^2 + (d + a_2)\lambda + e + b_2 = 0. \tag{14}$$

By the Routh-Hurwitz criterion, all roots of Eq. (14) have negative real parts if and only if (H3) holds. From Lemmas 1-4, we know that if (a)-(d) of (I) are not satisfied, then Eq. (6) has no roots with zero real part for all $\tau \geq 0$. If one of (a)-(d) holds, when $\tau \neq \tau_k^{(j)}$, $k = 1, \ldots, 5$; $j = 0, 1, \ldots$, Eq. (6) has no roots with zero real part and τ_0 is the minimum value of τ so that Eq. (6) has purely imaginary roots. Applying Lemma 5, we obtain the conclusion of the lemma. □

Let

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau) \tag{15}$$

be the root of Eq. (6) satisfying $\alpha(\tau_0) = 0$, $\omega(\tau_0) = \omega_0$. Then we have the following lemma:

Lemma 7. *Suppose that* $z_0 = \omega_0^2$, $h'(z_0) \neq 0$ *and* $a_2\omega_0 \neq 0$ *(or* $b_2 \neq 0$*). Then, at* $\tau = \tau_0$, $\pm i\omega_0$ *is a pair of simple purely imaginary roots of Eq.* (6)*. Moreover,*

$$\frac{dRe(\lambda(\tau_0))}{d\tau} \neq 0,$$

also, $\frac{dRe(\lambda(\tau_0))}{d\tau}$ *and* $h'(z_0)$ *have the same sign.*

Proof. Differentiating Eq. (6) with respect to τ, we can easily obtain:

$$[\frac{d\lambda}{d\tau}]^{-1} = \frac{(5\lambda^4 + 4a\lambda^3 + 3b\lambda^2 + 2c\lambda + d)e^{2\lambda\tau} + a_2}{2a_2\lambda^2 + 2b_2\lambda} - \frac{\tau}{\lambda}.$$

Then, we get

$$[\frac{dRe(\lambda(\tau_0))}{d\tau}]^{-1} = \frac{z_0}{K}h'(z_0),$$

where $K = 4a_2^2\omega_0^4 + 4b_2^2\omega_0^2$. Thus, we obtain

$$sign\{\frac{dRe(\lambda(\tau_0))}{d\tau}\} = sign\{[\frac{dRe(\lambda(\tau_0))}{d\tau}]^{-1}\} = sign\{\frac{z_0}{K}h'(z_0)\} \neq 0.$$

Since $K, z_0 > 0$, we conclude that the sign of $\frac{dRe(\lambda(\tau_0))}{d\tau}$ is determined by the sign of $h'(z_0)$. □

Now, we state the main theorem:

Theorem 1. *Suppose that (H1), (H2) and (H3) hold.*
(I) If the conditions (a)-(d) of Lemma 6 are all not satisfied, then the zero

solution of system (3) *is asymptotically stable for all* $\tau \geq 0$.
(II) If one of the conditions (a)-(d) of Lemma 6 is satisfied, then the zero solution of system (3) *is asymptotically stable for all* $\tau \in [0, \tau_0)$.
(III) If all the conditions as stated in (II) hold and $h'(z_0) \neq 0$, *then system* (3) *undergoes a Hopf bifurcation at the zero solution as* τ *passes through* τ_0.

Proof. By applying Lemmas 6, 7 and bifurcation theory, all parts can be easily proved. □

(For more information about bifurcation theory and Hopf bifurcation, see [23].)

3 Numerical simulation

In this section, we give a numerical simulation to support our theoretical analysis. We consider the following system

$$\begin{cases} \dot{x_1}(t) = -0.5x_1(t) + tanh(y_1(t-\tau_2)) \\ \qquad\quad -tanh(y_2(t-\tau_2)) + 2tanh(y_3(t-\tau_2)) \\ \dot{x_2}(t) = -x_2(t) + tanh(y_1(t-\tau_2)) \\ \qquad\quad +tanh(y_2(t-\tau_2)) + tanh(y_3(t-\tau_2)) \\ \dot{y_1}(t) = -2y_1(t) + tanh(x_1(t-\tau_1)) - tanh(x_2(t-\tau_1)) \\ \dot{y_2}(t) = -0.5y_2(t) + tanh(x_1(t-\tau_1)) + tanh(x_2(t-\tau_1)) \\ \dot{y_3}(t) = -0.5y_3(t) + tanh(x_1(t-\tau_1)) - tanh(x_2(t-\tau_1)) \end{cases} \tag{16}$$

which has $(0,0,0,0,0)$ as an equilibrium point. From section 2, by (5) and (8), we can compute $p = 9$, $q = 27.375$, $r = 32.3125$, $s = -24.43359375$ and $v = -0.984375$. Then Eq. (9) has a unique positive real root $z_0 = 0.5308$. Its easy to show that $\tau_0 = 0.885959203$, $h'(z_0) > 0$ and $sign\{\frac{d(Re\lambda(\tau_0))}{d\tau}\} = 1$. Here, we have chosen $\tau_1 = 0.3$ and $\tau_2 = 0.4$, Figure 1 shows that the origin is asymptotically stable. When τ passes through the critical value τ_0, a Hopf bifurcation occurs and a family of stable periodic solutions bifurcates from the origin. In Figure 2, the bifurcating periodic solutions are presented by choosing $\tau_1 = 0.4$ and $\tau_2 = 0.5$.

4 Conclusions

In this paper, we discussed the dynamics of a class of BAM neural network with two neurons in the X-layer, three neurons in the Y-layer and two time delays. We have proved that the zero solution loses its stability and Hopf bifurcation occurs. In fact, a family of periodic solutions bifurcate from the zero solution when τ passes through a critical value. Also, in the main

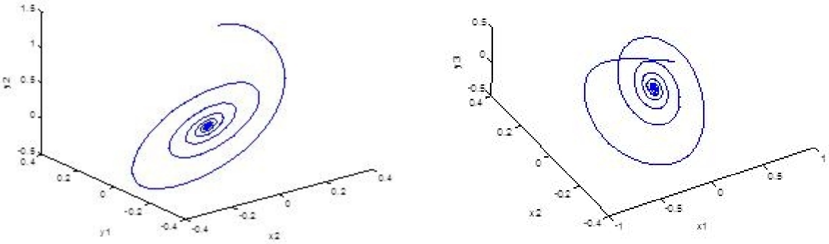

Figure 1: When $\tau_1 = 0.3$ and $\tau_2 = 0.4$, the origin is asymptotically stable.

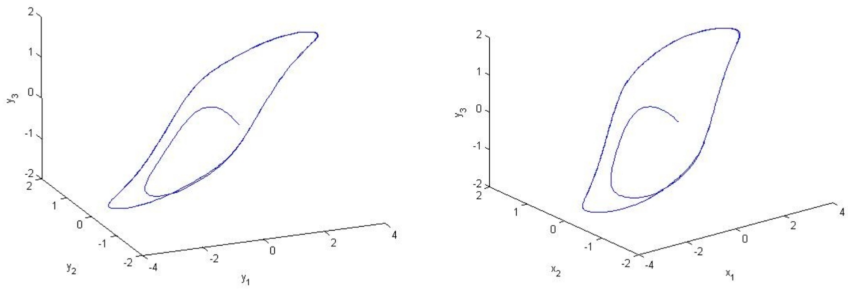

Figure 2: When $\tau_1 = 0.4$ and $\tau_2 = 0.5$, a family of periodic solutions bifurcates from the origin.

theorem, we resulted asymptotically stability of the zero solution in system (3) under some conditions. Finally, the results have been illustrated through numerical simulations.

At the end, for further research, we would like to point out that we discussed the dynamics of a special class of BAM neural network, but the complexity found in this case might be carried out to larger networks under some conditions.

References

1. Hopfield, J. J. *Neurons with graded response have collective computational properties like those of two-state neurons.* Proc. Nat. Acad. Sci. USA, 81 (1984), 3088–3092.

2. Marcus, C. M. and Westervelt, R. M. *Stability of analog neural network with delay.* Phys. Rev. A, 39 (1989), 347–359.

3. Cao, J. and Wang, J. *Absolute exponential stability of recurrent neural networks with Lipschitz continuous activation and time delays.* Neural Networks, 17 (2004), 379–390.

4. Cao, J. D. and Zhou, D. *Stability analysis of delayed cellular neural networks.* Neural Networks, 11 (1998), 1601–1605.

5. Guo, S. and Huang, L. *Hopf bifurcating periodic orbits in a ring of neurons with delays.* Physica D, 183 (2003), 19–44.

6. Hu, H. and Huang, L. *Stability and Hopf bifurcation analysis on a ring of four neurons with delays.* Applied Mathematics and Computation, 213 (2009), 587–599.

7. Liao, X., Wong, K. and Wu, Z. *Bifurcation analysis on a two-neuron system with distributed delays.* Physica D, 149 (2001), 123–141.

8. Li, Y., Chen, X. and Zhao, L. *Stability and existence of periodic solution to delayed Cohen-Grossberg BAM neural networks with impulses on time scales.* Neurocomputing, 72 (2009), 1621–1630.

9. Liu, X., Martin, R. R., Wu, M. and Tang, M. *Global exponential stability of bidirectional associative memory neural networks with time delays.* IEEE Trans. Neural Network, 19 (3) (2008), 397–407.

10. Xu, C., He, X. and Li, P. *Global existence of periodic solutions in a six-neuron BAM neural network model with discrete delays.* Neurocomputing, 74 (2011), 3257–3267.

11. Kosko, B. *Adaptive bidirectional associative memories.* Appl. Opt., 26 (1987), 4947–4960.

12. Gopalsamy, K. and He, X. *Delay-independent stability in bi-directional associative memory networks.* IEEE Trans. Neural Networks, 5 (1994), 998–1002.

13. Cao, J. and Xiao, M. *Stability and Hopf bifurcation in a simplified BAM neural network with two time delays.* IEEE Transaction on Neural Networks, 18 (2007), 416–430.

14. Ge, J. and Xu, J. *Synchronization and synchronized periodic solution in a simplified five-neuron BAM neural network with delays.* Neurocomputing, 74 (2011), 993–999.

15. Li, C., Liao, X. and Zhang, R. *Delay-dependent exponential stability analysis of bi-directional associative memory neural networks with time delay: an LMT approach.* Chaos Solitons Fractals, 24 (2005), 1119–1134.

16. Song, Y., Han, M. and Wei, J. *Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays.* Physica D, 200 (2005), 185–204.

17. Sun, C. and Han, M. *Global Hopf bifurcation analysis on a BAM neural network with delays.* Math. Comput. Modelling, 45 (2007), 61–67.

18. Xu, C. J., Tang, X. H. and Liao, M. X. *Frequency domain analysis for bifurcation in a simplified tri-neuron BAM network model with two delays.* Neural Networks, 23 (2010), 872–880.

19. Yang, Y. and Ye, J. *Stability and bifurcation in a simplified five-neuron BAM neural network with delays.* Chaos, Solitons and Fractals, 42 (2009), 2357–2363.

20. Zhang, T., Jiang, H. and Teng, Z. *On the distribution of the roots of a fifth degree exponential polynomial with application to a delayed neural network model.* Neurocomputing, 72 (2009), 1098–1104.

21. Javidmanesh, E., Afsharnezhad, Z. and Effati, S. *Existence and stability analysis of bifurcating periodic solutions in a delayed five-neuron BAM neural network model.* Nonlinear Dyn. 72 (2013), 149–164.

22. Ruan, S. and Wei, J. *On the zeros of transcendental functions with applications to stability of delay differential equations with two delays.* Dynamics of Continuous Discrete and Impulsive Systems Series A: Mathematics and Analytical 10 (2003), 863–874.

23. Hassard, B. D., Kazarinoff, N. D. and Wan, Y. H. *Theory and applications of Hopf bifurcation.* Cambridge University Press, Cambridge (1981).