Iranian Journal of Numerical Analysis and Optimization
Vol. 14, No. 3, 2024, pp 900–937
https://doi.org/ 10.22067/ijnao.2024.87373.1412
https://ijnao.um.ac.ir/



Research Article

# A numerical computation for solving delay and neutral differential equations based on a new modification to the Legendre wavelet method

N.M. El-Shazly* and M.A. Ramadan

**Abstract**

The goal of this study is to use our suggested generalized Legendre wavelet method to solve delay and equations of neutral differential form with proportionate delays of different orders. Delay differential equations have some application in the mathematical and physical modelling of real-world problems such as human body control and multibody control systems, electric circuits, dynamical behavior of a system in fluid mechanics, chemical engineering, infectious diseases, bacteriophage infection's spread, population dynamics, epidemiology, physiology, immunology, and neural networks.

*Corresponding author

Received 23 March 2024; revised 24 May 2024; accepted 31 May 2024

Naglaa M. El-Shazly
Mathematics and Computer Science Department, Faculty of Science, Menoufia University, Menoufia, Egypt. e-mail: Naglaamoh1@yahoo.com

Mohamed A. Ramadan
Mathematics and Computer Science Department, Faculty of Science, Menoufia University, Menoufia, Egypt. e-mail: ramadanmohamed13@yahoo.com

**How to cite this article**
El-Shazly, N.M. and Ramadan, M.A., A numerical computation for solving delay and neutral differential equations based on a new modification to the Legendre wavelet method. *Iran. J. Numer. Anal. Optim.*, 2024; 14(3): 900-937. https://doi.org/10.22067/ijnao.2024.87373.1412

The use of orthonormal polynomials is the key advantage of this method because it reduces computational cost and runtime. Some examples are provided to demonstrate the effectiveness and accuracy of the suggested strategy. The method's accuracy is reported in terms of absolute errors. The numerical findings are compared to other numerical approaches in the literature, particularly the regular Legendre wavelets method, and show that the current method is quite effective in order to solve such sorts of differential equations.

**AMS subject classifications (2020):** Primary 34K40; Secondary 65L05, 40C05.

**Keywords:** Generalized Legendre wavelets; orthonormal polynomials delay differential equations; neutral differential equations; accuracy.

## 1 Introduction

Delay differential equations are important in the mathematical and physical modelling of real-world problems such as human body control and multi-body control systems, electric circuits, the dynamical behavior of a system in fluid mechanics, chemical engineering [18, 33], infectious diseases, bacteriophage infection's spread [9], population dynamics, epidemiology, physiology, immunology, neural networks, and the application of Legendre wavelet for solving differential pharmacology.

For solving nonlinear differential equations with proportional delays, there are several numerical techniques, like the Runge–Kutta–Fehlberg methods [25], Adomian decomposition method [8, 14, 26], Hermite wavelet-based method [32], Aboodh transformation method [1], power series method [5], decomposition method [34], differential transform method [22], Iterative variational approach [19], Pade's series-based approach and power method [35], spectral method [2], variable multistep methods [21], quasilinearization technique [28], polynomial least squares method [10], homotopy perturbation method [31], first kind Bessel's functions [39, 40], Legendre polynomials of shifted form [41, 44], and the first Boubaker polynomial approach [12]. Moreover, solving equations of nonlinear ordinary differential type using collocation methods with the use of Bessel polynomials are studied in [38, 42, 43]. In addition, numerous numerical approaches, such as the One-leg-method [36] and Chebyshev polynomials [30], have been employed to approximate the solutions of neutral differential equations. Gümgüm, Özdek,. and Özaltun [16, 17] have presented Legendre wavelet solutions of both high order non-linear ordinary delay differential equations and neutral differential equations with proportional delays. Nisar et al. [24] presented the efficient and significant solutions to a nonlinear fractional model. Zhang et al. [45] investigated the multiple solitons, lump solitons, and interaction with two stripe soliton solutions, for the fractional gCBS-BK equation. Amer and Olorode [3] presented a numerical evaluation of a novel slot-drill enhanced oil recovery technology for tight rocks.

In this paper, we use our suggested generalized Legendre wavelet approach ($GLWM$) [13] to solve delay and neutral differential equations with proportionate delays of different orders in the following form:

$$\sum_{p=0}^{3}\sum_{q=0}^{Q} R_{pq}(t)y^{(p)}(t-\eta_{pq}(t)) + \sum_{r=0}^{2}\sum_{s=0}^{r} S_{rs}(t)y^{(r)}(\delta_{rs}t)y^{(s)}(\gamma_{rs}t) = h(t), \quad Q \leq 3, \tag{1}$$

with the initial conditions

$$y^{(p)}(0) = \alpha_p \qquad p = 0\ ,\ 1\ ,\ 2, \tag{2}$$

where the given continuous functions $R_{pq}(t), S_{rs}(t), h\ (t)$ and the variable delays $\eta_{pq}(t)$ on $0 \leq t < 1$, $\delta_{rs}$ and $\gamma_{rs}$ constants are assigned to indicate proportional delays.

The mechanism of our suggested method is reducing computational cost and runtime with the use of orthonormal polynomials. We provide various examples to demonstrate the effectiveness and accuracy of the suggested strategy. We reported the method's accuracy in terms of absolute errors. The numerical findings are compared to other numerical approaches in the literature, particularly the regular Legendre wavelets method ($RLWM$), and manifest that the current method is quite effective in order to solve such sorts of differential equations.

## 2 Definitions and preliminaries

### 2.1 Legendre wavelet and its properties

Considering a single function "mother wavelet" $\psi(t)$, from which wavelets represent a family of functions by dilating and transforming this single function. This family of continuous wavelets [15] has the following form:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\,\psi\left(\frac{t-b}{a}\right), \qquad a,\ b\ \in R\ ,\ \ a \neq 0. \tag{3}$$

The Legendre wavelets on the interval $[0\ ,\ 1)$ is defined by

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}} L_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \ \leq\ t\ <\ \frac{\hat{n}+1}{2^k}\ ; \\ 0 & otherwise, \end{cases} \tag{4}$$

for which $k$ is positive integer, $n = 1\ ,\ 2\ , \ldots,\ 2^{k-1}$ and $\hat{n} = 2n-1$, the order of the Legendre polynomial is denoted by $m = 0\ ,\ 1\ ,\ 2\ ,\ \ldots\ ,\ M$ and the normalized time is denoted by $t$. The Legendre polynomials acquired in the above definition are defined as follows:

$$\begin{aligned} &L_0(t) = 1\ \ , \\ &L_1(t) = t\ \ , \\ &L_{m+1}(t) = \frac{2m+1}{m+1}\,t\,L_m(t) - \frac{m}{m+1}\,L_{m-1}(t), \qquad m = 1,\ 2,\ 3,\ \ldots, \end{aligned} \tag{5}$$

which are orthogonal over $[-1,1]$ with weighting function $w(t) = 1$, for more details (see [4]). After shifting the Legendre polynomials by $t = 2x-1$, the shifted Legendre polynomials $L_m(x) = L^*_m(2x-1)$ that are orthogonal on $[0,1)$ can be denoted as follows:

$$L_m(x) = \sum_{s=0}^{m} (-1)^{m+s}\,\frac{(m+s)!x^s}{(m-s)!(s!)^2}.$$

## 2.2 Generalized Legendre wavelet expansion

In this subsection, we offer a generalization for the Legendre wavelets method, denoted by $GLWM$ [13], for solving delay and neutral differential equations with proportionate delays of different orders defined as in (1). The proposed $GLWM$ on the interval $[0, 1)$ are defined by

$$\psi^{(\mu)}{}_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\,\mu^{\frac{k}{2}} L_m(\mu^k t - \hat{n}), & \frac{\hat{n}-1}{\mu^k} \leq t < \frac{\hat{n}+1}{\mu^k}\ ; \\ 0 & otherwise \end{cases} \tag{6}$$

for which $k$ is positive integer, $n = 1\ ,\ 2\ ,\ldots,\ \mu^{k-1}\ ,\ \mu\ \geq\ 3$ and $\hat{n} = 2n-1$ and the order of the Legendre Polynomial is denoted by $m = 0\ ,\ 1\ ,\ 2\ ,\ \ldots\ ,\ M$ and the normalized time is denoted by $t$.

# 3 Discussion and results

## 3.1 Function approximation

A function $f(t)$ defined on $[0, 1)$ may be expanded as infinite series of the type seen below:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\ \psi^{\mu}{}_{n,m}, \tag{7}$$

where $c_{n,m} =< f,\ \psi^{\mu}{}_{n,m} > = \int_0^1 f(t)\ \psi^{\mu}{}_{n,m}(t)\ dt$ .

After trimming, (7) can be written as follows:

$$f(t) \approx \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M} c_{n,m}\ \psi^{\mu}{}_{n,m}(t) = C^T \Psi\ (t), \tag{8}$$

where

$$C = [c_{1,0}, c_{1,1}, \ldots, c_{1,M}, c_{2,0}, c_{2,1}, \ldots, c_{2,M}\ , \ldots, c_{\mu^{k-1},0} c_{\mu^{k-1},1}, \ldots, c_{\mu^{k-1},M}]^T$$

and

$$\Psi(t) =$$
$$\left[\psi^{\mu}_{1,0}\psi^{\mu}{}_{1,1}\dots\psi^{\mu}_{1,M}\dots\psi^{\mu}_{2,0}\psi^{\mu}_{2,1}\dots\psi^{\mu}_{2,M}\dots\psi^{\mu}_{\mu^{k-1},0}\psi^{\mu}_{\mu^{k-1},1}\dots\psi^{\mu}_{\mu^{k-1},M}\right]^T.$$

## 3.2 Generalized Legendre wavelet operational matrix of differentiation

The $q$th derivative of the vector $\Psi(t)$, defined in (6) can be obtained by

$$\frac{d^q}{dt^q}\Psi(t) = D^q\Psi(t)\,, \tag{9}$$

where $D^q$ is the $q$th power of the $\mu^{k-1}(M+1)\times\mu^{k-1}(M+1)$ operational matrix of differentiation $D$, defined in [23] as follows:

$$D = \begin{pmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & F \end{pmatrix},$$

where $F$ is a $(M+1)\times(M+1)$ submatrix of the type

$$F = \mu^k \times$$
$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \\ \sqrt{3} & 0 & 0 & \dots & 0 & 0 & \\ 0 & \sqrt{3}\sqrt{5} & 0 & \dots & 0 & 0 & \\ \sqrt{7} & 0 & \sqrt{5}\sqrt{7} & \dots & 0 & 0 & \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \\ \sqrt{2M+1} & 0 & \sqrt{5}\sqrt{2M+1} & \dots & \sqrt{2M-1}\sqrt{2M+1} & 0 & \}M\ is\ odd \\ 0 & \sqrt{3}\sqrt{2M+1} & 0 & \dots & \sqrt{2M-1}\sqrt{2M+1} & 0 & \}M\ is\ even \end{pmatrix}.$$

## 3.3 The use of the operational differentiation matrix

To address the problem presented in (1) and (2), we first find the approximated solution considering the truncated series in (8), utilizing generalized Legendre wavelets as

$$y(t) \approx \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M} c_{n,m} \ \psi^{\mu}{}_{n,m}(t) = C^T \Psi(t) \ , \tag{10}$$

where the coefficients $c_{n,m}$ are to be determined. Using (9) to approximate the $p$th derivative as

$$y^p(t) = C^T \frac{d^p}{dt^p} \Psi(t) \ = C^T D^p \Psi(t). \tag{11}$$

Substituting (10) and (11) into (1) implies that

$$\begin{aligned} &\sum_{p=0}^{3} \sum_{q=0}^{Q} R_{pq}(t) C^T D^p \Psi(t - \eta_{pq}(t)) \\ &+ \sum_{r=0}^{2} \sum_{s=0}^{r} S_{rs}(t) C^T D^r \Psi \ (\delta_{rs} t) \ C^T D^s \Psi(\gamma_{rs} t) \ = h(t) \ \ , \ \ Q \leq 3. \end{aligned} \tag{12}$$

We need $\mu^{k-1}(M+1)$ equations to determine the unbeknown coefficients $c_{n,m}$ of the vector $C$. The first three equations are derived using the initial conditions (2), (3), and (4) as

$$\begin{aligned} &y(0) = C^T D \ \Psi \ (0) \ , \\ &y^{(p)} \ (0) = C^T D^p \ \Psi \ (0) \ , \qquad p = 1, \ 2 \ . \end{aligned}$$

and $\mu^{k-1}(M + 1) - 3$ equations are obtained by substituting the first $(\mu^{k-1}(M + 1)) - 3$ roots of shifted Legendre polynomial $P_{\mu^{k-1}(M+1)}(t)$ in (12).

Then, using MATLAB, we can solve the obtained system of nonlinear equations and the approximated solution in (10) is obtained.

### 3.4 Convergence criteria of the proposed GLWM

In this subsection, we discuss the theoretical analysis of the convergence of our approach to solve (1).

We want to prove that $y(t) = \sum\limits_{n=1}^{\infty} \sum\limits_{m=0}^{\infty} c_{n,m} \psi^{\mu}_{n,m}(t)$ defined in (10) using the GLWM converges to $y(t)$.

Let $L^2(R)$ be the Hilbert space. We have shown that $\psi^{(\mu)}_{n,m}(t) = \sqrt{m+\frac{1}{2}}\mu^{\frac{k}{2}} L_m(\mu^k t - \hat{n})$ forms an orthonormal basis [13].

Let $y(t) = \sum_{i=0}^{M} h_{ni}\psi^{\mu}_{ni}(t)$ be a solution of (1) such that $h_{1i} = \langle y(t), \psi^{\mu}_{1i}(t)\rangle$ for $n = 1$ in which $\langle\ .\ ,\ .\ \rangle$ denotes the inner product.

Let we denote $\psi^{\mu}_{ni}(t) = \psi^{\mu}(t)$ and $\alpha_j = \langle y(t), \psi^{\mu}(t)\rangle$

$$y(t) = \sum_{i=1}^{M} \langle y(t), \psi^{\mu}_{1i}(t)\rangle \ \ \psi^{\mu}_{1i}(t).$$

Consider the sequences of partial sums

$$W_{n-1} = \sum_{j=1}^{n-1} \alpha_j \ \psi^{\mu}(t_j) \qquad \text{and} \qquad W_{m-1} = \sum_{j=1}^{m-1} \alpha_j \ .\psi^{\mu}(t_j)$$

Then,

$$\begin{aligned}\langle y(t), W_{n-1}\rangle = \left\langle y(t), \sum_{j=1}^{n-1} \alpha_j \ \psi^{\mu}(t_j)\right\rangle &= \sum_{j=1}^{n-1} \bar{\alpha}_j \ \ \langle y(t), \psi^{\mu}(t_j)\rangle \\ &= \sum_{j=1}^{n-1} \bar{\alpha}_j \ \alpha_j = \sum_{j=1}^{n-1} |\alpha_j|^2 .\end{aligned}$$

Moreover,

$$\begin{aligned}\| \ W_{n-1} - W_{m-1} \ \|^2 &= \left\| \sum_{j=m}^{n-1} \alpha_j \ \psi^{\mu}(t_j) \right\|^2 \\ &= \left\langle \sum_{i=m}^{n-1} \alpha_i \ \psi^{\mu}(t_i) \ , \ \sum_{j=m}^{n-1} \alpha_j \ \psi^{\mu}(t_j) \right\rangle \\ &= \sum_{i=m}^{n-1} \sum_{j=m}^{n-1} \alpha_i \ \bar{\alpha}_j \ \ \langle \psi^{\mu}(t_i) \ , \ \psi^{\mu}(t_j)\rangle = \sum_{i=m}^{n-1} | \ \alpha_i \ |^2 .\end{aligned}$$

As $n \to \infty$, by Bessel's inequality, we get that $\sum_{i=m}^{n-1} | \ \alpha_i \ |^2$ is convergent, it yields that $\{W_{n-1}\}$ is a Cauchy sequence and it converges to $W$ (say).

Now, we have

$$\begin{aligned} < W - y(t), \psi^{\mu}(t_j) > &= < W, \psi^{\mu}(t_j) > - < y(t), \psi^{\mu}(t_j) > \\ &= \ < \lim_{n\to\infty} W_{n-1} \ , \ \psi^{\mu}(t_j) > \ - \ \alpha_j \\ &= \lim_{n\to\infty} < W_{n-1} \ , \ \psi^{\mu}(t_j) > \ - \ \alpha_j \\ &= \lim_{n\to\infty} < \sum_{j=1}^{n-1} \alpha_j \ \psi^{\mu}(t_j) \ , \ \psi^{\mu}(t_j) > \ - \alpha_j \\ &= \ \alpha_j - \alpha_j = 0, \end{aligned}$$

which is satisfied only in the case if $y(t) = W$. Thus, $y(t) = \sum_{j=1}^{\infty} \alpha_j \ \psi^{\mu}(t_j)$.

### 3.5 Error bound

Suppose that the function $y(t)$ defined in $[0, \ 1]$ is $m$ times continuously differentiable function. Then there exists a mean error bound for the approximation of $\sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M} c_{n,m} \ \psi^{\mu}(t) = C^T \Psi^{\mu}(t)$ to $y(t)$ as follows [37]:

$$\left\| \ y - C^T \psi^{\mu}(t) \ \right\| \leq \frac{1}{m! \mu^{mk}} \sup_{t\in[0, \ 1]} \left| y^{(m)}(t) \ \right| \ .$$

We divide the interval $[0, \ 1]$ into subintervals $\left[\frac{\hat{n}-1}{\mu^k} \ , \ \frac{\hat{n}+1}{\mu^k}\right]$. So we can approximate $y(t)$ to the polynomial $C^T \Psi^{\mu}(t)$ of $m$th degree, taking into consideration a minimum error for these subintervals. Therefore, we can utilize the maximum error estimation for this polynomial that insets $y(t)$, that is,

$$\begin{aligned}
\left\| \, y - C^T\psi^{\mu}(t) \, \right\|^2 &= \int_0^1 \, [y(t) - C^T\psi^{\mu}(t)]^{\,2} \quad dt \\
&= \sum_{n=1}^{\mu^{k-1}} \int_{\frac{2n-2}{\mu^k}}^{\frac{2n}{\mu^k}} \, [\, y(t) - C^T\psi^{\mu}(t)]^{\,2} \, dt \\
&\leq \sum_{n=1}^{\mu^{k-1}} \int_{\frac{2n-2}{\mu^k}}^{\frac{2n}{\mu^k}} \, [y(t) - y^*(t)]^{\,2} \quad dt \\
&\leq \sum_{n=1}^{\mu^{k-1}} \int_{\frac{2n-2}{\mu^k}}^{\frac{2n}{\mu^k}} \, [\frac{1}{m!\mu^{mk}} \sup_{t\in[0,\,1]} \left| y^{(m)}(t) \, \right|]^2 \quad dt \\
&\leq \int_0^1 \, [\frac{1}{m!\mu^{mk}} \, \underset{t\in[0,\,1]}{sup} \left| y^{(m)}(t) \, \right|]^2 \, dt \\
&= \frac{1}{m!\mu^{mk}} \sup_{t\in[0,\,1]} \left| y^{(m)}(t) \, \right|^2 \; ,
\end{aligned}$$

where $y^*(t)$ denotes the $m$th order interpolation of $y(t)$. Taking the square roots of both sides yields the desired outcome.

## 4 Numerical examples

In this section, we demonstrate the advantage and high accuracy of our proposed GLWM by applying it to various conventional delay differential equations. All the numerical test examples of our program were carried out by MATLAB R2015a.

**Example 1.** Assume the equation of the second-order neutral differential form through proportional delays shown below [17]:

$$\begin{aligned}
&y''(t) = \frac{3}{4}y(t) + y(\frac{t}{2}) + y'(\frac{t}{2}) + \frac{1}{2}y''(\frac{t}{2}) - t^2 - t + 1, \qquad 0 \leq t \leq 1, \\
&y(0) = \; 0 \; , \qquad y'(0) = 0.
\end{aligned} \tag{13}$$

The exact solution of this initial value problem is given by $y(t) \; = t^2$.

We first apply the GLWM for $M = 2, k = 1, \mu = 3$.

For this choice of $M, k, \mu$, the function approximation for $y(t)$ will take the summation form,

$$y(t) \;\approx\; \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M} c_{n\,,\,m}\; \psi^{\mu}_{n,m}(t) \;=\; \sum_{n=1}^{1} \sum_{m=0}^{2} c_{n\,,\,m}\; \psi^{\mu}_{n,m}\;(t) \;=\; C^T \Psi\;, \quad (14)$$

where $C_{3\times 1} = [\; c_{1,0}\;\; c_{1,1}\;\; c_{1,2}\;]^T$ and $\Psi_{3\times 1}(t) \;=\; \left[\psi^{\mu}_{1,0}(t)\; \psi^{\mu}_{1,1}(t)\; \psi^{\mu}_{1,2}(t)\right]^T$, where the generalized Legendre wavelets $\psi^{\mu}_{1,m}(t)$ , $m = 0, 1, 2$ in this case, are given by

$$\psi^{\mu}_{1,0}(t) = \begin{cases} \frac{\sqrt{2}\sqrt{3}}{2}, & 0 \;\le\; t \;<\; \frac{2}{3}, \\ 0, & otherwise, \end{cases}$$

$$\psi^{\mu}_{1,1}(t) = \begin{cases} \frac{3\sqrt{2}}{2}(3t-1), & 0 \;\le\; t \;<\; \frac{2}{3}, \\ 0, & otherwise, \end{cases}$$

$$\psi^{\mu}_{1,2}(t) = \begin{cases} \frac{\sqrt{15}}{\sqrt{2}}\; (\frac{3}{2}(3t-1)^2 - \frac{1}{2}), & 0 \;\le\; t \;<\; \frac{2}{3}, \\ 0, & otherwise. \end{cases}$$

Thus, $y(t)$ and $y(\frac{t}{2})$ can be approximated as

$$y(t) \;=\; c_{1,0}\, \frac{\sqrt{2}\sqrt{3}}{2} \;+\; c_{1,1}\frac{3\sqrt{2}}{2}(3t-1) \;+\; c_{1,2}\, \frac{\sqrt{15}}{\sqrt{2}}\, (\frac{3}{2}(3t-1)^2 - \frac{1}{2}),$$

$$y(\frac{t}{2}) \;=\; c_{1,0}\, \frac{\sqrt{2}\sqrt{3}}{2} \;+\; c_{1,1}\frac{3\sqrt{2}}{2}(3\frac{t}{2}-1) \;+\; c_{1,2}\, \frac{\sqrt{15}}{\sqrt{2}}\, (\frac{3}{2}(3\frac{t}{2}-1)^2 - \frac{1}{2}).$$

To calculate the first and second derivatives of $y(t)$, we use the $3 \times 3$ operational matrix of differentiation$P$ and $P^2$ in the form

$$P \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 3\sqrt{3} & 0 & 0 \\ 0 & 3\sqrt{15} & 0 \end{pmatrix}, \qquad P^2 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9\sqrt{45} & 0 & 0 \end{pmatrix}$$

as follows:

$$\begin{aligned} y'(t) &= c_{1,1} 3\sqrt{3}\psi^{\mu}_{1,0} + c_{1,2} 3\sqrt{15}\psi^{\mu}_{1,1} \\ &= c_{1,1} 3\sqrt{3}\frac{\sqrt{2}\sqrt{3}}{2} + c_{1,2} 3\sqrt{15}\frac{3\sqrt{2}}{2}(3t-1), \end{aligned}$$

$$y''(t) \;=\; c_{1,2}\; 9\sqrt{45}\; \psi^{\mu}_{1,0} \;= c_{1,2}\; 9\sqrt{45}\; \frac{\sqrt{2}\sqrt{3}}{2}\;,$$

and hence $y''(\frac{t}{2}) \;= c_{1,2}\; 9\sqrt{45}\; \frac{\sqrt{2}\sqrt{3}}{2}$ .

Using these approximations, (13) takes the form

$$
\begin{aligned}
c_{1,2}9\sqrt{45}\frac{\sqrt{2}\sqrt{3}}{2} &= \frac{3}{4}(c_{1,0}\frac{\sqrt{2}\sqrt{3}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3t-1) + c_{1,2}\,\frac{\sqrt{15}}{\sqrt{2}}(\frac{3}{2}(3t-1)^2 - \frac{1}{2}) \\
&\quad + (c_{1,0}\frac{\sqrt{2}\sqrt{3}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3\frac{t}{2}-1) + c_{1,2}\frac{\sqrt{15}}{\sqrt{2}}(\frac{3}{2}(3\frac{t}{2}-1)^2 - \frac{1}{2}) \\
&\quad + c_{1,1}3\sqrt{3}\frac{\sqrt{2}\sqrt{3}}{2} + c_{1,2}3\sqrt{15}\frac{3\sqrt{2}}{2}(3\frac{t}{2}-1) \\
&\quad + \frac{1}{2}c_{1,2}\,9\sqrt{45}\frac{\sqrt{2}\sqrt{3}}{2} - t^2 - t + 1.
\end{aligned} \tag{15}
$$

It should be noted that in order to find the unknown coefficients, $c_{1,0}$ $c_{1,1}$ $c_{1,2}$, we need three equations. Two equations are obtained from the initial conditions in (13) as follows:

$$
c_{1,0}\,\frac{\sqrt{2}\sqrt{3}}{2} - c_{1,1}\frac{3\sqrt{2}}{2} + c_{1,2}\,\frac{\sqrt{15}}{\sqrt{2}} = 0\,,
$$

$$
c_{1,1}\,3\sqrt{3}\,\frac{\sqrt{2}\sqrt{3}}{2} - c_{1,2}\,3\sqrt{15}\,\frac{3\sqrt{2}}{2} = 0\,.
$$

We can gain the third equation by inserting the first root of third-order shifted generalized Legendre polynomial, given by $t = 0.07513$, in (15). Solving this 3×3 nonlinear system gives

$$
\begin{aligned}
C_{3\times 1} &= [c_{1,0} \quad c_{1,1} \quad c_{1,2}]^T \\
&= [0.12096245643373 \quad 0.104756560175784 \quad 0.027048027531119]^T.
\end{aligned}
$$

Hence, the approximate solution of [17, Example 1] using our proposed $GLWM$ is obtained as

$$
\begin{aligned}
y(t) &= C^T\Psi \\
&= [0.12096245643373 \quad 0.104756560175784 \quad 0.027048027531119]^T \times \\
&\quad \begin{bmatrix} \frac{\sqrt{2}\sqrt{3}}{2} \\ \frac{3\sqrt{2}}{2}(3t-1) \\ \frac{\sqrt{15}}{\sqrt{2}}\,(\frac{3}{2}(3t-1)^2 - \frac{1}{2}) \end{bmatrix}.
\end{aligned}
$$

Along with the absolute errors compared to the exact solution, the estimates of the approximation can be evaluated at the locations in the prescribed interval, $0 \le t < \frac{2}{3}$ and summarized in the table (Table 1) below.

Table 1: Approximate solution and the absolute error of in [17, Example 1] using our $GLWM$ for $M = 2; k = 1; \mu = 3$

| $t$ | **Exact solution** | **Approximate solution** | **Absolute error** |
|---|---|---|---|
| 0.1 | 0.01 | 0.009999999999990999 | 9.0015e-15 |
| 0.2 | 0.04 | 0.03999999999999143 | 8.5733e-15 |
| 0.3 | 0.09 | 0.08999999999999162 | 8.3786e-15 |
| 0.4 | 0.16 | 0.1599999999999916 | 8.4176e-15 |
| 0.5 | 0.25 | 0.2499999999999913 | 8.6902e-15 |
| 0.6 | 0.36 | 0.3599999999999908 | 9.1964e-15 |

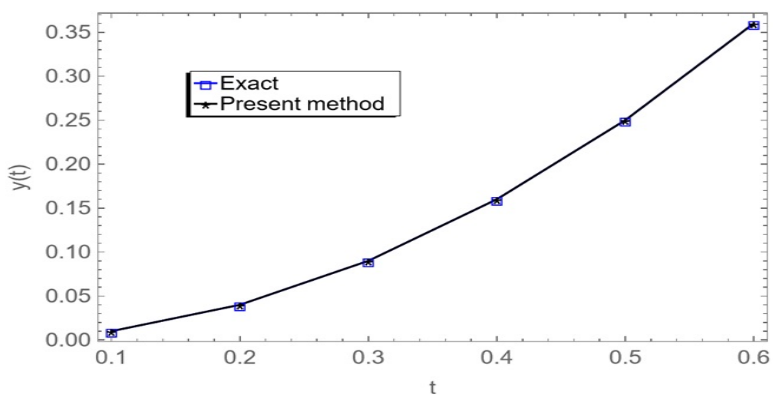


Figure 1: Approximate solution against the exact solution for Example 1

In Table 2 below, absolute error comparisons between the proposed approach $GLWM$ and the $RLWM$ for the same $M$ ($M = 2$) and other numerical methods, namely, the Runge–Kutta method of two-stage order-one case (RKM) [25], One-leg $\theta$ method with $\theta = 0.8$ [36], variational iteration method (VIM) with $n = 6$ [11], homotopy perturbation method (HPM) with $n = 6$ [7], reproducing kernel Hilbert space method (RKHSM) with $n = 100$ [20], Legendre–Gauss collocation method (LCM) with $n = 10$ [6], homotopy analysis method (HAM) with $n = 6$ [29] are provided. Also, we present solutions on this interval for comparison because the numerical approaches mentioned previously produced solutions in the same range. From Table 1, Figure 1, and Table 2, we can presume that the current method is more accurate, effective, and promising when compared to other numerical methods, particularly with the normal Legendre wavelet method.

Table 2: Comparison of the absolute error of the suggested method with other numerical methods

| $t$ | **Present method** $GLWM(M=2)$ | $RLWM$[**17**] $(M=2)$ | **One-leg** $\theta$ **Method**[36] | **RKM**[**25**] |
|---|---|---|---|---|
| 0.1 | 9.0015e-15 | 3.43e-11 | 6.10e-03 | 1.00e−03 |
| 0.2 | 8.5733e-15 | 7.79e-11 | 2.58e-02 | 2.02e−03 |
| 0.3 | 8.3786e-15 | 1.98e-10 | 6.47e-02 | 3.07e−03 |
| 0.4 | 8.4176e-15 | 3.26e-10 | 1.37e-01 | 4.17e−03 |
| 0.5 | 8.6902e-15 | 4.62e-10 | 2.81e-01 | 5.34e−03 |

| $t$ | **VIM** $n=6$ [11] | **RKHSM** $n=100$ [2] | **HPM** $n=6$ [7] | **HAM** $n=6$ [20] | **LCM** $n=10$ [6] |
|---|---|---|---|---|---|
| 0.1 | 1.67e−04 | 9.57e−06 | 1.67e−04 | 2.25e−08 | 6.59e−17 |
| 0.2 | 7.15e−04 | 1.95e−04 | 7.15e−04 | 9.81e−08 | 1.37e−17 |
| 0.3 | 1.73e−03 | 2.94e−04 | 1.72e−03 | 2.44e−07 | 5.67e−18 |
| 0.4 | 3.30e−03 | 3.93e−04 | 3.30e−03 | 4.90e−07 | 6.98e−17 |
| 0.5 | 5.55e−03 | 4.92e−04 | 5.55e−03 | 8.69e−07 | 2.13e−17 |

**Example 2.** Consider the following equation of the first order neutral differential form through proportional delay [17]:

$$y'(t) = -y(t) + 0.1y(0.8t) + 0.5y'(0.8t) + (0.32t - 0.5)e^{-0.8t} + e^{-t}, 0 \leq t \leq 1$$
$$y(0) = 0. \tag{16}$$

The exact solution of this initial value problem is given by $y(t) = te^{-t}$.

We first apply the GLWM for $M = 4, k = 1, \mu = 3$.

For this choice of $M, k, \mu$, the function approximation for $y(t)$ will take the summation form:

$$y(t) \approx \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M} c_{n,m} \psi^{\mu}_{n,m}(t) = \sum_{n=1}^{1} \sum_{m=0}^{4} c_{n,m} \psi^{\mu}_{n,m}(t) = C^T \Psi, \tag{17}$$

where $C_{5\times 1} = [c_{1,0} \quad c_{1,1} \quad c_{1,2} \;\; c_{1,\,3} \quad c_{1,\,4}]^T$ and

$$\Psi_{5\times 1}(t) \;=\; \left[\psi^{\mu}_{1,0}(t)\; \psi^{\mu}_{1,1}(t)\; \psi^{\mu}_{1,2}(t) \quad \psi^{\mu}_{1,3}(t) \;\; \psi^{\mu}_{1,4}(t)\;\right]^T,$$

where the generalized Legendre wavelets $\psi^{\mu}_{1,m}(t), m = 0,1,2,3,4$, which in this case, are given by

$$\psi^{\mu}_{1,0}(t) = \begin{cases} \frac{\sqrt{6}}{2}, & 0 \le t < \frac{2}{3}, \\ 0, & otherwise, \end{cases}$$

$$\psi^{\mu}_{1,1}(t) = \begin{cases} \frac{3\sqrt{2}}{2}(3t-1), & 0 \le t < \frac{2}{3}, \\ 0, & otherwise, \end{cases}$$

$$\psi^{\mu}_{1,2}(t) = \begin{cases} \frac{\sqrt{30}}{4}\;(3(3t-1)^2-1), & 0 \le t < \frac{2}{3}, \\ 0, & otherwise, \end{cases}$$

$$\psi^{\mu}_{1,3}(t) = \begin{cases} \frac{\sqrt{42}}{4}\;(3t-1)\;(5(3t-1)^2-3), & 0 \le t < \frac{2}{3}, \\ 0, & otherwise, \end{cases}$$

$$\psi^{\mu}_{1,4}(t) = \begin{cases} \frac{3\sqrt{6}}{16}\;(35(3t-1)^4-30(3t-1)^2+3), & 0 \le t < \frac{2}{3}, \\ 0, & otherwise. \end{cases}$$

So, we can approximate $y(t)$ and $y(0.8t)$ as

$$\begin{aligned} y(t) =& c_{1,0}\frac{\sqrt{6}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3t-1) + c_{1,2}\frac{\sqrt{30}}{4}(3(3t-1)^2-1) \\ &+ c_{1,3}\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3) \\ &+ c_{1,4}\frac{3\sqrt{6}}{16}(35(3t-1)^4-30(3t-1)^2+3), \end{aligned}$$

$$\begin{aligned} y(0.8t) =& c_{1,0}\frac{\sqrt{6}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3(0.8t)-1) + c_{1,2}\frac{\sqrt{30}}{4}\;(3(3(0.8t)-1)^2-1) \\ &+ c_{1,3}\frac{\sqrt{42}}{4}(3(0.8t)-1)(5(3(0.8t)-1)^2-3) \\ &+ c_{1,4}\frac{3\sqrt{6}}{16}(35(3(0.8t)-1)^4-30(3(0.8t)-1)^2+3). \end{aligned}$$

In order to approximate the first derivative of $y(t)$, we use the $5 \times 5$ operational matrix of differentiation $P$ in the form

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 3\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 3\sqrt{15} & 0 & 0 & 0 \\ 3\sqrt{7} & 0 & 3\sqrt{35} & 0 & 0 \\ 0 & 3\sqrt{27} & 0 & 3\sqrt{63} & 0 \end{pmatrix},$$

as follows:

$$\begin{aligned} y'(t) &= \left(c_{1,1}3\sqrt{3} + c_{1,3}3\sqrt{7}\right)\psi^{\mu}_{1,0} + \left(c_{1,2}3\sqrt{15} + c_{1,4}3\sqrt{27}\right)\psi^{\mu}_{1,1} \\ &\quad + c_{1,3}3\sqrt{35}\psi^{\mu}_{1,2} + c_{1,4}3\sqrt{63}\psi^{\mu}_{1,3}. \end{aligned}$$

Using these approximations, (16) takes the form

$$\begin{aligned} &(c_{1,1}3\sqrt{3} + c_{1,3}3\sqrt{7})\frac{\sqrt{6}}{2} + (c_{1,2}3\sqrt{15} + c_{1,4}3\sqrt{27})\,(\frac{3\sqrt{2}}{2}(3t-1)) \\ &+c_{1,3}3\sqrt{35}\left(\frac{\sqrt{30}}{4}(3(3t-1)^2-1)\right) + c_{1,4}3\sqrt{63}\left(\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3)\right) \\ &\quad = -\begin{bmatrix} c_{1,0}\frac{\sqrt{6}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3t-1) + c_{1,2}\frac{\sqrt{30}}{4}(3(3t-1)^2-1) \\ +c_{1,3}\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3) \\ +c_{1,4}\frac{3\sqrt{6}}{16}\,(35(3t-1)^4 - 30(3t-1)^2+3) \end{bmatrix} \\ &\qquad +0.1\begin{bmatrix} c_{1,0}\frac{\sqrt{6}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3(0.8t)-1) + c_{1,2}\frac{\sqrt{30}}{4}(3(3(0.8t)-1)^2-1) \\ +c_{1,3}\frac{\sqrt{42}}{4}(3(0.8t)-1)(5(3(0.8t)-1)^2-3) \\ +c_{1,4}\frac{3\sqrt{6}}{16}(35(3(0.8t)-1)^4 - 30(3(0.8t)-1)^2+3) \end{bmatrix} \\ &\qquad +0.5\begin{bmatrix} \left(c_{1,1}3\sqrt{3} + c_{1,3}3\sqrt{7}\right)\frac{\sqrt{6}}{2} + \left(c_{1,2}3\sqrt{15} + c_{1,4}3\sqrt{27}\right)\left(\frac{3\sqrt{2}}{2}(3(0.8t)-1)\right) \\ +c_{1,3}3\sqrt{35}\,\left(\frac{\sqrt{30}}{4}(3(3(0.8t)-1)^2-1)\right) \\ +c_{1,4}3\sqrt{63}\left(\frac{\sqrt{42}}{4}(3(0.8t)-1)\,(5(3(0.8t)-1)^2-3)\right) \end{bmatrix} \\ &\qquad + (0.32t-0.5)e^{-0.8t} + e^{-t}. \end{aligned} \tag{18}$$

Note that in order to determine the unknown coefficients $c_{1,0}$ $c_{1,1}$ $c_{1,2}$ $c_{1,3}$ $c_{1,4}$, we need five equations. One equation is obtained from the initial conditions in (16) as follows:

$$c_{1,0}\frac{\sqrt{6}}{2} - c_{1,1}\frac{3\sqrt{2}}{2} + c_{1,2}\frac{\sqrt{30}}{2} - c_{1,3}\frac{\sqrt{42}}{2} + c_{1,4}\frac{3\sqrt{6}}{2} = 0.$$

The second, third, fourth, and fifth equations are obtained by inserting the smaller four roots of the sixth-order shifted generalized Legendre polynomial, that are given by $t_1 = 0.03127, \ \ t_2 = 0.1538, \ \ t_3 = 0.3333, \ \ t_4 = 0.5128$ , in (18).

Solving this nonlinear $5 \times 5$ system gives

$$
\begin{aligned}
C_{5\times 1} &= [\ c_{1,0} \quad c_{1,1} \quad c_{1,2} \ \ c_{1,3} \quad c_{1,4}\ ]^T \\
&= [0.17673294249913 \quad 0.07841472133705 \quad 0.01644085313212 \\
&\qquad 0.00146577426114 \quad 0.00009124056578]^T.
\end{aligned}
$$

Hence, the approximate solution of Example 2 using our proposed $GLWM$ is obtained as

$$
\begin{aligned}
y(t) &= C^T \Psi \\
&= [0.17673294249913 \quad 0.07841472133705 \quad 0.01644085313212 \\
&\qquad 0.00146577426114 \quad 0.00009124056578]^T \\
&\quad * \begin{bmatrix} \frac{\sqrt{6}}{2} \\ \frac{3\sqrt{2}}{2}(3t-1) \\ \frac{\sqrt{30}}{4}\ (3(3t-1)^2-1) \\ \frac{\sqrt{42}}{4}\ (3t-1)\ (5(3t-1)^2-3) \\ \frac{3\sqrt{6}}{16}\ (35(3t-1)^4 - 30(3t-1)^2+3) \end{bmatrix}.
\end{aligned}
$$

Along with the absolute errors compared to the exact solution, we can evaluate the approximation at the locations in the prescribed interval, $0 \leq t < \frac{2}{3}$ and summarized in the table (Table 3) below.

Table 3: Comparison of the absolute error for Example 2 of the suggested method with other numerical methods

| $t$ | **suggested method** $GLWM$ $(M=4), \mu=3$ | $RLWM$ [17] $(M=4)$ | **One-leg** $\theta$ **Method**[36] | **Two-stage order-one Runge-Kutta method** [25] | **Variational iteration method** $n=6$ [11] | **RKHSM** $n=100$ [7] | **HPM** $n=6$ [7] |
|---|---|---|---|---|---|---|---|
| 0.1 | **6.44e-07** | 1.19e−05 | 4.65e−03 | 8.68e−04 | 1.30e−03 | 1.42e−04 | 1.06e−03 |
| 0.2 | **3.78e-06** | 2.01e−05 | 1.45e−02 | 1.49e−03 | 2.14e−03 | 1.17e−04 | 1.35e−03 |
| 0.3 | **2.50e-06** | 2.40e−05 | 2.57e−02 | 1.90e−03 | 2.63e−03 | 9.45e−04 | 1.18e−03 |
| 0.4 | **3.33e-06** | 2.15e−06 | 3.60e−02 | 2.16e−03 | 2.84e−03 | 7.59e−04 | 7.61e−04 |
| 0.5 | **5.88e-06** | 2.76e−05 | 4.43e−02 | 2.28e−03 | 2.83e−03 | 6.03e−04 | 2.32e−04 |
| 0.6 | **1.35e-05** | 2.13e−05 | 5.03e−02 | 2.31e−03 | 2.67e−03 | 4.73e−04 | 2.98e−04 |

In Table 3 and Figure 2, absolute error comparisons between the proposed approach $GLWM$ and the $RLWM$ for the same $M$ $(M = 2)$ and other

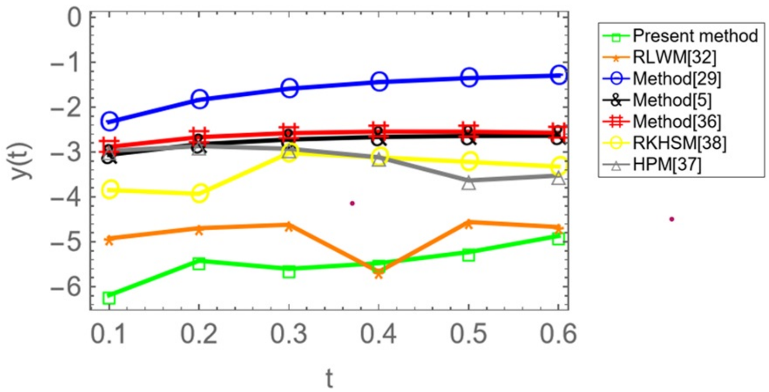


Figure 2: Absolute error for Example 2 using the presented method against the other methods listed in Table 3

numerical methods, Runge–Kutta method of two-stage order-one case (RKM) [**26**], One-leg $\theta$ method with $\theta = 0.8$ [36], Variational iteration method (VIM) with $n = 6$ [11], Homotopy perturbation method (HPM) with $n = 6$ [7], Reproducing Kernel Hilbert space method (RKHSM) with $n = 100$ [20], Legendre–Gauss collocation method (LCM) with $n = 10$ [6], Homotopy analysis method(HAM) with $n = 6$ [29] are provided.

We can presume that the current method is more effective and promising when compared to other numerical solutions, particularly with the normal Legendre wavelet method. Moreover, the absolute errors compared to the exact solution, we can evaluate the approximation at the locations in the prescribed interval, $0 \leq t < \frac{1}{2}$, for two values of $\mu = 3$ , $\mu = 4$ and summarized in the next table, Table 4, given as.

As, one can see the absolute error is improved as we increase the values of the parameter $\mu$ .

**Example 3.** Consider the following third order nonlinear equation with proportional delay [16]:

$$\begin{aligned} &y'''(t) + 1 - 2y^2(\frac{t}{2}) = 0 , \qquad 0 \leq t < 1 . \\ &y(0) = 0 , \ y'(0) = 1 , \ y''(0) = 0, \end{aligned} \tag{19}$$

with the exact solution of the type $y(t) = \sin(t)$.

Table 4: Comparison of the absolute error for Example 2 of our proposed method GLWM in two cases $\mu = 3$ , $\mu = 4$.

| $t$ | **Present method** $GLWM(M=4)$ , $\mu=3$ | **Present method** $GLWM(M=4)$ , $\mu=4$ |
|---|---|---|
| 0.1 | **6.44e-07** | **7.52 e-07** |
| 0.2 | **3.78e-06** | **1.49e-08** |
| 0.3 | **2.50e-06** | **8.55e-07** |
| 0.4 | **3.33e-06** | **1.64e-06** |
| 0.5 | **5.88e-06** | **2.43e-05** |

We first apply the GLWM for $M = 5, k = 1, \mu = 3$.

For this choice of $M$, $k$ , $\mu$, the function approximation for $y(t)$ will take the summation form:

$$y(t) \approx \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M} c_{n\,,\,m}\, \psi^{\mu}_{n,m}(t) = \sum_{n=1}^{1} \sum_{m=0}^{5} c_{n\,,\,m}\, \psi^{\mu}_{n,m}(t) = C^T \Psi \,, \quad (20)$$

where $C_{6\times 1} = [\, c_{1,0} \quad c_{1,1} \quad c_{1,2} \;\; c_{1,\,3} \quad c_{1,\,4} \quad c_{1,\,5}]^T$ and

$$\Psi_{6\times 1}(t) = \left[ \psi^{\mu}_{1,0}(t)\; \psi^{\mu}_{1,1}(t)\; \psi^{\mu}_{1,2}(t) \quad \psi^{\mu}_{1,3}(t) \;\; \psi^{\mu}_{1,4}(t) \quad \psi^{\mu}_{1,5}(t) \right]^T ,$$

where the generalized Legendre wavelets are $\psi^{\mu}_{1,m}(t)$ , $m = 0, 1, 2, 3, 4, 5.$

So, we can approximate $y(t)$ and $y(t/2)$ as

$$\begin{aligned} y(t) =& c_{1,0}\frac{\sqrt{6}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3t-1) + c_{1,2}\frac{\sqrt{30}}{4}(3(3t-1)^2-1) \\ &+ c_{1,3}\frac{\sqrt{42}}{4}(3t-1)\,(5(3t-1)^2-3) \\ &+ c_{1,4}\frac{3\sqrt{6}}{16}(35(3t-1)^4-30(3t-1)^2+3) \\ &+ c_{1,5}\frac{\sqrt{66}}{16}\left(63(3t-1)^5-70(3t-1)^3+15(3t-1)\right), \end{aligned}$$

$$\begin{aligned}
y(t/2) =& c_{1,0}\frac{\sqrt{6}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3(t/2)-1) + c_{1,2}\frac{\sqrt{30}}{4}\left(3(3(t/2)-1)^2-1\right)\\
&+ c_{1,3}\frac{\sqrt{42}}{4}(3(t/2)-1)(5(3(t/2)-1)^2-3)\\
&+ c_{1,4}\frac{3\sqrt{6}}{16}(35(3(t/2)-1)^4-30(3(t/2)-1)^2+3)\\
&+ c_{1,5}\frac{\sqrt{66}}{16}\left(63(3(t/2)-1)^5-70(3(t/2)-1)^3+15(3(t/2)-1)\right).
\end{aligned}$$

To approximate the first, second, and third derivatives of $y(t)$, we use the $6\times 6$ operational matrix of differentiation $P$ in the form

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 3\sqrt{3} & 0 & 0 & 0 & 0 & 0\\ 0 & 3\sqrt{15} & 0 & 0 & 0 & 0\\ 3\sqrt{7} & 0 & 3\sqrt{35} & 0 & 0 & 0\\ 0 & 3\sqrt{27} & 0 & 3\sqrt{63} & 0 & 0\\ 3\sqrt{11} & 0 & 3\sqrt{55} & 0 & 3\sqrt{99} & 0 \end{pmatrix},$$

$$P^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 9\sqrt{45} & 0 & 0 & 0 & 0 & 0\\ 0 & 9\sqrt{15}\sqrt{35} & 0 & 0 & 0 & 0\\ 0 & 0 & 9\sqrt{35}\sqrt{63} & 0 & 0 & 0\\ 0 & 9\sqrt{15}\sqrt{55}+9\sqrt{27}\sqrt{99} & 0 & 9\sqrt{63}\sqrt{99} & 0 & 0 \end{pmatrix},$$

and

$$P^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 27\sqrt{35}\sqrt{45} & 0 & 0 & 0 & 0 & 0\\ 0 & 27\sqrt{15}\sqrt{35}\sqrt{63} & 0 & 0 & 0 & 0\\ 27\sqrt{45}\sqrt{55}+27\sqrt{81}\sqrt{99}+27\sqrt{7}\sqrt{63}\sqrt{99} & 0 & 27\sqrt{35}\sqrt{63}\sqrt{99} & 0 & 0 & 0 \end{pmatrix},$$

as follows:

$$
\begin{aligned}
y'(t) &= \left(c_{1,1}3\sqrt{3}+c_{1,3}3\sqrt{7}+c_{1,5}3\sqrt{11}\right)\psi^{\mu}_{1,0}+\left(c_{1,2}3\sqrt{15}+c_{1,4}3\sqrt{27}\right)\psi^{\mu}_{1,1}\\
&\quad+\left(c_{1,3}3\sqrt{35}+c_{1,5}3\sqrt{55}\right)\ \psi^{\mu}_{1,2}+c_{1,4}3\sqrt{63}\ \psi^{\mu}_{1,3}+c_{1,5}\ 3\sqrt{99}\psi^{\mu}_{1,4}\\
&=\left(c_{1,1}3\sqrt{3}+c_{1,3}3\sqrt{7}+c_{1,5}3\sqrt{11}\right)\frac{\sqrt{6}}{2}\\
&\quad+\left(c_{1,2}3\sqrt{15}+c_{1,4}3\sqrt{27}\right)\left(\frac{3\sqrt{2}}{2}(3t-1)\right)\\
&\quad+\left(c_{1,3}3\sqrt{35}+c_{1,5}3\sqrt{55}\right)\left(\frac{\sqrt{30}}{4}(3(3t-1)^2-1)\right)\\
&\quad+c_{1,4}3\sqrt{63}\left(\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3)\right)\\
&\quad+c_{1,5}3\sqrt{99}\left(\frac{3\sqrt{6}}{16}(35(3t-1)^4-30(3t-1)^2+3)\right),
\end{aligned}
$$

$$
\begin{aligned}
y''(t) &= \left(c_{1,2}9\sqrt{45}\right)\psi^{\mu}_{1,0}+\left(c_{1,3}9\sqrt{35}\sqrt{15}+c_{1,5}\left(9\sqrt{55}\sqrt{15}+9\sqrt{27}\sqrt{99}\right)\right)\psi^{\mu}_{1,1}\\
&\quad+\left(c_{1,4}9\sqrt{63}\sqrt{35}\right)\psi^{\mu}_{1,2}+c_{1,5}9\sqrt{99}\sqrt{63}\psi^{\mu}_{1,3}\\
&=\left(c_{1,2}9\sqrt{45}\right)\frac{\sqrt{6}}{2}+\left(c_{1,3}9\sqrt{35}\sqrt{15}+c_{1,5}\left(9\sqrt{55}\sqrt{15}+9\sqrt{27}\sqrt{99}\right)\right)\\
&\quad\times\left(\frac{3\sqrt{2}}{2}(3t-1)\right)+\left(c_{1,4}9\sqrt{63}\sqrt{35}\right)\left(\frac{\sqrt{30}}{4}(3(3t-1)^2-1)\right)\\
&\quad+c_{1,5}9\sqrt{99}\sqrt{63}\left(\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3)\right),
\end{aligned}
$$

$$
\begin{aligned}
y'''(t) &= \left(c_{1,3}27\sqrt{35}\sqrt{45}+c_{1,5}\left(27\sqrt{55}\sqrt{45}+27\sqrt{81}\sqrt{99}+27\sqrt{7}\sqrt{63}\sqrt{99}\right)\right)\psi^{\mu}_{1,0}\\
&\quad+\left(c_{1,4}27\sqrt{63}\sqrt{35}\sqrt{15}\right)\psi^{\mu}_{1,1}+\left(c_{1,5}27\sqrt{63}\sqrt{35}\sqrt{99}\right)\psi^{\mu}_{1,2}\\
&=\left(c_{1,3}27\sqrt{35}\sqrt{45}+c_{1,5}\left(27\sqrt{55}\sqrt{45}+27\sqrt{81}\sqrt{99}+27\sqrt{7}\sqrt{63}\sqrt{99}\right)\right)\frac{\sqrt{6}}{2}\\
&\quad+\left(c_{1,4}27\sqrt{63}\sqrt{35}\sqrt{15}\right)\left(\frac{3\sqrt{2}}{2}(3t-1)\right)\\
&\quad+\left(c_{1,5}27\sqrt{63}\sqrt{35}\sqrt{99}\right)\ \left(\frac{\sqrt{30}}{4}(3(3t-1)^2-1)\right).
\end{aligned}
$$

Using these approximations, (19) takes the form

$$
\begin{aligned}
&\left(c_{1,3}27\sqrt{35}\sqrt{45}+c_{1,5}\left(27\sqrt{55}\sqrt{45}+27\sqrt{81}\sqrt{99}+27\sqrt{7}\sqrt{63}\sqrt{99}\right)\right)\frac{\sqrt{6}}{2}\\
&+\left(c_{1,4}27\sqrt{63}\sqrt{35}\sqrt{15}\right)\left(\frac{3\sqrt{2}}{2}(3t-1)\right)\\
&+\left(c_{1,5}27\sqrt{63}\sqrt{35}\sqrt{99}\right)\left(\frac{\sqrt{30}}{4}(3(3t-1)^2-1)\right)+1\\
&-2\begin{pmatrix} c_{1,0}\frac{\sqrt{6}}{2}+c_{1,1}\frac{3\sqrt{2}}{2}(3(t/2)-1)+c_{1,2}\frac{\sqrt{30}}{4}(3(3(t/2)-1)^2-1)\\ +c_{1,3}\frac{\sqrt{42}}{4}(3(t/2)-1)(5(3(t/2)-1)^2-3)\\ +c_{1,4}\frac{3\sqrt{6}}{16}(35(3(t/2)-1)^4-30(3(t/2)-1)^2+3)\\ +c_{1,5}\frac{\sqrt{66}}{16}\left(63(3(t/2)-1)^5-70(3(t/2)-1)^3+15(3(t/2)-1)\right)\end{pmatrix}^2\\
&=0.
\end{aligned} \tag{21}
$$

Note that in order to determine the unknown coefficients $c_{1,0}$ $c_{1,1}$ $c_{1,2}$ $c_{1,3}$ $c_{1,4}$ $c_{1,5}$ , we need six equations. Three equations are obtained from the initial conditions in (19) as follows:

$$
c_{1,0}\frac{\sqrt{6}}{2}-c_{1,1}\frac{3\sqrt{2}}{2}+c_{1,2}\frac{2\sqrt{30}}{4}-c_{1,3}\frac{2\sqrt{42}}{4}+c_{1,4}\frac{24\sqrt{6}}{16}-c_{1,5}\frac{8\sqrt{66}}{16}=0,
$$

$$
\begin{aligned}
&\left(c_{1,1}3\sqrt{3}+c_{1,3}3\sqrt{7}+c_{1,5}3\sqrt{11}\right)\frac{\sqrt{6}}{2}-\left(c_{1,2}3\sqrt{15}+c_{1,4}3\sqrt{27}\right)\left(\frac{3\sqrt{2}}{2}\right)\\
&+\left(c_{1,3}3\sqrt{35}+c_{1,5}3\sqrt{55}\right)\left(\frac{2\sqrt{30}}{4}\right)-c_{1,4}3\sqrt{63}\left(\frac{2\sqrt{42}}{4}\right)\\
&+c_{1,5}3\sqrt{99}\left(\frac{24\sqrt{6}}{16}\right)=1,
\end{aligned}
$$

$$
\begin{aligned}
&\left(c_{1,2}9\sqrt{45}\right)\frac{\sqrt{6}}{2}-\left(c_{1,3}9\sqrt{35}\sqrt{15}+c_{1,5}\left(9\sqrt{55}\sqrt{15}+9\sqrt{27}\sqrt{99}\right)\right)\left(\frac{3\sqrt{2}}{2}\right)\\
&+\left(c_{1,4}9\sqrt{63}\sqrt{35}\right)\left(\frac{2\sqrt{30}}{4}\right)-c_{1,5}9\sqrt{99}\sqrt{63}\left(\frac{2\sqrt{42}}{4}\right)=0.
\end{aligned}
$$

The fourth, fifth, and sixth equations are obtained by inserting the smaller three roots of the seventh order shifted generalized Legendre polynomial, $t_1=0.02251, t_2=0.1129, t_3=0.2538$, in (21).

Solving this nonlinear $6\times 6$ system gives

$$C_{6\times 1} = [\ c_{1,0}\quad c_{1,1}\quad c_{1,2}\ \ c_{1,3}\quad c_{1,4}\ \ c_{1,5}\ ]^T$$
$$= [0.26223389730166\quad 0.14684295578193\quad -0.00438943865859$$
$$-0.00071526649488\quad 0.00001058352043\ \ 0.00000106184998\ \ ]^T.$$

Hence, the approximate solution of Example 3 using our proposed $GLWM$ is obtained as

$$\begin{aligned} y(t) &= C^T\Psi \\ &= [0.26223389730166\quad 0.14684295578193\quad -0.00438943865859 \\ &\quad -0.00071526649488\quad 0.00001058352043\quad 0.00000106184998]^T \\ &\quad * \begin{bmatrix} \frac{\sqrt{6}}{2} \\ \frac{3\sqrt{2}}{2}(3t-1) \\ \frac{\sqrt{30}}{4}\ (3(3t-1)^2-1) \\ \frac{\sqrt{42}}{4}\ (3t-1)\,(5(3t-1)^2-3) \\ \frac{3\sqrt{6}}{16}\,(35(3t-1)^4-30(3t-1)^2+3) \\ \frac{\sqrt{66}}{16}\ \left(63(3t-1)^5-70(3t-1)^3+15(3t-1)\ \right) \end{bmatrix}. \end{aligned}$$

Along with the absolute errors compared to the exact solution, we can evaluate the approximation at the locations in the prescribed interval, $0 \leq t < \frac{2}{3}$, and summarized in the table (Table 5) below.

Table 5: Numerical results and the absolute error for **Example 3** for our proposed method GLWM using fifth- and sixth-order polynomials ($M = 5\ ,\ 6$)

| t | Exact solution | Approximate solution $M=5$ **Present method** $GLWM$ ($M=5, \mu=3, k=1$) | Absolute Error | Approximate solution $M=6$ **Present method** $GLWM$ ($M=6, \mu=3, k=1$) | Absolute Error |
|---|---|---|---|---|---|
| 0.1 | 0.09983341665 | 0.09983341651 | 1.369e-10 | 0.09983341665 | 1.969e-12 |
| 0.2 | 0.1986693308 | 0.198669332 | 1.2394e-09 | 0.1986693307 | 9.529e-11 |
| 0.3 | 0.2955202067 | 0.2955202026 | 4.0622e-09 | 0.2955202072 | 5.532e-10 |
| 0.4 | 0.3894183423 | 0.3894183507 | 8.4262e-09 | 0.3894183404 | 1.956e-9 |
| 0.5 | 0.4794255386 | 0.4794258702 | 3.3162e-07 | 0.4794255601 | 2.152e-8 |
| 0.6 | 0.5646424734 | 0.5646445308 | 2.0574e-06 | 0.5646427814 | 3.08e-7 |

In Table 6, absolute error comparisons between the proposed approach $GLWM$ and the $RLWM$ and other numerical methods is shown.

**Example 4.** Assume the following equation of the second order nonlinear differential form through proportional delay [16]:

$$\begin{aligned} &y''(t) + 2\,y(t) - y^2(t) + y(t^3/8) = \sin\ t - \sin^2 t + \sin(t^3/8), \quad 0 \leq t \leq 1, \\ &y(0) = 0, \qquad y'(0) = 1, \end{aligned} \tag{22}$$

Table 6: Comparison of the absolute errors with other numerical methods

| t | Absolute Error for present method $GLWM$ $(M=5)$ $(\mu=3,k=1)$ | Absolute Error for present method $GLWM$ $(M=6)$ $(\mu=3,k=1)$ | Absolute Error for Legendre wavelets method $RLWM$ $(M=5)$ | Absolute Error for Legendre wavelets method $RLWM$ $(M=6)$ | Decomposition Method $E13$ [34] | Adomian decomposition Method $E9$ [14] |
|---|---|---|---|---|---|---|
| 0.1 | 1.369e-10 | 1.969e-12 | 2.54e-09 | 5.37e-10 | 0.0 | 1.02e-15 |
| 0.2 | 1.2394e-09 | 9.529e-11 | 3.24e-09 | 1.39e-09 | 0.0 | 5.28e-13 |
| 0.3 | 4.0622e-09 | 5.532e-10 | 2.11e-08 | 1.59e-09 | 0.0 | 2.02e-11 |
| 0.4 | 8.4262e-09 | 1.956e-9 | 1.44e-08 | 7.06e-09 | 0.0 | 2.69e-10 |
| 0.5 | 3.3162e-07 | 2.152e-8 | 1.21e-07 | 3.52e-09 | 2.61e-09 | 2.00e-09 |
| 0.6 | 2.0574e-06 | 3.08e-7 | 1.42e-07 | 3.27e-08 | 1.04e-08 | 1.03e-08 |

with the exact solution of the form $y(t) = \sin\ t$. Comparison between approximate solution and the absolute error of [16, Example 3] using our GLWM for $M = 5, 6; k = 1; \mu = 3$ is listed (in Table 7) below. Also, comparison between the absolute error for Example 4 of the present method with the RLWM of [16, Example 3] is listed in Table 8.

Table 7: Approximate solution and the absolute error of [16, Example 3] using our $GLWM$ for $M$=5, 6; $k = 1$; $\mu = 3$

| $t$ | Exact Solution | Approximate solution $M$=5; $k = 1$; $\mu = 3$ | Approximate solution $M$=6; $k = 1$; $\mu = 3$ |
|---|---|---|---|
| 0.1 | 0.09983341665 | 0.09983341629 | 0.09983341679 |
| 0.2 | 0.1986693308 | 0.1986693294 | 0.1986693306 |
| 0.3 | 0.2955202067 | 0.2955202132 | 0.2955202065 |
| 0.4 | 0.3894183423 | 0.3894183346 | 0.3894183434 |
| 0.5 | 0.4794255386 | 0.4794255315 | 0.4794255352 |
| 0.6 | 0.5646424734 | 0.5646429917 | 0.5646425129 |

Along with the absolute errors compared to the exact solution, we can evaluate the approximation at the locations in the prescribed interval, $0 \leq t < \frac{1}{2}$, for two values of $\mu = 3, \mu = 4$ and summarized in the table (Table 9) below.

**Example 5.** Consider the following third order nonlinear differential equation with proportional delay [27]:

Table 8: Comparison of the absolute error for Example 4 of the present method with the RLWM of [16, Example 3]

| $t$ | **Present method** $GLWMM = 5,$ $\mu = 3, k = 1$ | $RLWM$ [13] $M = 6,$ $\mu = 3,\ k = 1$ | **RLWM [16]** $M = 5, k = 0$ | **RLWM [16]** $M = 6, k = 0$ |
|---|---|---|---|---|
| 0.1 | **3.562e-10** | **1.3936e-10** | 8.963065387e-09 | 3.389353381e-10 |
| 0.2 | **1.414e-9** | **1.846e-10** | 2.720358344e-08 | 3.618011279e-09 |
| 0.3 | **6.549e-9** | **1.6084e-10** | 2.394278514e-08 | 3.060617093e-09 |
| 0.4 | **7.734e-9** | **1.1379e-9** | 6.937304025e-08 | 7.998320783e-09 |
| 0.5 | **7.082e-9** | **3.3891e-9** | 1.053035117e-07 | 7.465058682e-09 |
| 0.6 | **5.183e-7** | **3.9459e-8** | 1.310158346e-07 | 1.884611267e-08 |

Table 9: Comparison of the absolute error for Example 4 of our proposed method GLWM in two cases $\mu = 3$ , $\mu = 4$

| $t$ | **Present method** $GLWM$ $(M = 5), \mu = 3$ | **Present method** $GLWM$ $(M = 5), \mu = 4$ | **Present method** $GLWM$ $(M = 6), \mu = 3$ | **Present method** $GLWM$ $(M = 5), \mu = 4$ |
|---|---|---|---|---|
| 0.1 | **3.562e-10** | **2.158e-10** | **1.3936e-10** | **2.9452e-11** |
| 0.2 | **1.414e-9** | **5.474e-10** | **1.846e-10** | **6.4737e-11** |
| 0.3 | **6.549e-9** | **1.051e-9** | **1.6084e-10** | **1.5656e-10** |
| 0.4 | **7.734e-9** | **7.761e-9** | **1.1379e-9** | **2.8361e-10** |
| 0.5 | **7.082e-9** | **2.619e-7** | **3.3891e-9** | **3.3032e-8** |

$$\begin{aligned} &y'''(t) = -y(t) - y(t-0.3) \ + e^{(-t+0.3)}, \qquad 0 \le t \le 1, \\ &y(0) = 0 \ \ , \ \ y'(0) = -1, \ \ y''(0) = 1, \end{aligned} \tag{23}$$

with the exact solution of the form $y(t) \ = e^{-t}$.

We first apply the GLWM for $M = 9, k = 1, \mu = 3$.

For this choice of $M$,$k$,$\mu$, the function approximation for $y(t)$ will take the summation form,

$$y(t) \ \approx \ \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M} c_{n\ ,\ m} \ \psi^{\mu}_{n,m}(t) \ = \ \sum_{n=1}^{1} \sum_{m=0}^{9} c_{n\ ,\ m} \ \psi^{\mu}_{n,m} \ (t) \ = \ C^T \Psi \ , \tag{24}$$

where $C_{10\times 1} = [\, c_{1,0} \quad c_{1,1} \quad c_{1,2} \;\; c_{1,\,3} \quad c_{1,\,4} \quad c_{1,5} \;\; c_{1,6} \;\; c_{1,7} \;\; c_{1,8} \;\; c_{1,9} \,]^T$ and

$$\Psi_{10\times 1}(t) \;=\; \begin{bmatrix} \psi^{\mu}_{1,0}(t) & \psi^{\mu}_{1,1}(t) & \psi^{\mu}_{1,2}(t) & \psi^{\mu}_{1,3}(t) & \psi^{\mu}_{1,4}(t) \\ \psi^{\mu}_{1,5}(t) & \psi^{\mu}_{1,6}(t) & \psi^{\mu}_{1,7}(t) & \psi^{\mu}_{1,8}(t) & \psi^{\mu}_{1,9}(t) \end{bmatrix}^T,$$

where the generalized Legendre wavelets are $\psi^{\mu}_{1,m}(t), m = \; 0, 1, 2, \dots, 9.$

Thus, $y(t)$ and $y(t-0.3)$ can be approximated as

$$\begin{aligned} y(t) =& c_{1,0}\frac{\sqrt{6}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3t-1) + c_{1,2}\frac{\sqrt{30}}{4}(3(3t-1)^2-1) \\ &+ c_{1,3}\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3) \\ &+ c_{1,4}\frac{3\sqrt{6}}{16}\,(35(3t-1)^4-30(3t-1)^2+3) \\ &+ c_{1,5}\frac{\sqrt{66}}{16}\left(63(3t-1)^5-70(3t-1)^3+15(3t-1)\right) \\ &+ c_{1,6}\frac{\sqrt{78}}{32}\left(231(3t-1)^6-315(3t-1)^4+105(3t-1)^2-5\right) \\ &+ c_{1,7}\frac{3\sqrt{10}}{32}\left(429(3t-1)^7-693(3t-1)^5+315(3t-1)^3-35(3t-1)\right) \\ &+ c_{1,8}\frac{\sqrt{102}}{256}(6435(3t-1)^8-12012(3t-1)^6+6930(3t-1)^4 \\ &- 1260(3t-1)^2+35) + c_{1,9}\Bigg(\frac{\sqrt{114}}{256}\left(12155(3t-1)^9-25740(3t-1)^7\right. \\ &\left.+18018(3t-1)^5-4620(3t-1)^3+315(3t-1)\right)\Bigg), \end{aligned}$$

$$\begin{aligned} y(t-0.3) =& c_{1,0}\frac{\sqrt{6}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3(t-0.3)-1) + c_{1,2}\frac{\sqrt{30}}{4}(3(3(t-0.3)-1)^2-1) \\ &+ c_{1,3}\frac{\sqrt{42}}{4}(3(t-0.3)-1)(5(3(t-0.3)-1)^2-3) \\ &+ c_{1,4}\frac{3\sqrt{6}}{16}(35(3(t-0.3)-1)^4-30(3(t-0.3)-1)^2+3) \\ &+ c_{1,5}\frac{\sqrt{66}}{16}\left(63(3(t-0.3)-1)^5-70(3(t-0.3)-1)^3+15(3(t-0.3)-1)\right) \\ &+ c_{1,6}\frac{\sqrt{78}}{32}\left(231(3(t-0.3)-1)^6-315(3(t-0.3)-1)^4\right. \\ &\left.+105(3(t-0.3)-1)^2-5\,\right) \end{aligned}$$

$$
\begin{aligned}
&+ c_{1,7}\frac{3\sqrt{10}}{32}\big(429(3(t-0.3)-1)^7\\
&-693(3(t-0.3)-1)^5+315(3(t-0.3)-1)^3-35(3(t-0.3)-1)\big)\\
&+ c_{1,8}\frac{\sqrt{102}}{256}\big(6435(3(t-0.3)-1)^8-12012(3(t-0.3)-1)^6\\
&+6930(3(t-0.3)-1)^4-1260(3(t-0.3)-1)^2+35\big)\\
&+ c_{1,9}\left(\frac{\sqrt{114}}{256}\left(\begin{array}{l}12155(3(t-0.3)-1)^9-25740(3(t-0.3)-1)^7+\\ 18018(3(t-0.3)-1)^5-4620(3(t-0.3)-1)^3\\ +315(3(t-0.3)-1)\end{array}\right)\right),
\end{aligned}
$$

In order to approximate the first, second, and third derivatives of $y(t)$, we use the $10\times 10$ operational matrix of differentiation $P$ in the form

$$
P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
3\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 3\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
3\sqrt{7} & 0 & 3\sqrt{35} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 3\sqrt{27} & 0 & 3\sqrt{63} & 0 & 0 & 0 & 0 & 0 & 0\\
3\sqrt{11} & 0 & 3\sqrt{55} & 0 & 3\sqrt{99} & 0 & 0 & 0 & 0 & 0\\
0 & 3\sqrt{39} & 0 & 3\sqrt{91} & 0 & 3\sqrt{143} & 0 & 0 & 0 & 0\\
3\sqrt{15} & 0 & 3\sqrt{75} & 0 & 3\sqrt{135} & 0 & 3\sqrt{195} & 0 & 0 & 0\\
0 & 3\sqrt{51} & 0 & 3\sqrt{119} & 0 & 3\sqrt{187} & 0 & 3\sqrt{255} & 0 & 0\\
3\sqrt{19} & 0 & 3\sqrt{95} & 0 & 3\sqrt{171} & 0 & 3\sqrt{247} & 0 & 3\sqrt{323} & 0
\end{pmatrix},
$$

$$
P^2 = 1.0e+03 * \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0.060 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0.2062 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0.2700 & 0 & 0.4226 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0.7238 & 0 & 0.7108 & 0 & 0 & 0 & 0 & 0 & 0\\
0.6814 & 0 & 1.3061 & 0 & 1.0708 & 0 & 0 & 0 & 0 & 0\\
0 & 1.6301 & 0 & 2.0289 & 0 & 1.5029 & 0 & 0 & 0 & 0\\
1.3359 & 0 & 2.7382 & 0 & 2.8944 & 0 & 2.0069 & 0 & 0 & 0\\
0 & 2.9897 & 0 & 4.0479 & 0 & 3.9033 & 0 & 2.5829 & 0 & 0
\end{pmatrix},
$$

and

$$P^{3} = 1.0e+05 * \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0107 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0491 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0940 & 0 & 0.1261 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.3187 & 0 & 0.2550 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3953 & 0 & 0.6945 & 0 & 0.4486 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.1453 & 0 & 1.2636 & 0 & 0.7200 & 0 & 0 & 0 & 0 \\ 1.1651 & 0 & 2.2579 & 0 & 2.0655 & 0 & 1.0821 & 0 & 0 & 0 \end{pmatrix},$$

as follows:

$$\begin{aligned} y'(t) =&(c_{1,1}3\sqrt{3} + c_{1,3}3\sqrt{7} + c_{1,5}3\sqrt{11} + c_{1,7}3\sqrt{15} + c_{1,9}3\sqrt{19})\psi^{\mu}_{1,0} \\ &+(c_{1,2}3\sqrt{15} + c_{1,4}3\sqrt{27} + c_{1,6}\,3\sqrt{39} + c_{1,8}3\sqrt{51})\psi^{\mu}_{1,1} \\ &+(c_{1,3}3\sqrt{35} + c_{1,5}3\sqrt{55} + c_{1,7}3\sqrt{75} + c_{1,9}3\sqrt{95})\psi^{\mu}_{1,2} \\ &+(c_{1,4}3\sqrt{63} + c_{1,6}3\sqrt{91} + c_{1,8}3\sqrt{119})\psi^{\mu}_{1,3} \\ &+(c_{1,5}\,3\sqrt{99} + c_{1,7}3\sqrt{135} + c_{1,9}3\sqrt{171})\psi^{\mu}_{1,4} \\ &+(c_{1,6}3\sqrt{143} + c_{1,8}3\sqrt{187})\psi^{\mu}_{1,5} + (c_{1,7}3\sqrt{195} + c_{1,9}3\sqrt{247})\psi^{\mu}_{1,6} \\ &+(c_{1,8}3\sqrt{255})\psi^{\mu}_{1,7} + (c_{1,9}3\sqrt{323})\psi^{\mu}_{1,8} \\ =&(c_{1,1}3\sqrt{3} + c_{1,3}3\sqrt{7} + c_{1,5}3\sqrt{11} + c_{1,7}\,3\sqrt{15} + c_{1,9}3\sqrt{19})\frac{\sqrt{6}}{2} \\ &+(c_{1,2}\,3\sqrt{15} + c_{1,4}3\sqrt{27} + c_{1,6}\,3\sqrt{39} + c_{1,8}3\sqrt{51})(\frac{3\sqrt{2}}{2}(3t-1)) \\ &+(c_{1,3}3\sqrt{35} + c_{1,5}3\sqrt{55} + c_{1,7}\,3\sqrt{75} + c_{1,9}3\sqrt{95})(\frac{\sqrt{30}}{4}(3(3t-1)^2-1)) \\ &+(c_{1,4}3\sqrt{63} + c_{1,6}3\sqrt{91} + c_{1,8}3\sqrt{119})\,(\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3)) \\ &+\left(c_{1,5}3\sqrt{99} + c_{1,7}3\sqrt{135} + c_{1,9}3\sqrt{171}\right)\left(\frac{3\sqrt{6}}{16}(35(3t-1)^4 - 30(3t-1)^2 + 3)\right) \\ &+\left(c_{1,6}3\sqrt{143} + c_{1,8}3\sqrt{187}\right)\frac{\sqrt{66}}{16}\left(63(3t-1)^5 - 70(3t-1)^3 + 15(3t-1)\right) \\ &+\left(c_{1,7}3\sqrt{195} + c_{1,9}3\sqrt{247}\right)\frac{\sqrt{78}}{32}\left(231(3t-1)^6 - 315(3t-1)^4 + 105(3t-1)^2 - 5\right) \\ &+\left(c_{1,8}3\sqrt{255}\right)\frac{3\sqrt{10}}{32}\left(429(3t-1)^7 - 693(3t-1)^5 + 315(3t-1)^3 - 35(3t-1)\right) \\ &+\left(c_{1,9}3\sqrt{323}\right)\frac{\sqrt{102}}{256}\left(6435(3t-1)^8 - 12012(3t-1)^6 \right. \\ &\left. +6930(3t-1)^4 - 1260(3t-1)^2 + 35\right) \end{aligned}$$

$$
\begin{aligned}
y''(t) = &\left(c_{1,2}27\sqrt{5}+c_{1,4}270+c_{1,6}189\sqrt{13}+c_{1,8}324\sqrt{17}\right)\frac{\sqrt{6}}{2}\\
&+\left(c_{1,3}45\sqrt{21}+c_{1,5}126\sqrt{33}+c_{1,7}243\sqrt{45}+c_{1,9}396\sqrt{57}\right)\left(\frac{3\sqrt{2}}{2}(3t-1)\right)\\
&+\left(c_{1,4}189\sqrt{5}+c_{1,6}162\sqrt{65}+c_{1,8}297\sqrt{85}\right)\left(\frac{\sqrt{30}}{4}(3(3t-1)^2-1)\right)\\
&+\left(c_{1,5}81\sqrt{77}+c_{1,7}198\sqrt{105}+c_{1,9}351\sqrt{133}\right)\left(\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3)\right)\\
&+\left(c_{1,6}99\sqrt{117}+c_{1,8}234\sqrt{153}\right)\left(\frac{3\sqrt{6}}{16}(35(3t-1)^4-30(3t-1)^2+3)\right)\\
&+\left(c_{1,7}117\sqrt{165}+c_{1,9}270\sqrt{209}\right)\frac{\sqrt{66}}{16}\left(63(3t-1)^5-70(3t-1)^3+15(3t-1)\right)\\
&+\left(c_{1,8}135\sqrt{221}\right)\frac{\sqrt{78}}{32}\left(231(3t-1)^6-315(3t-1)^4+105(3t-1)^2-5\right)\\
&+\left(c_{1,9}153\sqrt{285}\right)\frac{3\sqrt{10}}{32}\left(429(3t-1)^7-693(3t-1)^5+315(3t-1)^3-35(3t-1)\right)
\end{aligned}
$$

$$
\begin{aligned}
y'''(t) =&1.0e+05\\
&*\begin{pmatrix}
(0.0107c_{1,3}+0.0940c_{1,5}+0.3953c_{1,7}+1.1651c_{1,9})\frac{\sqrt{6}}{2}\\
+(0.0491c_{1,4}+0.3187c_{1,6}+1.1453c_{1,8})\left(\frac{3\sqrt{2}}{2}(3t-1)\right)\\
+(0.1261c_{1,5}+0.6945c_{1,7}+2.2579c_{1,9})\left(\frac{\sqrt{30}}{4}(3(3t-1)^2-1)\right)\\
+(0.2550c_{1,6}+1.2636c_{1,8})\left(\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3)\right)\\
+(0.4486c_{1,7}+2.0655c_{1,9})\left(\frac{3\sqrt{6}}{16}(35(3t-1)^4-30(3t-1)^2+3)\right)\\
+(0.7200c_{1,8})\frac{\sqrt{66}}{16}\left(63(3t-1)^5-70(3t-1)^3+15(3t-1)\right)\\
+(1.0821c_{1,9})\frac{\sqrt{78}}{32}\left(231(3t-1)^6-315(3t-1)^4+105(3t-1)^2-5\right)
\end{pmatrix}
\end{aligned}
$$

Using these approximations, (24) takes the form

$$
\begin{aligned}
&1.0e+05\\
&*\begin{pmatrix}
(0.0107c_{1,3}+0.0940c_{1,5}+0.3953c_{1,7}+1.1651c_{1,9})\frac{\sqrt{6}}{2}\\
+(0.0491c_{1,4}+0.3187c_{1,6}+1.1453c_{1,8})\left(\frac{3\sqrt{2}}{2}(3t-1)\right)\\
+(0.1261c_{1,5}+0.6945c_{1,7}+2.2579c_{1,9})\left(\frac{\sqrt{30}}{4}(3(3t-1)^2-1)\right)\\
+(0.2550c_{1,6}+1.2636c_{1,8})\left(\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3)\right)\\
+(0.4486c_{1,7}+2.0655c_{1,9})\left(\frac{3\sqrt{6}}{16}(35(3t-1)^4-30(3t-1)^2+3)\right)\\
+(0.7200c_{1,8})\frac{\sqrt{66}}{16}\left(63(3t-1)^5-70(3t-1)^3+15(3t-1)\right)\\
+(1.0821c_{1,9})\frac{\sqrt{78}}{32}\left(231(3t-1)^6-315(3t-1)^4+105(3t-1)^2-5\right)
\end{pmatrix}
\end{aligned}
$$

$$
= -\begin{pmatrix}
c_{1,0}\frac{\sqrt{6}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3t-1) + c_{1,2}\frac{\sqrt{30}}{4}(3(3t-1)^2-1) \\
+ c_{1,3}\frac{\sqrt{42}}{4}(3t-1)(5(3t-1)^2-3) \\
+ c_{1,4}\frac{3\sqrt{6}}{16}(35(3t-1)^4-30(3t-1)^2+3) \\
+ c_{1,5}\frac{\sqrt{66}}{16}\left(63(3t-1)^5-70(3t-1)^3+15(3t-1)\right) \\
+ c_{1,6}\frac{\sqrt{78}}{32}\left(231(3t-1)^6-315(3t-1)^4+105(3t-1)^2-5\right) \\
+ c_{1,7}\frac{3\sqrt{10}}{32}\left(429(3t-1)^7-693(3t-1)^5+315(3t-1)^3-35(3t-1)\right) \\
+ c_{1,8}\frac{\sqrt{102}}{256}\left(6435(3t-1)^8-12012(3t-1)^6+6930(3t-1)^4\right. \\
\left.-1260(3t-1)^2+35\right) + c_{1,9}\left(\frac{\sqrt{114}}{256}\left(12155(3t-1)^9-25740(3t-1)^7\right.\right. \\
\left.\left.+18018(3t-1)^5-4620(3t-1)^3+315(3t-1)\right)\right)
\end{pmatrix}
$$

$$
-\begin{pmatrix}
c_{1,0}\frac{\sqrt{6}}{2} + c_{1,1}\frac{3\sqrt{2}}{2}(3(t-0.3)-1) + c_{1,2}\frac{\sqrt{30}}{4}(3(3(t-0.3)-1)^2-1) \\
+ c_{1,3}\frac{\sqrt{42}}{4}(3(t-0.3)-1)(5(3(t-0.3)-1)^2-3) \\
+ c_{1,4}\frac{3\sqrt{6}}{16}(35(3(t-0.3)-1)^4-30(3(t-0.3)-1)^2+3) \\
+ c_{1,5}\frac{\sqrt{66}}{16}\left(63(3(t-0.3)-1)^5-70(3(t-0.3)-1)^3+15(3(t-0.3)-1)\right) \\
+ c_{1,6}\frac{\sqrt{78}}{32}\left(231(3(t-0.3)-1)^6-315(3(t-0.3)-1)^4\right. \\
\left.+105(3(t-0.3)-1)^2-5\right) + c_{1,7}\frac{3\sqrt{10}}{32}\left(429(3(t-0.3)-1)^7\right. \\
\left.-693(3(t-0.3)-1)^5+315(3(t-0.3)-1)^3-35(3(t-0.3)-1)\right) \\
+ c_{1,8}\frac{\sqrt{102}}{256}\left(6435(3(t-0.3)-1)^8-12012(3(t-0.3)-1)^6\right. \\
\left.+6930(3(t-0.3)-1)^4-1260(3(t-0.3)-1)^2+35\right) \\
+ c_{1,9}\left(\frac{\sqrt{114}}{256}\begin{pmatrix} 12155(3(t-0.3)-1)^9-25740(3(t-0.3)-1)^7+ \\ 18018(3(t-0.3)-1)^5-4620(3(t-0.3)-1)^3+ \\ 315(3(t-0.3)-1) \end{pmatrix}\right)
\end{pmatrix}
$$

$$
+e^{(-t+0.3)}, \tag{25}
$$

Note that to determine the unknown coefficients $c_{1,0},\ c_{1,1},\ c_{1,2},\ c_{1,\,3},\ c_{1,\,4},\ c_{1,5},\ c_{1,6},\ c_{1,7},\ c_{1,8},\ c_{1,9}$, we need ten equations.

Three equations are obtained using the initial conditions in (23) as follows:

$$c_{1,0}\tfrac{\sqrt{6}}{2}-c_{1,1}\tfrac{3\sqrt{2}}{2}+c_{1,2}\tfrac{2\sqrt{30}}{4}-c_{1,3}\tfrac{2\sqrt{42}}{4}+c_{1,4}\tfrac{24\sqrt{6}}{16}-c_{1,5}\tfrac{8\sqrt{66}}{16}$$
$$+c_{1,6}\tfrac{26\sqrt{78}}{32}-c_{1,7}\tfrac{48\sqrt{10}}{32}+c_{1,8}\tfrac{128\sqrt{102}}{256}-c_{1,9}\tfrac{128\sqrt{114}}{256}=0,$$

$$\begin{aligned}
&\left(c_{1,1}3\sqrt{3}+c_{1,3}3\sqrt{7}+c_{1,5}3\sqrt{11}+c_{1,7}3\sqrt{15}+c_{1,9}3\sqrt{19}\right)\tfrac{\sqrt{6}}{2}\\
&-\left(c_{1,2}3\sqrt{15}+c_{1,4}3\sqrt{27}+c_{1,6}3\sqrt{39}+c_{1,8}3\sqrt{51}\right)\left(\tfrac{3\sqrt{2}}{2}\right)\\
&+\left(c_{1,3}3\sqrt{35}+c_{1,5}3\sqrt{55}+c_{1,7}3\sqrt{75}+c_{1,9}3\sqrt{95}\right)\left(\tfrac{2\sqrt{30}}{4}\right)\\
&-\left(c_{1,4}3\sqrt{63}+c_{1,6}3\sqrt{91}+c_{1,8}3\sqrt{119}\right)\left(\tfrac{2\sqrt{42}}{4}\right)\\
&+\left(c_{1,5}3\sqrt{99}+c_{1,7}3\sqrt{135}+c_{1,9}3\sqrt{171}\right)\left(\tfrac{24\sqrt{6}}{16}\right)\\
&-\left(c_{1,6}3\sqrt{143}+c_{1,8}3\sqrt{187}\right)\tfrac{8\sqrt{66}}{16}+\left(c_{1,7}3\sqrt{195}+c_{1,9}3\sqrt{247}\right)\tfrac{16\sqrt{78}}{32}\\
&-\left(c_{1,8}3\sqrt{255}\right)\tfrac{48\sqrt{10}}{32}+\left(c_{1,9}3\sqrt{323}\right)\tfrac{128\sqrt{102}}{256}=-1,
\end{aligned}$$

$$\begin{aligned}
&\left(c_{1,2}27\sqrt{5}+c_{1,4}270+c_{1,6}189\sqrt{13}+c_{1,8}324\sqrt{17}\right)\tfrac{\sqrt{6}}{2}\\
&-\left(c_{1,3}45\sqrt{21}+c_{1,5}126\sqrt{33}+c_{1,7}243\sqrt{45}+c_{1,9}396\sqrt{57}\right)\left(\tfrac{3\sqrt{2}}{2}\right)\\
&+\left(c_{1,4}189\sqrt{5}+c_{1,6}162\sqrt{65}+c_{1,8}297\sqrt{85}\right)\left(\tfrac{2\sqrt{30}}{4}\right)\\
&-\left(c_{1,5}81\sqrt{77}+c_{1,7}198\sqrt{105}+c_{1,9}351\sqrt{133}\right)\left(\tfrac{2\sqrt{42}}{4}\right)\\
&+\left(c_{1,6}99\sqrt{117}+c_{1,8}234\sqrt{153}\right)\left(\tfrac{24\sqrt{6}}{16}\right)-\left(c_{1,7}117\sqrt{165}+c_{1,9}270\sqrt{209}\right)\tfrac{8\sqrt{66}}{16}\\
&+\left(c_{1,8}135\sqrt{221}\right)\tfrac{16\sqrt{78}}{32}-\left(c_{1,9}153\sqrt{285}\right)\tfrac{48\sqrt{10}}{32}=1.
\end{aligned}$$

The reminder seven equations are obtained by inserting the smaller three roots of the 11th-order shifted Legendre polynomial, $t_1=0.008698$,$t_2=0.04498$, $t_3=0.1069, t_4=0.1889$,$t_5=0.2837$,$t_6=0.383$,$t_7=0.4778$, in (25).

Solving this nonlinear $10\times 10$ system gives

$$\begin{aligned}
C_{10\times 1}&=[\,c_{1,0}\quad c_{1,1}\quad c_{1,2}\quad c_{1,\,3}\quad c_{1,\,4}\quad c_{1,5}\quad c_{1,6}\quad c_{1,7}\quad c_{1,8}\quad c_{1,9}\,]^T\\
&=[\,0.59593988797141\quad -0.113848030223\quad 0.00976752484854\\
&\quad -0.00054936985920\quad 0.00002304551981\quad -0.00000077145558\\
&\quad 0.00000002149144\quad -0.00000000051302\quad 0.00000000001061\\
&\quad 0.00000000000022\,]^T
\end{aligned}$$

Hence, the approximate solution of Example 3 using our proposed $GLWM$ is obtained as

$$
\begin{aligned}
y(t) &= C^T\Psi \\
&= [\,0.59593988797141 \quad -0.113848030223 \quad 0.00976752484854 \\
&-0.00054936985920 \quad 0.00002304551981 \quad -0.00000077145558 \\
&\ 0.00000002149144 \quad -0.00000000051302 \quad 0.00000000001061 \quad 0.00000000000022\,]^T
\end{aligned}
$$

$$
* \begin{bmatrix}
\frac{\sqrt{6}}{2} \\
\frac{3\sqrt{2}}{2}(3t-1) \\
\frac{\sqrt{30}}{4}\left(3(3t-1)^2-1\right) \\
\frac{\sqrt{42}}{4}(3t-1)\left(5(3t-1)^2-3\right) \\
\frac{3\sqrt{6}}{16}\left(35(3t-1)^4-30(3t-1)^2+3\right) \\
\frac{\sqrt{66}}{16}\left(63(3t-1)^5-70(3t-1)^3+15(3t-1)\right) \\
\frac{\sqrt{78}}{32}\left(231(3t-1)^6-315(3t-1)^4+105(3t-1)^2-5\right) \\
\frac{3\sqrt{10}}{32}\left(429(3t-1)^7-693(3t-1)^5+315(3t-1)^3-35(3t-1)\right) \\
\frac{\sqrt{102}}{256}\left(6435(3t-1)^8-12012(3t-1)^6+6930(3t-1)^4-1260(3t-1)^2+35\right) \\
\frac{\sqrt{114}}{256}\left(12155(3t-1)^9-25740(3t-1)^7+18018(3t-1)^5-4620(3t-1)^3+315(3t-1)\right)
\end{bmatrix}.
$$

Along with the absolute errors compared to the exact solution, we can evaluate the approximation at the locations in the prescribed interval, $0 \le t < \frac{2}{3}$ and summarized in the table (Table 10) below.

Table 10: Approximate solutions of [27, Example 3] using the $RLWM$ and $GLWM$ for $M = 9$

| $t$ | **Exact Solution** | **Approximate solution of** $RLWM$ $M=9;k=0$ | **Approximate solution of** $GLWM$ $M=9;k=1;\mu=3$ |
|---|---|---|---|
| 0.1 | 0.9048374180359596 | 0.9048374180282546 | 0.9048374180356493 |
| 0.2 | 0.8187307530779818 | 0.8187307530802117 | 0.8187307530782875 |
| 0.3 | 0.7408182206817179 | 0.740818220690352 | 0.7408182206816675 |
| 0.4 | 0.6703200460356393 | 0.6703200460269125 | 0.6703200460363143 |
| 0.5 | 0.6065306597126334 | 0.6065306597153067 | 0.6065306597129507 |
| 0.6 | 0.5488116360940264 | 0.5488116361078827 | 0.5488116360935682 |

In Table 11, absolute error comparisons for [27, Example 3] of the present method with the RLWM, Hermite Polynomial Collocation Method, H-CLSM, H-DLSM, Chebyshev Polynomial Collocation Method, C-CLSM and C-DLSM are as follows:

Table 11: Comparison of the absolute error for [27, Example 3] of the present method with the RLWM, Hermite Polynomial Collocation Method, H-CLSM, H-DLSM, Chebyshev Polynomial Collocation Method, C-CLSM and C-DLSM.

| t | Absolute error of $GLWM$ $M = 9$, $\mu = 3$, $k = 1$ | Absolute error of $RLWM$ $M = 9$, $k = 0$ | Absolute error of Hermite Polynomial Collocation Method | Absolute error of H-CLSM | Absolute error of H-DLSM | Absolute error of Chebyshev Polynomial Collocation Method | Absolute error of C-CLSM | Absolute error of C-DLSM |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 3.06e-13 | 2.23e-12 | 6.20e-09 | 3.38e-10 | 1.38e-12 | 3.70e-07 | 3.05e-09 | 3.53e-12 |
| 0.4 | 6.75e-13 | 8.73e-12 | 5.76e-08 | 4.85e-09 | 7.33e-12 | 2.38e-06 | 9.42e-09 | 5.78e-11 |
| 0.6 | 4.58e-13 | 1.39e-11 | 1.79e-07 | 1.07e-08 | 1.77e-11 | 5.97e-06 | 2.68e-08 | 1.78e-10 |

## 5 Conclusion

As demonstrated in this study, the current method produces more accurate findings than the other methods, especially the regular Legendre wavelets method. This method has a substantially lower maximum absolute error than the other numerical and semi-analytical ones for, simply solving, the delay and neutral differential equations with proportion at delays of different orders using our suggested $GLWM$, as demonstrated in this paper. We hope to see the same accuracy in the author's future research of fractional differential equations based on the accurate results derived from these polynomials in this work.

## Acknowledgment

We express our sincere thanks to the anonymous referees for valuable suggestions that improved the final manuscript.

## References


[1] Aboodh, K.S., Farah, R.A., Almardy, I.A. and Osman, A.K. *Solving delay differential equations by Aboodh transformation method*, International Journal of Applied Mathematics & Statistical Sciences, 7(2) (2018), 55–64.

[2] Ali, I., Brunner, H. and Tang, T. *A spectral method for pantograoh-type delay differential equations and its convergence analysis*, J. Comput. Math. 27(2-39) (2009), 254–265.

[3] Amer, H. and Olorode, O. *Numerical evaluation of a novel Slot-Drill Enhanced Oil Recovery Technology for Tight Rocks*, SPE J. 27(4) (2022), 2294–2317.

[4] Balaji, S. *Legendre wavelet operational matrix method for solution of fractional order Riccati differential equation*, J. Egypt. Math. Soc., 23 (2) (2015), 263–270.

[5] Benhammouda, B., Leal, H.V. and Martinez, L.H. *Procedure for exact solutions of nonlinear Pantograph delay differential equations*, British Journal of Mathematics and Computer Science, 4(19) (2014), 2738–2751.

[6] Bhrawy, A.H., Assas, L.M., Tohidi, E. and A. Alghamdi, M. *Legendre–Gauss collocation method for neutral functional-differential equations with proportional delays*, Adv. Differ. Eq., 63, (2013) 1–16.

[7] Biazar, J. and Ghanbari, B. *The homotopy perturbation method for solving neutral functional-differential equations with proportional delays*, J. King Saud Univ. Sci., 24 (2012), 33–37.

[8] Blanco-Cocom, L., Estrella, A.G. and Avila-Vales, E. *Solving delaydifferential systems with history functions by the Adomian decomposition method*, Appl. Math. Comput. 218 (2012), 5994–6011.

[9] Bocharova, G.A. and Rihanb, F.A. *Numerical modeling in biosciences using delay differential equations*, J. Comput. Appl. Math. 125 (2000), 183–199.

[10] Căruntu, B. and Bota, C. *Analytical approximate solutions for a general class of nonlinear delay differential equations*, Sci. World J. (2014), 631416.

[11] Chen, X. and Wang, L. *The variational iteration method for solving a neutral functional- differential equation with proportional delays*, Comput. Math. Appl., 59 (2010), 2696–2702.

[12] Davaeifar, S. and Rashidinia, J. *Solution of a system of delay differential equations of multipantograph type*, J. Taibah Univ. Sci. 11 (2017), 1141–1157.

[13] El-Shazly, N.M., Ramadan, M.A. and Radwan, T. *Generalized Legendre wavelets,definition, properties and their applications for solving linear differential equations*, Egyptian Journal of Pure and Applied Science, 62(1) (2024), 20–32.

[14] Evans, D.J. and Raslan, K.R. *The Adomian decomposition method for solving delay differential equations*, Int. J. Comput. Math., 82(1) (2005), 49–54.

[15] Gu, J.S. and Jiang, W.S. *The Haar wavelets operational matrix of integration*, Int. J. Syst. Sci., 27 (7) (1996), 623–628.

[16] Gümgüm, S., Özdek, D.E. and Özaltun, G. *Legendre wavelet solution of high order nonlinear ordinary delay differential equations*, Turk. J. Math. 43 (2019), 1339–1352.

[17] Gümgüm, S., Özdek, D., Özaltun, E.G. and Bildik, N. *Legendre wavelet solution of neutral differential equations with proportional delays*,J. Appl. Math. Comput. 61 (2019), 389–404.

[18] Ha, P. *Analysis and numerical solutions of delay differential-algebraic equations*, Ph. D, Technical University Of Berlin, Berlin, Germany, 2015.

[19] Khader, M.M. *Numerical and theoretical treatment for solving linear and nonlinear delay differential equations using variational iteration method*, Arab Journal of Mathematical Sciences, 19(2) (2013), 243–256.

[20] Lv, X. and Gao, Y. *The RKHSM for solving neutral functional-differential equations with proportional delays*, Math. Methods Appl. Sci., 36 (2013), 642–649.

[21] Martin, J.A. and Garcia, O. *Variable multistep methods for delay differential equations*, Math. Comput. Model. 35(2002), 241–257.

[22] Mirzaee, F. and Latifi, L. *Numerical solution of delay differential equations by differential transform method*, Journal of Sciences (Islamic Azad University), 20(78/2) (2011), 83–88.

[23] Mohammadi, F. and Hosseini, M.M. *A new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations*, J. Frankl. Inst., 348 (2011), 1787–1796.

[24] Nisar, K.S., Ilhan, O. A., Manafian, J., Shahriari, M. and Soybaş, D. *Analytical behavior of the fractional Bogoyavlenskii equations with conformable derivative using two distinct reliable methods*, Results i Phys. 22 (2021), 103975.

[25] Oberle, H.J. and Pesch, H.J. *Numerical treatment of delay differential equations by Hermite interpolation*, Numer. Math. 37 (1981), 235–255.

[26] Ogunfiditimi, F.O. *Numerical solution of delay differential equations using the Adomian decomposition method*, nt. J. Eng. Sci. 4(5)(2015), 18–23.

[27] Pushpam, A.E.K. and Kayelvizhi, C. *Solving delay differential equations using least square method based on successive integration technique*, Mathematical Statistician and Engineering Applications, 72(1) (2023), 1104–1115.

[28] Ravi-Kanth, A.S.V. and Kumar, P.M.M. *A numerical technique for solving nonlinear singularly perturbed delay differential equations*, Math. Model. Anal. 23(1) (2018), 64–78.

[29] Sakar, M.G. *Numerical solution of neutral functional-differential equations with proportional delays*, Int. J. Optim. Control Theor. Appl., 7(2) (2017), 186–194.

[30] Sedaghat, S., Ordokhani, Y. and Dehghan, M. *Numerical solution of delay differential equations of pantograph type via Chebyshev polynomials*, Commun. Nonlinear Sci. Numer. Simul., 17 (2012), 4815–4830.

[31] Shakeri, F. and Dehghan, M. *Solution of delay differential equations via a homotopy perturbation method*, Math. Comput. Model. 48(2008), 486–498.

[32] Shiralashetti, S.C., Hoogarand, B.S. and Kumbinarasaiah, S. *Hermite wavelet based‘ method for the numerical solution of linear and nonlinear delay differential equations*, International Journal of Engineering Science and Mathematics, 6(8) (2017), 71–79.

[33] Stephen, A. G. and Kuang, Y. *A delay reaction-diffusion model of the spread of bacteriophage infection*, Society for Industrial and Applied Mathematics, 65(2) (2005), 550–566,

[34] Taiwo, O.A. and Odetunde, O.S. *On the numerical approximation of delay differential equations by a decomposition method* , Asian Journal of Mathematics & Statistics, 3(4)(2010), 237–243.

[35] Vanani, S.K. and Aminataei, A. *On the numerical solution of nonlinear delay differential equations*, Journal of Concrete and Applicable Mathematics, 8(4)(2010), 568–576.

[36] Wang, W. and Li, S. *On the one-leg-methods for solving nonlinear neutral functional differential equations*, Appl. Math. Comput., 193(1) (2007), 285–301.

[37] Yousefi, S.A. *Legendre scaling function for solving generalized Emden–Fowler equations*, Int. J. Inf. Syst. Sci. 3 (2007), 243–250.

[38] Yüzbaşı, Ş. *A numerical approach for solving a class of the nonlinear Lane-Emden type equations arising in astrophysics*, Math. Method. Appl. Sci. 34 (2011), 2218–2230.

[39] Yüzbaşı, Ş. *An efficient algorithm for solving multi-pantograph equation system*, Comput. Math. Appl. 64 (2012), 589–603.

[40] Yüzbaşı, Ş. *A numerical approximation based on the Bessel functions of first kind for solutions of Riccati type differential-difference equations*, Comput. Math. Appl., 64 (2012), 1691–1705.

[41] Yüzbaşı, Ş. *Shifted Legendre method with residual error estimation for delay linear Fredholm integro-differential equations*, J. Taibah Uni. Sci. 11(2)(2017), 344–352.

[42] Yüzbaşı, Ş. *A numerical scheme for solutions of a class of nonlinear differential equations*, J. Taibah Uni. Sci. 11 (2017), 1165–1181.

[43] Yüzbaşı, Ş. and Şahin, N. *On the solutions of a class of nonlinear ordinary differential equations by the Bessel polynomials*, J. Numer. Math. 20(1) (2012), 55–79.

[44] Yüzbaşı, Ş. and Sezer, M. *Shifted Legendre approximation with the residual correction to solve pantograph-delay type differential equations*, Appl. Math. Model. 39 (2015), 6529–6542.

[45] Zhang, M., Xie, X., Manafian, J., Ilhan, O.A. and Singh, G. *Characteristics of the new multiple rogue wave solutions to the fractional generalized CBS-BK equation*, J. Adv. Res. 38 (2022), 131–142.