

# Numerical study of the nonlinear Cauchy diffusion problem and Newell-Whitehead equation via cubic B-spline quasi-interpolation

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## Abstract

In this article, a numerical approximation to the solution of the Newell-Whitehead equation (NWE) and Cauchy problem of ill-posed non-linear diffusion equation have been studied. The presented scheme is obtained by using the derivative of the cubic B-spline quasi-interpolation (BSQI) to approximate the spatial derivative of the dependent variable and first order forward difference to approximate the time derivative of the dependent variable. Some numerical experiments are provided to illustrate the method. The results of numerical experiments are compared with analytical solutions. The main advantage of the scheme is that the algorithm is very simple and very easy to implement.

**Keywords:** B-spline quasi-interpolation; convection-diffusion equation; difference schemes.

## 1 Introduction

The use of spline function and its approximation plays an important role for the formation of stable numerical methods. Usually, a spline is a piecewise polynomial function defined in region, such that there exists a decomposition of  $D$  into subregions in each of which the function is a polynomial of some degree  $d$ . Also, the function, as a rule, is continuous in  $D$ , together with its derivatives of order up to  $(d - 1)$ . As the piecewise polynomial, spline, especially B-spline, have become a fundamental tool for numerical methods to get the solution of the differential equations [9, 13, 15, 16, 26]. The numerical

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solutions of partial differential equations by B-spline quasi-interpolation are introduced in [2, 5, 17, 20, 25].

Nonlinear equations play an important role in various filed of sciences. The world around us is nonlinear, so these kinds of equations arise naturally in a variety of models from theoretical physics, chemistry, and biology. The diffusion equation, one of these nonlinear equations, describes density dynamics in a material undergoing diffusion. It is also used to describe processes exhibiting diffusive-like behaviour, for instance the diffusion of alleles in a population in population genetics. It has also a great deal of application in different branches of sciences which have found a considerable amount of interest in recent years [1, 3, 4, 11, 14, 18, 23, 24].

Consider the nonlinear Cauchy diffusion equation as the following

$$Au = \phi(x, t), \quad x \in (a, b), \quad t > 0 \quad (1)$$

with initial condition

$$u(x, 0) = f(x), \quad x \in [a, b] \quad (2)$$

and boundary conditions of the form

$$u(a, t) = g_0(t), \quad u(b, t) = g_1(t), \quad t \geq 0 \quad (3)$$

$$A(u(x, t)) = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( (\kappa(t)u(x, t) + \omega(t)) \frac{\partial u}{\partial x} \right) \quad (4)$$

such that  $\kappa(t)u(x, t) + \omega(t)$  is positive [3, 11, 14, 23],  $a, b$  are constants,  $g_0(t), g_1(t), \kappa(t), \omega(t), f(x)$  and  $\phi(x, t)$  are known functions and  $\phi(x, t)$  be a smooth function.

The Newell-Whitehead equation models the interaction of the effect of the diffusion term with the nonlinear effect of the reaction term. For instance an equation to describe nearly 1D traveling-wave patterns is put forward in the form of a dispersive generalization of the Newell-Whitehead equation. The Newell-Whitehead equation is written as:

$$v_t = v_{xx} + \alpha v + \beta v^n, \quad x \in [a, b], \quad t \geq 0 \quad (5)$$

where  $\alpha, \beta$  are arbitrary constants,  $n$  is a positive integer and subscripts  $x$  and  $t$  denote differentiation.

Initial and boundary conditions are

$$v(x, 0) = f_1(x), \quad x \in [a, b] \quad (6)$$

$$v(a, t) = g_2(t), \quad v(b, t) = g_3(t), \quad t \geq 0 \quad (7)$$

where  $f_1(x), g_2(t), g_3(t)$  are known functions. The rest of this paper is organized as follows. In Section 2, we obtain the numerical schemes using cubic B-spline interpolation to solve the nonlinear Cauchy diffusion equation and

Newell-Whitehead equation. Some numerical examples are solved to assess the accuracy of the technique and the maximum absolute errors will be presented in Section 3. The conclusion appears in Section 4.

## 2 B-spline quasi-interpolant applied to the Cauchy problem and Newell-Whitehead equation

Assume that an interval  $I = [a, b]$  is given, denoted by  $S_d(X_n)$  the space of splines of degree  $d$  and class  $C^{d-1}$  on the uniform partition  $X_n = \{x_i = a + ih, i = 0, 1, \dots, n\}$  with meshlength  $h = (b - a)/n$ . Let a basis of  $S_d(X_n)$  be  $\{B_{j,d,r}, j = 1, 2, \dots, n + d\}$  where  $B_{j,d,r}$  is the  $j$ th B-spline of degree  $d$  for the knot sequence  $r := (r_i)_{i=-d}^{n+d}$  where  $r_{-d} = r_{-d+1} = \dots = r_{-1} = a$ ,  $r_n = r_{n+1} = \dots = r_{n+d} = b$  and  $r_i = x_i$   $0 \leq i \leq n$ . Since the cubic spline has become the most commonly used spline and we need the second order derivatives we use cubic B-spline quasi-interpolation in this paper.

From nonlinear differential equation (1) we have

$$u_t = \phi(x, t) + \kappa(t) (u_x^2 + uu_{xx}) + \omega(t)u_{xx} \quad (8)$$

and from discretizing this equation in time, we get

$$u_i^{k+1} = \tau \left( \phi(x_i, t_k) + \kappa(t_k) \left( (u_x)_i^k \right)^2 + u_i^k (u_{xx})_i^k \right) + \omega(t_k) (u_{xx})_i^k \quad (9)$$

where  $u_i^k, (u_x)_i^k, (u_{xx})_i^k$  are the approximation of the values  $u(x, t), u_x(x, t), u_{xx}(x, t)$  at  $(x_i, t_k), t_k = k\tau$ , and  $\tau$  is the time step. For fixed  $k$ , we can get the cubic quasi-interpolation as follows [19]:

$$Q_3 u^k = \sum_{j=1}^{n+3} \mu_j(u^k) B_{j,3,r}(x) \quad (10)$$

where  $u^k = u(x, t_k)$  and the coefficient functionals are respectively:

$$\begin{aligned} \mu_1(u^k) &= u_0^k, \\ \mu_2(u^k) &= \frac{1}{18} (7u_0^k + 18u_1^k - 9u_2^k + 2u_3^k) \\ \mu_j(u^k) &= \frac{1}{6} (-u_{j-3}^k + 8u_{j-2}^k - u_{j-1}^k), \quad 3 \leq j \leq n+1 \\ \mu_{n+2}(u^k) &= \frac{1}{18} (2u_{n-3}^k - 9u_{n-2}^k + 18u_{n-1}^k + 7u_n^k), \\ \mu_{n+3}(u^k) &= u_n^k. \end{aligned} \quad (11)$$

Using the de Boor-Cox formula [12, 21], the cubic B-spline basis  $B_{j,3,r}(x)$ , and his derivatives can be computed.

For  $u^k \in C^4(I)$  we have the error estimate [19] as

$$\|u^k - Q_3 u^k\|_\infty = O(h^4) \quad (12)$$

For approximate  $(u_x)_i^k, (u_{xx})_i^k$  by derivatives of the cubic B-spline quasi-interpolant (10) up to the order  $h^3$  we can evaluate the value of  $u^k$  at  $x_i$  by:

$$(Q_3 u_i^k)' = \sum_{j=1}^{n+3} \mu_j(u^k) B_j'(x_i), \quad (Q_3 u_i^k)'' = \sum_{j=1}^{n+3} \mu_j(u^k) B_j''(x_i). \quad (13)$$

We set

$$U^k = (u_0^k, u_1^k, \dots, u_n^k)^T, \quad U_x^k = ((u_0^k)', (u_1^k)', \dots, (u_n^k)'), \quad (14)$$

$$U_{xx}^k = ((u_0^k)'', (u_1^k)'', \dots, (u_n^k)''),$$

where

$$(u_i^k)' = (Q_3 u_i^k)', \quad (u_i^k)'' = (Q_3 u_i^k)'', \quad i = 0, 1, \dots, n. \quad (15)$$

By (15) we obtain

$$U_x^k = \frac{1}{h} D_1 U^k, \quad U_{xx}^k = \frac{1}{h^2} D_2 U^k \quad (16)$$

where  $D_1, D_2 \in \mathbb{R}^{(n+1) \times (n+1)}$  are obtain as follows:

$$D_1 = \begin{bmatrix} -11/6 & 3 & -3/2 & 1/3 & 0 & 0 & \dots & 0 & 0 \\ -1/3 & -1/2 & 1 & -1/6 & 0 & 0 & \dots & 0 & 0 \\ 1/12 & -2/3 & 0 & 2/3 & -1/12 & 0 & \dots & 0 & 0 \\ 0 & 1/12 & -2/3 & 0 & 2/3 & -1/12 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1/12 & -2/3 & 0 & 2/3 & -1/12 & 0 \\ 0 & 0 & \dots & 0 & 1/12 & -2/3 & 0 & 2/3 & -1/12 \\ 0 & 0 & \dots & 0 & 0 & 1/6 & -1 & 1/2 & 1/3 \\ 0 & 0 & \dots & 0 & 0 & -1/3 & 3/2 & -3 & 11/6 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 2 & -5 & 4 & -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1/6 & 5/3 & -3 & 5/3 & -1/6 & 0 & \dots & 0 & 0 \\ 0 & -1/6 & 5/3 & -3 & 5/3 & -1/6 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1/6 & 5/3 & -3 & 5/3 & -1/6 & 0 \\ 0 & 0 & \dots & 0 & -1/6 & 5/3 & -3 & 5/3 & -1/6 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 4 & -5 & 2 \end{bmatrix}$$

From the initial conditions (2) and boundary conditions (3), we can compute the numerical solution of (1) step by step using the scheme (9) and formulas (16). For implementation of this method from (2) we have  $U^0 = (f(x_0), f(x_1), \dots, f(x_n))^T$  and from (16), (9) and (3) the following algorithm is obtained

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 $U^0 \leftarrow (f(x_0), f(x_1), \dots, f(x_n))^T;$ 
for  $k = 0, 1, \dots, m$  do
   $U_x^k \leftarrow \frac{1}{h} D_1 U^k;$ 
   $U_{xx}^k \leftarrow \frac{1}{h^2} D_2 U^k;$ 
   $u_0^{k+1} \leftarrow g_0(t_{k+1});$ 
  for  $i = 1, 2, \dots, n - 1$  do
     $u_i^{k+1} \leftarrow \tau \left( f(x_i, t_k) + k(t_k) \left( \left( (U_x^k)_i \right)^2 + u_i^k (U_{xx}^k)_i \right) \right)$ 
     $+ \tau w(t_k) (U_{xx}^k)_i + u_i^k;$ 
  end
   $u_n^{k+1} \leftarrow g_1(t_{k+1});$ 
   $U^{k+1} \leftarrow (u_0^{k+1}, u_1^{k+1}, u_2^{k+1}, \dots, u_{n-1}^{k+1}, u_n^{k+1});$ 
end.

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Considering a maximum time like  $T$  that  $0 \leq t \leq T$  we have  $m = T/\tau$ .

Similarly from discretizing the Newell-Whitehead equation (5), we get

$$v_i^{k+1} = \tau \left( (v_{xx})_i^k + \alpha v_i^k + \beta (v_i^k)^n \right) + v_i^k \quad (17)$$

where  $v_i^k, (v_{xx})_i^k$  are the approximation of the values  $v(x, t), v_{xx}(x, t)$  at  $(x_i, t_k), t_k = k\tau$ , and  $\tau$  is the time step. For approximation of  $(v_{xx})_i^k$ , in relations (10), (11) and (13)-(16) we set  $v^k = v(x, t_k)$  and replacing  $v_i^k, V, V_{xx}, i = 0, 1, \dots, n$  respectively. Then from the initial conditions (6) and boundary conditions (7), we can compute the numerical solution of (5) step by step.

### 3 Numerical examples

In this section, two examples of the nonlinear Cauchy diffusion equation and Newell-Whitehead equation are considered and will be solved by B-spline quasi-interpolation method. To show the accuracy of the present method for our examples in comparison with the exact solutions, the amounts of errors is given in some mesh points and we report error norm which is defined by

$$|e|_1 = \frac{1}{n} \sum_{i=1}^{n-1} \frac{|u_i^{\text{exact}} - u_i^{\text{numerical}}|}{|u_i^{\text{exact}}|} \quad (18)$$

For the computational work we select the following examples from [7, 8, 10, 22].

**Example 1.** Let us consider the following nonlinear differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( \left( \frac{1}{6} e^{-t} u + (t+5) e^{-t} \right) \frac{\partial u}{\partial x} \right) = -\frac{7}{3} t - 9, \quad (x, t) \in [0, 1] \times [0, 1] \quad (19)$$

which has the exact solution  $u(x, t) = x^2 e^t + t$ . In (19)  $\phi(x, t) = -\frac{7}{3} t - 9$ ,  $\kappa(t) = \frac{1}{6} e^{-t}$ ,  $\omega(t) = (t+5) e^{-t}$ . In Table1, relative errors at different time levels are compared with the relative errors obtained by Zakeri et al. in [10]. In Figures 1 and 2 exact and numerical solutions are depicted.

**Example 2.** Relative errors at different time levels are compared with the relative errors obtained by Nourazar et al. [8]. for Eq. (5) with  $\alpha = 3, \beta = -4, n = 3, a = 0, b = 1$  and  $t = 1$  in Table 2. The exact solution of this example is  $v(x, t) = \sqrt{\frac{3}{4}} \frac{e^{\sqrt{6}x}}{e^{\sqrt{6}x} + e^{\left(\frac{\sqrt{6}}{2}x - \frac{9}{2}t\right)}}$ . The graph of the exact and numerical solution, are shown in Figures 3 and 4.

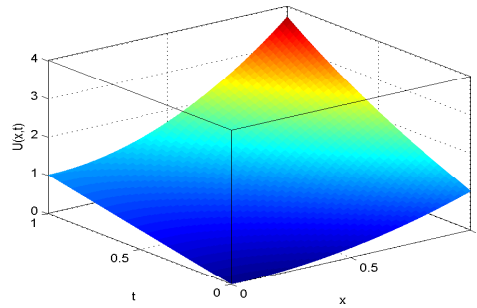
**Table 1:** Comparison of relative errors obtained from proposed method and method in [10].

$x$	Relative errors of proposed method			Relative errors obtained in [10]		
	$t = 0.25$	$t = 0.50$	$t = 0.75$	$t = 0.25$	$t = 0.50$	$t = 0.75$
0.2	8.0999e-08	6.6631e-08	6.9815e-08	2.7500e-08	4.7700e-09	3.4500e-08
0.4	1.0121e-07	9.3251e-08	1.0112e-07	4.8100e-07	4.3300e-07	2.8500e-07
0.6	8.1254e-08	8.1761e-08	9.1375e-08	2.2800e-06	2.2700e-06	1.9100e-06
0.8	4.4684e-08	4.7547e-08	5.4249e-08	6.7100e-06	6.7700e-06	5.8700e-06
$ e _1$	6.4079e-08	5.9977e-08	6.5621e-08	-	-	-

From the test examples, we can say that the BSQI scheme is feasible and the accuracy is better than the multi-quadric quasi-interpolation (MQQI) method [6]. Moreover, MQQI method has very close relation to the shape

**Table 2:** Comparison of errors of Example 2 with the errors obtained in [8]. ( $h = 0.02, \tau = 0.0001$ )

$x$	Relative errors of proposed method				Relative errors obtained in [8]		
	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 1$	$t = 0.1$	$t = 0.15$	$t = 0.2$
0.2	4.7533e-06	6.0414e-06	6.6041e-06	5.2440e-07	4.9987e-06	5.6384e-05	3.1193e-04
0.4	6.8110e-06	8.4592e-06	9.1422e-06	7.1097e-07	6.3997e-06	6.8760e-05	3.6460e-04
0.8	4.7680e-06	5.4354e-06	5.6195e-06	4.0217e-07	3.6819e-06	3.7324e-05	1.8700e-04
$ e _1$	4.8486e-06	5.8747e-06	6.2722e-06	4.7741e-07	-	-	-

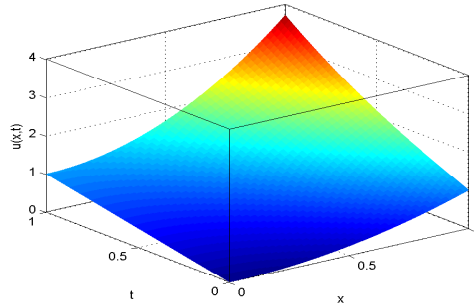


**Figure 1:** The exact solution of Example 1 for  $h = 0.02, \tau = 0.00001$

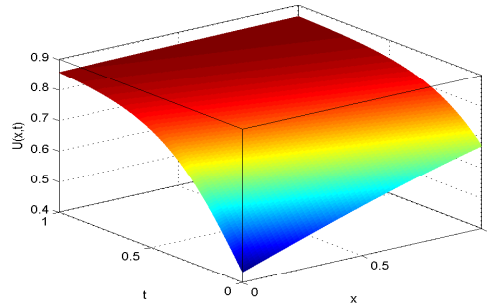
parameter  $c$  in MQ. In fact, the choice of the shape parameter is still a pending question. Furthermore, the MQQI is required to calculate derivatives of MQ quasi interpolant once for all, which is not easy to compute when  $h$  is small. Although the accuracy of BSQI is not better than that of other methods, we know that, at each time step, the complexity of BSQI is lower than theirs. The proposed method is an acceptable and valid scheme. Moreover, it can be implemented very easily.

### 4 Conclusions

In this article, we have applied the cubic B-spline quasi-interpolation method for solving the nonlinear Cauchy diffusion problem and Newell-Whitehead equation. The results have been compared with the exact solutions and demonstrated the good performance of the schemes. This method offers several advantages in reducing computational costs. On the other hand, this method is very simple to apply and to make an algorithm. Thus, this method may be reckoned as a simple and accurate solver for PDEs and it is worthy to note that this method can be utilized as an accurate algorithm to solve linear and nonlinear functional equations arising in physics and other



**Figure 2:** The numerical results of Example 1 for  $h = 0.02, \tau = 0.00001$



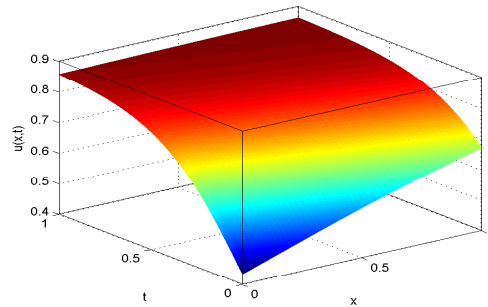
**Figure 3:** The exact solution of Example 2 for  $h = 0.02, \tau = 0.0001$

fields of applied mathematics. The computations associated with the examples in this article were performed using MATLAB R2013a.

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**Figure 4:** The numerical results of Example 2 for  $h = 0.02$ ,  $\tau = 0.0001$

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## مطالعه عددی مسأله غیر خطی کوشی و معادله نیویل-وایتهد با روش شبه درونیاب بی-اسپلاین مکعبی

حسین امینی خواه و جواد علوی

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**چکیده :** این مقاله به مطالعه یک تقریب عددی از معادله نیویل-وایتهد و معادله بدووضع انتشار کوشی می‌پردازد. در طرح ارائه شده از مشتق بی-اسپلاین شبه درونیاب برای تقریب مشتق متغیرهای وابسته و از تفاضل پیشرو مرتبه اول برای تقریب مشتق زمان استفاده می‌شود. مثال‌هایی برای تشریح روش بیان شده و نتایج عددی مثال‌ها با جواب‌های دقیق مقایسه شده‌اند. مزیت اصلی این روش در الگوریتم و پیاده سازی ساده آن است.

**کلمات کلیدی :** شبه درونیاب بی-اسپلاین مکعبی؛ معادله انتشار- انتقال؛ طرح تفاضلی.