Mashhad R. J. Math. Sci., Vol.1(1)(2008) 51-58.

A numerical solution for an inverse heat conduction problem

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Abstract

In this paper, we demonstrate the existence and uniqueness a semi-analytical solution of an inverse heat conduction problem (IHCP) in the form : $u_t = u_{xx}$ in the domain $D = \{(x,t)|\ 0 < x < 1,\ 0 < t \leq T\}$, $u(x,T) = f(x)$, $u(0,t) = g(t)$, and $u_x(0,t) = p(t)$, for any $0 \leq t \leq T$. Some numerical experiments are given in the final section.

Keywords and phrases: Inverse heat conduction problem, semi-analytical solution, finite difference method

AMS Subject Classification 2000: Primary 35R30; Secondary 18G10.

1 Introduction

The procedure to solve an IHCP is very important in determining unknown temperature histories and heat flux from known values in the body, which are usually

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measured as a function of space and time. Especially, a direct measurement of the heat flux or temperature in a boundary or initial time of hot body is almost impossible. Therefore, recent studies of IHCPs have been numerically treated and extended of multiple dimensions with the help of computing architecture. Some numerical and theoretical approaches to IHCPs are summarized in [4], [2]. Such a procedure using an exact solution are given by Burggraf [3]. It has been shown that, if an error is made in known boundary condition, then there will be some errors in unknown heat flux of other boundary. A lower bound of this error can be estimated by $\frac{1}{\sqrt{\Delta t}}\sinh(\frac{1}{\sqrt{\Delta t}})$. These results are consistent with earlier observation that small values of time Δt can produce large error in surface flux. In this paper, we apply a finite difference method of semi-implicit type for $\frac{\partial u}{\partial t}$ and use a parameter θ_M for driving a stable and convergent solution to the IHCPs.

Now, suppose that for any given t, $0 \le t \le T,\ u(x,t) \in C^4[0,1]$ and satisfying

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t),\ in\ \ D = \{(x,t)|\ 0 < x < 1,\ 0 < t \le T\}, && (1.1)\\
u(x,T) &= f(x), \qquad 0 \le x \le 1, && (1.2)\\
u(0,t) &= g(t), \qquad 0 \le t \le T, && (1.3)\\
u_x(0,t) &= p(t), \qquad 0 \le t \le T, && (1.4)\\
u_x(1,t) &= h(t), \qquad 0 \le t \le T, && (1.5)\\
u(x,0) &= \phi(x), \qquad 0 \le x \le 1, && (1.6)
\end{aligned}
$$

where $f(x),\ g(t)$, and $p(t)$ are piecewise-continuous known functions, T is a given positive constant number and $h(t),\ \phi(x)$, and $u(x,t)$ are unknown functions, which remain to be determined.

In the next section, we discrete the variable t and reduce $(1.1) - (1.4)$ to a system of linear, nonhomogenous second order differential equations. Stability and convergency of this method is studied in section 3. Some numerical result and discussion are given in section 4.

2 A Numerical Solution

In this section, we discrete the variable t to approximate the solution of (1.1) − (1.4).

Let $M \in \mathbf{N}$, $\Delta t_M = \frac{T}{M}$, and $t_i = i\Delta t$, for $i = 0,\ 1,\ ...,\ M$. For the solution u, we define $u_i(x) = u(x, t_i)$, for $i = 0,\ 1,\ ...,\ M$. Similarly for a given sequence of functions $\{u_i(x)|\ i = 0,\ 1,\ ...,\ M\}$ we use $\hat{u}_i(x)$ instead of the approximate of $u_i(x)$. Putting

$$\hat{u}_{i+1}(x) = \hat{u}_i(x) + (\theta_M \frac{\partial \hat{u}(x, t_i)}{\partial t} - (\theta_M - 1)\frac{\partial \hat{u}(x, t_{i+1})}{\partial t})\Delta t_M, \tag{2.1}$$

for $\theta_M \geq 0$, then by using (2.1) into (1.1) − (1.4) we obtain a system of linear nonhomogenous second order differential equations with given initial conditions in the form

$$\theta_M \Delta t_M \hat{u}_i''(x) + \hat{u}_i(x) = \hat{u}_{i+1}(x) + (\theta_M - 1)\Delta t_M \hat{u}_{i+1}''(x), \tag{2.2}$$

$$\hat{u}_i(0) = g(t_i) = g_i, \tag{2.3}$$

$$\hat{u}_i'(0) = p(t_i) = p_i, \tag{2.4}$$

for $i = 0,\ 1,\ ...,\ M-1$, where $\hat{u}_0(x)$ $(0 < x \leq 1)$ and $\hat{u}_i(1), i = 0,\ 1,\ ...,\ M-1$ are unknown.

Clearly $\hat{u}_M(x) = u_M(x) = f(x)$. Using these assumptions, the problem (2.2)–(2.4) has a solution of the following form

$$\hat{u}_i(x) = g_i \cos\frac{x}{\sqrt{\theta_M \Delta t_M}} + \frac{\hat{u}_i'(1) - \hat{V}_i'(1)}{\hat{W}_i'(1)}\ p_i \sin\frac{x}{\sqrt{\theta_M \Delta t_M}} + F_i(x), \tag{2.5}$$

for $i = 0,\ 1,\ ...,\ M-1$, where

$$\begin{aligned} F_i(x) &= \frac{1}{\theta_M\sqrt{\theta_M \Delta t_M}} \int_0^x \hat{u}_{i+1}(s) \sin\frac{x-s}{\sqrt{\theta_M \Delta t_M}}\ ds \\ &- \frac{\theta_M - 1}{\theta_M}\sqrt{\Delta t_M}\ p_{i+1} \sin\frac{x}{\sqrt{\theta_M \Delta t_M}} + \frac{\theta_M - 1}{\theta_M}\hat{u}_{i+1}(x) \\ &- \frac{\theta_M - 1}{\theta_M}\ g_{i+1} \cos\frac{x}{\sqrt{\theta_M \Delta t_M}},\quad i = 0,\ 1,\ ...,\ M-1, \end{aligned} \tag{2.6}$$

$\hat{V}_i$ and $\hat{W}_i$ are the solutions of the following problems, respectively,

$$\theta_M \Delta t_M \hat{V}_i''(x) + \hat{V}_i(x) = \hat{u}_{i+1}(x) + (\theta_M - 1)\Delta t_M \hat{u}_{i+1}''(x),$$

$$\begin{aligned}\hat{V}_i(0) &= g_i,\\ \hat{V}_i'(0) &= p_i, \qquad i = 0,\ 1,\ ...,\ M-1, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\theta_M \Delta t_M \hat{W}_i''(x) + \hat{W}_i(x) &= 0,\\ \hat{W}_i(0) &= 0,\\ \hat{W}_i'(0) &= 1, \qquad i = 0,\ 1,\ ...,\ M-1. \end{aligned} \tag{2.8}$$

Clearly, the heat flux $\hat{u}_x(1,t_i)$ may be obtained from (2.6) of the form

$$\hat{u}_x(1,t_i) = p_i\ \hat{W}_i'(1) + \hat{V}_i'(1), \qquad i = 0,\ 1,\ ...,\ M-1. \tag{2.9}$$

Now, for each $n \in \mathbf{N}$, if $\theta_M \Delta t_M \neq (n\pi)^{-2}$ and $f''(x)$ is a piecewise-continuous function in $[0,1]$, then the system of solutions (2.5) are unique [1].

The above result may be summarized in the following statement.

Theorem 2.1. *If, for each $n \in \mathbf{N}$, $\theta_M \Delta t_M \neq (n\pi)^{-2}$, and f'' is a piecewise-continuous function in $[0,1]$, then the system of differential equations (2.2)-(2.4) has a unique solution.*

Proof. See the analysis preceding the of above theorem.

3 Stability and convergency of solution

In the next theorem, the convergency of the solution (2.5) to the unique solution will be shown.

Theorem 3.1. *If θ_M, Δt_M and f satisfy the assumptions of Theorem 2.1 and $|\frac{\partial^2 u(x,t)}{\partial t^2}| \leq C < \infty$ for any $t \in [0,1]$, where C is a positive constant number and $\theta_M = \beta_M \Delta t^{-\alpha_M}$ such that $\beta_M > 0$, $1 + \frac{\ln(2\beta_M)}{\ln(\Delta t_M)} \leq \alpha_M \leq 1$ for any $M \geq 3$, then the solution of the system (2.2)-(2.4) is convergent to the unique solution of the problem (1.1)-(1.4), for any $0 \leq x \leq 1$.*

Proof. For all $i = 0,\ 1,\ ...,\ M-1$, we have

$$\theta_M \Delta t_M\ u_i''(x) +\ u_i(x) = u_{i+1}(x) + (\theta_M - 1)\Delta t_M\ u_{i+1}''(x)$$

$$- \quad \frac{1}{2}\Delta t_M^2(\theta_M^2 \ \frac{\partial^2 u(x,\xi_i)}{\partial t^2} - (\theta_M - 1)^2 \ \frac{\partial^2 u(x,\eta_i)}{\partial t^2}), \qquad (3.1)$$

where $t_i < \xi_i < t_i + \theta_M \Delta t_M$ and $t_{i+1} < \eta_i < t_{i+1} + (\theta_M - 1)\Delta t_M$.
Now, if we put

$$e_i(x) = \hat{u}_i(x) - u_i(x) \qquad for \quad i = \ 0, \ 1, \ ..., \ M, \qquad (3.2)$$

then $e_i(x)$ satisfies.

$$\begin{aligned} \theta_M \Delta t_M \ e_i''(x) + \ e_i(x) \ &= \ e_{i+1}(x) + (\theta_M - 1)\Delta t_M \ e_{i+1}''(x) \\ &- \ \frac{1}{2}\Delta t_M^2(\theta_M^2 \ \frac{\partial^2 u(x,\xi_i)}{\partial t^2} - (\theta_M - 1)^2 \ \frac{\partial^2 u(x,\eta_i)}{\partial t^2}), \ (3.3) \end{aligned}$$

and $e_i(0) = e_i'(0) = 0$, from which, we conclude that

$$\begin{aligned} e_i(x) \ &= \ \frac{1}{\theta_M\sqrt{\theta_M \Delta t_M}} \int_0^x e_{i+1}(s) \sin\frac{x-s}{\sqrt{\theta_M \Delta t_M}} ds + \frac{\theta_M - 1}{\theta_M} \ e_{i+1}(x) \\ &- \ \frac{1}{2}\theta_M^{-1/2}\Delta t_M^{3/2} \int_0^x (\theta_M^2 \frac{\partial^2 u(s,\xi_i)}{\partial t^2} - (\theta_M - 1)^2 \frac{\partial^2 u(s,\eta_i)}{\partial t^2}) \sin\frac{x-s}{\sqrt{\theta_M \Delta t_M}} ds \\ &= \ I_{i,1}(x) + I_{i,2}(x) + I_{i,3}(x), \qquad i = 0, \ 1, \ ..., \ M-1. \qquad (3.4) \end{aligned}$$

Clearly, the integrand in $I_{i,3}(x)$ denotes the truncation error and the other terms $I_{i,1}(x)$ and $I_{i,2}(x)$ show that, errors of initial and boundary data for the problem (1.1) how to propagate. In remaining of the proof, we consider two cases:
Case I. If

$$\begin{aligned} e_i(0) = e_i'(0) \ &= \ 0, \qquad i = 0, \ 1, \ ..., \ M-1, \\ e_M(x) \ &= \ 0, \qquad 0 < x < 1, \qquad (3.5) \end{aligned}$$

then clearly one may conclude that

$$|I_{i,3}(x)| \le C(\theta_M \Delta t_M)^{3/2}, \quad for \quad i = 0, \ 1, \ ..., \ M-1, \qquad (3.6)$$

where C is defined in Theorem 2.2. Thus, $e_{M-1}(x) = I_{M-1,3}(x)$ and one may show that

$$|e_{M-i}(x)| \ \le \ (\frac{1}{2}\theta_M^{-2}\Delta t_M^{-1} + 1 - \theta_M^{-1})|e_{M-i-1}(x)| + I_{M-i,3}(x)$$

$$\begin{aligned} &= \sum_{k=0}^{i-1}(1-\theta_M^{-1}+\frac{1}{2}\theta_M^{-2}\Delta t_M^{-1})^k |I_{M-i-k,3}(x)| \\ &\leq C(\theta_M \Delta t_M)^{(3/2)} \sum_{k=0}^{i-1}(\frac{1}{2}\theta_M^{-2}\Delta t_M^{-1}+1-\theta_M^{-1})^k, \quad i=1,\ 2,\ ...,\ M. \quad (3.7) \end{aligned}$$

Clearly, for any fixed value θ_M, limit $|e_{M-i}(x)|$ will not be zero, when Δt_M is vanished, which implies the divergency of the solution. If we choose $\theta_M = \beta_M \Delta t_M^{-\alpha_M}$ for $0 < \alpha_M \leq 1$ and $\beta_M \geq 0$, then for each $M \geq 3$, and α_M which satisfies in the following inequality

$$1 + \frac{\ln(2\beta_M)}{\ln \Delta t_M} \leq \alpha_M \leq 1,$$

we obtain that

$$|e_{M-i}(x)| \leq C_1 \beta_M^{3/2} \Delta t_M^{3/2(1-\alpha_M)}, \quad (3.8)$$

where

$$C_1 = C \sum_{k=0}^{i-1}(1-\theta_M^{-1}+\frac{1}{2}\theta_M^{-2}\Delta t_M^{-1})^k.$$

Consequently, if Δt_M tends to zero, then $|e_{M-i}(x)|$ is vanished and $\hat{u}_i(x)$ for $i = 1,\ 2,\ ...,\ M$, convergences to the exact unique solution of the problem (1.1)-(1.4).

Case 2. In this case, let us suppose

$$\begin{aligned} \hat{g}_i &= g_i + \epsilon_{i,1}, \\ \hat{p}_i &= p_i + \epsilon_{i,2}, \quad for \quad i = 0,\ 1,\ ...,\ M-1, \end{aligned}$$

and $\hat{f}(x) = f(x) + \epsilon(x)$, then by using (2.5), (2.6), (2.7) and (2.8), we obtain

$$\begin{aligned} e_i(x) &= \epsilon_{i,1} \cos \frac{x}{\sqrt{\theta_M \Delta t_M}} + \epsilon_{i,2} \sqrt{\theta_M \Delta t_M} \sin \frac{x}{\sqrt{\theta_M \Delta t_M}} \\ &+ \frac{1}{\theta_M \sqrt{\theta_M \Delta t_M}} \int_0^x e_{i+1}(s) \sin \frac{x-s}{\sqrt{\theta_M \Delta t_M}} ds \\ &- \frac{\theta_M - 1}{\theta_M} \sqrt{\Delta t_M}\ \epsilon_{i+1,2} \sin \frac{x}{\sqrt{\theta_M \Delta t_M}} \\ &- \frac{\theta_M - 1}{\theta_M} \sqrt{\Delta t_M}\ \epsilon_{i+1,1} \cos \frac{x}{\sqrt{\theta_M \Delta t_M}} + I_{i,3}(x), \quad (3.9) \end{aligned}$$

where $e_i(x)$ is a global error for $i = 0,\ 1,\ ...,\ M$, and $e_M(x) = \epsilon(x)$.
Now if $|\epsilon(x)|,\ |\epsilon_{i,1}|$, and $|\epsilon_{i,2}|$ tend to zero for each $i = 0,\ 1,\ ...,\ M$, then

$$|e_{M-i}(x)| \leq \sum_{k=0}^{i-1} (1 - \theta_M^{-1} + \frac{1}{2}\theta_M^{-2} \Delta t_M^{-1})^k\ |I_{M-i+k,3}(x)|,$$

and

$$|I_{M-i+k,3}(x)| \leq C\beta_M^{3/2}\ \Delta t_M^{3/2(1-\alpha_M)} \to 0.$$

Finally, $|e_{M-i}(x)|$ vanishes for all $i = 0,\ 1,\ ...,\ M$. ■

4 Numerical Examples

In this section we will present simulated cases to evaluate the capability of the proposed robust input estimation scheme.

Example 4.1. Assume that

$$\begin{aligned} f(x) &= x^2 + 2, \\ g(t) &= 2t, \\ p(t) &= 2, \\ T &= 1. \end{aligned}$$

Obviously, $u(x,t) = x^2 + 2t$ is an exact solution of the problem. Now, we use our numerical method to this problem. For $x = 1,\ \Delta t_M = 0.02,\ \alpha_M = 0.8$, and $\beta_M = 0.9$, the result are given in the following table.

t	$\hat{u}(1,t)$	$u(1,t)$	$\frac{\lvert u(1,t)-\hat{u}(1,t)\rvert}{\lvert u(1,t)\rvert}$	$u_x(1,t)$	$\hat{u}_x(1,t)$	$\frac{\lvert u_x(1,t)-\hat{u}_x(1,t)\rvert}{\lvert u_x(1,t)\rvert}$
0	0.993614	1	0.0063	2	1.98329	0.0083
0.2	1.38377	1.4	0.019	2	1.96382	0.018
0.4	1.77821	1.8	0.012	2	1.95272	0.023
0.6	2.1803	2.2	0.0089	2	1.95764	0.021
0.8	2.58875	2.6	0.0043	2	1.97603	0.011

Table 1. *Exact and estimate of the temperature and heat flux in the above problem*

One can see from the data in table 1, the relation errors generated through the computation show that the approximate and the exact solutions are vanished.

Example 4.2. Suppose that

$$\begin{aligned} f(x) &= x^3 + 6x, \\ g(t) &= 0, \\ p(t) &= 6t, \\ T &= 1. \end{aligned}$$

Clearly, the exact solution to this problem is $u(x,t) = x^3 + 6xt$.
Now, for $x = 1$, $\Delta t_M = \frac{1}{30}$, $\alpha_M = 0.9$, and $\beta_M = 5$, we obtain the following result given in table 2.

t	$\hat{u}(1,t)$	$u(1,t)$	$\frac{\lvert u(1,t)-\hat{u}(1,t)\rvert}{\lvert u(1,t)\rvert}$	$u_x(1,t)$	$\hat{u}_x(1,t)$	$\frac{\lvert u_x(1,t)-\hat{u}_x(1,t)\rvert}{\lvert u_x(1,t)\rvert}$
0	0.992943	1	0.007	3	2.9757	0.008
0.2	2.17963	2.2	0.009	4.2	4.13599	0.001
0.4	3.37199	3.4	0.008	5.4	5.31314	0.001
0.6	4.57445	4.6	0.005	6.6	6.52107	0.001
0.8	5.7853	5.8	0.002	7.8	7.75477	0.005

Table 2. *Exact and estimate of the temperature and heat flux in Example 4.2*

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