Iranian Journal of Numerical Analysis and Optimization
Vol. 9, No. 1, (2019), pp 127–150
DOI:10.22067/ijnao.v9i1.63182

Regularization technique and numerical analysis of the mixed system of first and second-kind Volterra–Fredholm integral equations

S. Pishbin* and J. Shokri

Abstract

It is important to note that mixed systems of first and second-kind Volterra–Fredholm integral equations are ill-posed problems, so that solving discretized system of such problems has a lot of difficulties. We will apply the regularization method to convert this mixed system (ill-posed problem) to system of the second kind Volterra–Fredholm integral equations (well-posed problem). A numerical method based on Chebyshev wavelets is suggested for solving the obtained well-posed problem, and convergence analysis of the method is discussed. For showing efficiency of the method, some test problems, for which the exact solution is known, are considered.

Keywords: Mixed systems of first and second-kind Volterra–Fredholm integral equations; Regularization method; Chebyshev wavelets; Convergence analysis.

1 Introduction

The Volterra–Fredholm integral equations [4, 5, 7, 13] arise from parabolic boundary value problems, from the mathematical modeling of the spatiotemporal development of an epidemic, mathematical population dynamics, and from various physical and biological models.

*Corresponding author
Received 8 March 2017; revised 29 August 2018; accepted 15 October 2018
S. Pishbin
Department of Mathematics, Faculty of Sciences, Urmia University, Urmia, P. O. Box 165, Iran. e-mail: s.pishbin@urmia.ac.ir

J. Shokri
Department of Mathematics, Faculty of Sciences, Urmia University, Urmia, P. O. Box 165, Iran. j.shokri@urmia.ac.ir

In this paper, we consider mixed system of Volterra–Fredholm integral equations (VFIEs) consisting of first and second-kind VFIEs as follows:

$$F(t,x) = AU(t,x) + \int_0^t \int_\Omega \mathbf{K}(t,\eta,x,s)U(\eta,s)dsd\eta, \quad t \in I = [0,T], \quad x \in \Omega, \tag{1}$$

where $F(t,x) = [f(t,x), g(t,x)]^T,\ U(t,x) = [u_1(t,x), u_2(t,x)]^T$ and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{K}(t,\eta,x,s) = \begin{bmatrix} k_{11}(t,\eta,x,s) & k_{12}(t,\eta,x,s) \\ k_{21}(t,\eta,x,s) & k_{22}(t,\eta,x,s) \end{bmatrix}.$$

Here, Ω denotes a (closed) bounded region in $\mathbb{R}^d$ $(d = 1, 2, 3)$ with the (piecewise) smooth boundary $\partial\Omega$.

The reformulation of the initial-boundary-value problem for the linear heat equation in a two-dimensional spatial domain Ω with the boundary $\partial\Omega$ by single-layer techniques leads to a mixed system of Volterra–Fredholm integral equations. Mixed systems of VFIEs are considered as the ill-posed problems. However, we will first apply the method of regularization that received a considerable amount of interest, especially in solving first kind integral equations. The method transforms mixed system to the system of second kind integral equations. The method of regularization was established independently by Phillips [12] and Tikhonov [14]. The method of regularization consists of replacing ill-posed problem by well-posed problem. Some numerical methods have been proposed for solving Volterra–Fredholm integral equations of the second kind; see, for example, [2, 6, 8, 10, 11, 15, 16]. In this paper, the wavelet collocation method is developed for a mixed system of first and second kind Volterra–Fredholm integral equations.

The paper is organized as follows. In Section 2, we consider some applications of Volterra–Fredholm integral equation and in Section 3, we introduce regularization technique to transform mixed system (1) into the system of second kind integral equations. In Section 4, a numerical method based on Chebyshev wavelets is applied for solving the obtained well-posed problem. The convergence analysis is given in Section 5 and the numerical experiments are carried out in Section 6, which will be used to verify the theoretical results.

2 Some applications

We can consider Volterra–Fredholm integral equation of the first kind as a spacial case of system (1)(let $u_1 = 0, k_{12} = 0$). The reformulation of the initial-boundary-value problem for the linear heat equation in a two-dimensional spatial domain Ω with boundary $\partial\Omega$ by singlelayer techniques leads to a Volterra–Fredholm integral equation of the first kind in the following form [3]:

$$g_\Gamma(t,\theta) = \int_0^t \int_0^1 \mathbf{K}(t-s, X(\theta)-X(s))U(s,\phi)d\phi ds,$$

where $X(\theta)$ is a smooth 1-periodic parametric representation of the boundary curve $\Gamma = \partial\Omega$, and g_Γ represents the function describing the given boundary condition on $I \times \partial\Omega$.

For the another example, you can consider the following integral equation [1]

$$f(x,t) = (\Lambda(t) - f_*(x)) = \int_0^t \int_\Omega F(t,\tau)k(x,y)\Phi(y,t)dyd\tau,$$

$$(x = \bar{x}(x_1,x_2,x_3), y = \bar{y}(y_1,y_2,y_3), (x,y) \in \Omega, t \in [0,T],)$$

under the condition

$$\int_\Omega \Phi(x,t)dx = P(t),$$

can be investigated from the contact problem of a rigid surface (G,ν) having an elastic material occupying the domain Ω where $f_*(x)$ describing the surface of stamp. This stamp impressed into an elastic layer surface (plane) by a variable known force with respect to time $P(t)$ whose eccentricity of application $e(t), (t \in [0,T])$ that case rigid displacement $\Lambda(t)$. Here G is the displacement magnitude and ν is Poisson's coefficient.

3 Regularization technique

The linear operator defined by second equation in system (1) has not generally a continuous inverse, so that it is difficult to obtain a precise numerical solution by classical discretization methods. Thus regularization techniques could be used instead to transform integral equations such as second equation in system (1) into second-kind integral equations. More precisely, we consider the following integral equations:

$$\begin{cases} f(t,x) = u_1(t,x) + \int_0^t \int_\Omega k_{11}(t,\eta,x,s)u_1(\eta,s)dsd\eta \\ \qquad\qquad + \int_0^t \int_\Omega k_{12}(t,\eta,x,s)u_{2,\varepsilon}(\eta,s)dsd\eta, \\ g(t,x) = \varepsilon u_{2,\varepsilon}(t,x) + \int_0^t \int_\Omega k_{21}(t,\eta,x,s)u_1(\eta,s)dsd\eta \\ \qquad\qquad + \int_0^t \int_\Omega k_{22}(t,\eta,x,s)u_{2,\varepsilon}(\eta,s)dsd\eta, \end{cases} \tag{2}$$

where ε is a fixed positive number.

Under a hypothesis that will be clarified later, it can be proved that the solution $u_{2,\varepsilon}(x,t)$ of equation (2) converges (when $\varepsilon \to 0$) to the solution $u_2(x,t)$ of equation (1). Now, we consider the matrix form of equation (2) as the following system of integral equations:

$$F(t,x) = EU_\varepsilon(t,x) + \int_0^t \int_\Omega \mathbf{K}(t,\eta,x,s)U_\varepsilon(\eta,s)dsd\eta, \tag{3}$$

where $E = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$ and $U_\varepsilon(x,t) = [u_1(x,t), u_{2,\varepsilon}(x,t)]^T$.

We need the following definition and lemma to regularization technique.

Definition 1. *A self-adjoint operator $\kappa : H \to H$, where H is a real Hilbert space, is called coercive if there exists a constant $c > 0$ such that*

$$\langle \kappa x, x \rangle \geq c\|x\|^2, \quad \textit{for all } x \in H.$$

Lemma 1. *Suppose that the integral operator of system (3) is continuous and coercive in a Hilbert space, where F, U, U_ε are defined. Then*

- *$\|U_\varepsilon\|$ is bounded independently of ε;*
- *$\|U_\varepsilon - U\|$ tends to 0 when $\varepsilon \to 0$.*

Proof. From (3) we conclude the following:

$$\|E\|\|U_\varepsilon\| = \|F - \int_0^t \int_\Omega \mathbf{K}U_\varepsilon dsd\eta\| \geqslant -\|F\| + \|\int_0^t \int_\Omega \mathbf{K}U_\varepsilon dsd\eta\|, \tag{4}$$

The coercivity property of the integral operator implies

$$\|\int_0^t \int_\Omega \mathbf{K}U_\varepsilon dsd\eta\| \geqslant \alpha\|U_\varepsilon\|, \tag{5}$$

where α is the coercivity constant.

From (4) and (5) we deduce

$$\|E\|\|U_\varepsilon\| \geqslant \alpha\|U_\varepsilon\| - \|F\|; \tag{6}$$

therefore it is obtained:

$$(\alpha - \|E\|)\|U_\varepsilon\| \leqslant \|F\|,$$

which proves the first part of the lemma.

Now, for proving the second result, by using (1) and (3), we have

$$E(U_\varepsilon - U) = -\int_0^t \int_\Omega \mathbf{K}(t,\eta,x,s)[U_\varepsilon(\eta,s) - U(\eta,s)]dsd\eta + \tilde{U}_1 - EU, \tag{7}$$

where $\tilde{U}_1(x,t) = [u_1(x,t), 0]^T$. By rearranging (7), we can write

$$\int_0^t \int_\Omega \mathbf{K}(t,\eta,x,s)[U_\varepsilon(\eta,s) - U(\eta,s)]dsd\eta = -E(U_\varepsilon - U) - \varepsilon\tilde{U}_2, \quad (8)$$

where, by using vectors operations $EU - \tilde{U}_1 = \varepsilon\tilde{U}_2$ is obtained, which $\tilde{U}_2 = [0, u_2]^T$. Taking the norm from the both side of (8) and using the coercivity property imply

$$\alpha\|U_\varepsilon - U\| \leqslant \|E(U_\varepsilon - U) - \varepsilon\tilde{U}_2\| \leqslant \|E\|\|U_\varepsilon - U\| + \varepsilon\|\tilde{U}_2\|. \quad (9)$$

From $\|\tilde{U}_2\| = \|u_2\|$, we deduce that

$$(\alpha - \|E\|)\|U_\varepsilon - U\| \leqslant \varepsilon\|u_2\|, \quad (10)$$

and now from $\|E\| = 1 + \varepsilon \to 1$ when $\varepsilon \to 0$ and $\alpha \gg 1$ then it is concluded that $\|U_\varepsilon - U\| \to 0$ when $\varepsilon \to 0$. This completes the proof. □

The conditions of existence and uniqueness of solutions related to the VFIEs (3) can be investigated by considering the theorem about existence and uniqueness of solution of the second-kind Volterra–Fredholm integral equation in [3].

Theorem 1. *Assume that*
1. $F \in C(I \times \Omega)$
2. $\mathbf{K} \in C(D \times \Omega^2)$ *where* $D = \{(t,\eta), 0 \le \eta \le t \le T\}$.
Then the mixed system (3) possesses a unique solution $U_\varepsilon \in C(I \times \Omega)$.

4 Numerical treatment

4.1 Chebyshev wavelets

Dilations and translations of the "Mother function," or "analyzing wavelet" $\Phi(x)$ define an orthogonal basis, our wavelet basis:

$$\Phi_{(s,l)}(t) = 2^{\frac{-s}{2}}\Phi(2^{-s}t - l),$$

the variables s and l are integers that scale and dilate the mother function Φ to generate wavelets, such as a Daubechies wavelet family. The scale index s indicates wavelet's width, and the location index l gives its position. Notice that the mother functions are rescaled, or "dilated" by powers of two, and translated by integers. What makes wavelet bases especially interesting is the self-similarity caused by the scales and dilations. Once we know about the mother functions, we know everything about the basis. Chebyshev wavelets

$\phi_{(n,m)} = \phi(k,n,m,t)$ have four arguments, $n = 1, 2, \ldots, 2^{k-1}$, where k can assume any positive integer and m is the degree of Chebyshev polynomials of the first kind. They are defined on the interval $[0,1)$ as

$$\phi_{(n,m)}(t) = \begin{cases} 2^{\frac{k}{2}}\hat{T}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & otherwise, \end{cases} \tag{11}$$

where

$$\hat{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}}T_m(t), & m > 0, \end{cases}$$

and $m = 0, 1, \ldots, M-1, \quad n = 1, 2, \ldots, 2^{k-1}$. Also $T_m(t)$ are Chebyshev polynomials of degree m which are orthogonal with respect to the function $w(t) = \frac{1}{\sqrt{1-t^2}}$ on the interval $[-1, 1]$. We consider the two-dimensional Chebyshev wavelet $\phi_{(n_1,m_1,n_2,m_2)}(x, y)$ as follows:

$$\begin{cases} 2^{\frac{k_1+k_2}{2}}\hat{T}_{m_1}(2^{k_1}x - 2n_1 + 1)\hat{T}_{m_2}(2^{k_2}y - 2n_2 + 1), & \frac{n_1-1}{2^{k_1-1}} \le x < \frac{n_1}{2^{k_1-1}}, \\ & \frac{n_2-1}{2^{k_2-1}} \le y < \frac{n_2}{2^{k_2-1}}, \\ 0, & otherwise, \end{cases} \tag{12}$$

where $m_1 = 0, 1, \ldots, M_1 - 1$, $m_2 = 0, 1, \ldots, M_2 - 1$, $n_1 = 1, \ldots, 2^{k_1-1}$, and $n_2 = 1, \ldots, 2^{k_2-1}$.

A function $u(x, y) \in L^2([0, 1) \times [0, 1))$ may be expanded as

$$u(x, y) = \sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty} u_{n_1m_1n_2m_2}\phi_{(n_1,m_1,n_2,m_2)}(x, y). \tag{13}$$

If the infinite series in equation (13) is truncated, then

$$u(x, y) \approx \sum_{n_1=1}^{2^{k_1-1}}\sum_{m_1=0}^{M_1-1}\sum_{n_2=1}^{2^{k_2-1}}\sum_{m_2=0}^{M_2-1} u_{n_1m_1n_2m_2}\phi_{(n_1,m_1,n_2,m_2)}(x, y) = \Phi(x)^T U\Phi(y). \tag{14}$$

where $\Phi(x)$ and $\Phi(y)$ are $(2^{k_1-1})M_1 \times 1$ and $(2^{k_2-1})M_2 \times 1$ matrices, respectively, such that

$$\Phi(x) = [\phi_{(1,0)}(x), \ldots, \phi_{(1,M_1-1)}(x), \ldots, \phi_{(2^{k_1-1},0)}(x), \ldots, \phi_{(2^{k_1-1},M_1-1)}(x)]^T,$$

$$\Phi(y) = [\phi_{(1,0)}(y), \ldots, \phi_{(1,M_2-1)}(y), \ldots, \phi_{(2^{k_2-1},0)}(y), \ldots, \phi_{(2^{k_2-1},M_2-1)}(y)]^T.$$

Also U is a $(2^{k_1-1})M_1 \times (2^{k_2-1})M_2$ matrix whose elements can be calculated as

$$u_{n_1m_1n_2m_2} = \int_0^1\int_0^1 \phi_{(n_1,m_1)}(x)\phi_{(n_2,m_2)}(y)u(x, y)w_{n1}(x)w_{n2}(y)dydx,$$

where $n_1 = 1, \dots, 2^{k_1-1}$, $m_1 = 0, 1, \dots, M_1 - 1$, $n_2 = 1, \dots, 2^{k_2-1}$, $m_2 = 0, 1, \dots, M_2 - 1$ and

$$w_n(t) = \begin{cases} w(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & otherwise \end{cases}$$

Now, consider system (2) with $I = \Omega = [0, 1]$ and approximate the solution of this system using (13) as

$$\hat{u}_1(t, x) = \Phi(t)^T U_1 \Phi(x). \tag{15}$$

$$\hat{u}_{2,\varepsilon}(t, x) = \Phi(t)^T U_{2,\varepsilon} \Phi(x). \tag{16}$$

Also, we approximate the kernels in system (2) respect to two variables η and s as follows:

$$k_{ij}(t, \eta, x, s) \approx \hat{k}_{ij}(t, \eta, x, s) = \Phi(\eta)^T K_{ij} \Phi(s) \quad (i, j = 1, 2). \tag{17}$$

Inserting (15), (16), (17), and the following collocation points

$$\begin{aligned} t_i &= \frac{i}{M_1 2^{k_1-1} + 1}, \quad i = 1, 2, \dots, M_1 2^{k_1-1}, \\ x_j &= \frac{j}{M_1 2^{k_2-1} + 1}, \quad j = 1, 2, \dots, M_2 2^{k_2-1}, \end{aligned} \tag{18}$$

into system (2), we have the following linear system of algebraic equations for the unknown coefficients U_1 and $U_{2,\varepsilon}$:

$$\begin{cases} f(t_i, x_j) = \Phi(t_i)^T U_1 \Phi(x_j) + \int_0^{t_i} \int_0^1 \hat{k}_{11}(t_i, \eta, x_j, s) \Phi(\eta)^T U_1 \Phi(s) ds d\eta \\ \qquad + \int_0^{t_i} \int_0^1 \hat{k}_{12}(t_i, \eta, x_j, s) \Phi(\eta)^T U_{2,\varepsilon} \Phi(s) ds d\eta, \\ g(t_i, x_j) = \varepsilon \Phi(t_i)^T U_{2,\varepsilon} \Phi(x_j) + \int_0^{t_i} \int_0^1 \hat{k}_{21}(t_i, \eta, x_j, s) \Phi(\eta)^T U_1 \Phi(s) ds d\eta \\ \qquad + \int_0^{t_i} \int_0^1 \hat{k}_{22}(t_i, \eta, x_j, s) \Phi(\eta)^T U_{2,\varepsilon} \Phi(s) ds d\eta, \\ \qquad i = 1, 2, \dots, M_1 2^{k_1-1}, \quad j = 1, 2, \dots, M_2 2^{k_2-1}. \end{cases} \tag{19}$$

4.2 Normalization

In the language of optimization theory, for the linear bounded operator $K : X \to Y$ and $Kx = y$, determine x_ε that minimizes the Tikhonov functional

$$J_\varepsilon(x) = \|Kx - y\|^2 + \varepsilon\|x\|^2, \quad \text{for all } x \in X.$$

We can consider the following theorem from [9].

Theorem 2. *Let $K : X \to Y$ be a linear bounded operator between Hilbert spaces and let $\epsilon > 0$. Then the Thikhonov functional J_ε has a unique minimum $x_\varepsilon \in X$. This minimum x_ε is the unique solution of the normal equation*

$$\varepsilon x_\varepsilon + K^* K x_\varepsilon = K^* y.$$

Here, $K^* : Y \to X$ denotes the adjoint of K.

By using Theorem 2, system (2) with $I = \Omega = [0,1]$ can be written in the normal form as follows:

$$\begin{cases} f(p,q) = u_1(p,q) + \displaystyle\int_0^p \int_0^1 k_{11}(p,\eta,q,s) u_1(\eta,s) ds d\eta \\ \qquad\qquad\qquad + \displaystyle\int_0^p \int_0^1 k_{12}(p,\eta,q,s) u_{2,\varepsilon}(\eta,s) ds d\eta, \\ \displaystyle\int_p^1 \int_0^1 k_{22}(p,t,q,x) g(t,x) dx dt \\ \qquad = \varepsilon u_{2,\varepsilon}(p,q) \\ \qquad\quad + \displaystyle\int_p^1 \int_0^1 \int_0^t \int_0^1 k_{22}(p,t,q,x) k_{21}(t,\eta,x,s) u_1(\eta,s) ds d\eta dx dt \\ \qquad\quad + \displaystyle\int_p^1 \int_0^1 \int_0^t \int_0^1 k_{22}(p,t,q,x) k_{22}(t,\eta,x,s) u_{2,\varepsilon}(\eta,s) ds d\eta dx dt, \end{cases} \tag{20}$$

Now, we can consider the numerical method based on Chebyshev wavelets from the previous subsection for the approximate solution of system (20).

5 Convergence analysis

In this section, we investigate the convergence analysis of the proposed Chebyshev wavelet collocation method, using polynomial approximation theory.

Lemma 2. *Assume that $u(x,y) \in L^2([0,1) \times [0,1))$ can be expanded in the form of series (13) and that $\hat{u}(t,x)$ is the approximation of $u(x,y)$ which is defined by (14). Then $\hat{u}(t,x)$ converges to $u(t,x)$.*

Proof. We recall the series $u(t,x)$ from (13) and the truncated series $\hat{u}(t,x)$ from (14), respectively, as follows:

$$u(x,y) = \sum_{n_1=1}^{\infty} \sum_{m_1=0}^{\infty} \sum_{n_2=1}^{\infty} \sum_{m_2=0}^{\infty} u_{n_1 m_1 n_2 m_2} \phi_{(n_1,m_1,n_2,m_2)}(x,y),$$

and

$$\hat{u}(t,x) = \sum_{n_1=1}^{2^{k_1-1}} \sum_{m_1=0}^{M_1-1} \sum_{n_2=1}^{2^{k_2-1}} \sum_{m_2=0}^{M_2-1} u_{n_1 m_1 n_2 m_2} \phi_{(n_1,m_1,n_2,m_2)}(t,x),$$

where $m_1 = 0,1,\ldots,M_1-1,\ m_2 = 0,1,\ldots,M_2-1,\ n_1 = 1,\ldots,2^{k_1-1}$, and $n_2 = 1,\ldots,2^{k_2-1}$. The functions $\phi_{(n_1,m_1,n_2,m_2)}(t,x)$ are the Chebyshev wavelet. Let $L^2([0,1)\times[0,1))$ be the Hilbert space and let

$$\phi_{(n_1,m_1,n_2,m_2)}(t,x) = 2^{\frac{-(n_1+n_2)}{2}} \phi(2^{\frac{-n_1}{2}} t - m_1)\phi(2^{\frac{-n_2}{2}} x - m_2),$$

where $\phi_{(n_1,m_1,n_2,m_2)}(t,x)$ form a basis of $L^2([0,1)\times[0,1))$ as in (12).
From (13), we consider

$$u(t,x) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} u_{1m_1 1m_2} \phi_{(1,m_1,1,m_2)}(t,x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \phi_{(i,j)}(t,x),$$

where $c_{ij} =< u(t,x), \phi_{(i,j)}(t,x) >$ for $k_1 = 1, k_2 = 1$ and $< .,. >$ represents an inner product. Let us denote $\phi_{(i,j)}(t,x) = \phi(t,x)$ and $\alpha_{ij} =< u(t,x), \phi(t,x) >$.

Define the sequence of partial sums $S_{n,m}$, $n > m$ of $\{\alpha_{ij}\phi(t_i,x_j)\}$. Let S_{n_1,m_1} and S_{n_2,m_2} be arbitrary partial sums with $n_1 > n_2$. We show that $S_{n,m}$ is a Cauchy sequence in a Hilbert space.

Let $S_{n,m} = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \alpha_{ij}\phi(t_i,x_j)$. From

$$< u(t,x), \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \alpha_{ij}\phi(t_i,x_j) >= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} |\alpha_{ij}|^2,$$

we can show that $\|S_{n_1,n_2} - S_{m_1,m_2}\|^2 = \sum_{i=m_1+1}^{n_1} \sum_{j=m_2+1}^{n_2} |\alpha_{ij}|^2$ for $n_1 > m_1,\ n_2 > m_2$.

We can write

$$\begin{aligned} &\| \sum_{i=m_1+1}^{n_1} \sum_{j=m_2+1}^{n_2} \alpha_{ij}\phi(t_i,x_j)\|^2 \\ &\quad =< \sum_{i=m_1+1}^{n_1} \sum_{j=m_2+1}^{n_2} \alpha_{ij}\phi(t_i,x_j), \sum_{i=m_1+1}^{n_1} \sum_{j=m_2+1}^{n_2} \alpha_{ij}\phi(t_i,x_j) > \\ &\quad = \sum_{i=m_1+1}^{n_1} \sum_{j=m_2+1}^{n_2} |\alpha_{ij}|^2, \end{aligned}$$

for $n_1 > m_1, n_2 > m_2$. Then it is concluded that

$$\| \sum_{i=m_1+1}^{n_1} \sum_{j=m_2+1}^{n_2} \alpha_{ij}\phi(t_i, x_j)\|^2 = \sum_{i=m_1+1}^{n_1} \sum_{j=m_2+1}^{n_2} |\alpha_{ij}|^2,$$

for $n_1 > m_1, n_2 > m_2$.
Now from Bessel's inequality, we deduce that $\sum_{i=m_1+1}^{n_1} \sum_{j=m_2+1}^{n_2} |\alpha_{ij}|^2$ is convergent and therefore $\| \sum_{i=m_1+1}^{n_1} \sum_{j=m_2+1}^{n_2} \alpha_{ij}\phi(t_i, x_j)\|^2 \to 0$,
that is, $\| \sum_{i=m_1+1}^{n_1} \sum_{j=m_2+1}^{n_2} \alpha_{ij}\phi(t_i, x_j)\| \to 0$ and $\{S_{n,m}\}$ is a Cauchy sequence thus it converges to a real number like 's'.
In the continuance, we prove that $u(t, x) = s$.

$$\begin{aligned} < s - u(t,x), \phi(t_i, x_i) > &=< s, \phi(t_i, x_i) > - < u(t,x), \phi(t_i, x_i) > \\ &=< \lim_{n\to\infty} S_{n,m}, \phi(t_i, x_i) > -\alpha_{ij} \\ &= \lim_{n\to\infty} < S_{n,m}, \phi(t_i, x_i) > -\alpha_{ij} = \alpha_{ij} - \alpha_{ij} = 0; \end{aligned}$$

then it is concluded that $u(t, x) = s$ or $\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \alpha_{ij}\phi(t_i, x_j)$ converges to $u(t, x)$, and by induction it is evident that this result is established for any integer numbers of k_1 and k_2, so for $k_1 \to \infty$ and $k_2 \to \infty$. In the other hand, we can consider $\hat{u}(t, x)$ as $S_{n,m}$, which means $\hat{u}(t, x)$ converges to $u(t, x)$ and this completes the proof. □

Theorem 3. *Consider a function $u(x, y) \in L^2([0, 1] \times [0, 1])$, with bounded forth partial derivations, $|\frac{\partial^4 u(x,y)}{\partial^2 x \partial^2 y}| < B$. Then the wavelet coefficient, $u_{n_1 m_1 n_2 m_2}$ in (14) , decay as follows:*

$$|u_{n_1 m_1 n_2 m_2}| \leq \frac{\pi B}{2^4 (n_1 n_2)^{5/2} (m_1^2 - 1)(m_2^2 - 1)}. \tag{21}$$

Proof. From (14) it follows that

$$\begin{aligned} u_{n_1 m_1 n_2 m_2} &= \int_0^1 \int_0^1 u(x,y)\phi_{(n_1,m_1)}(x)\phi_{(n_2,m_2)}(y) w_{n_1}(x) w_{n_2}(y) dy dx \\ &= \int_0^1 \phi_{(n_1,m_1)}(x) w_{n_1}(x) \Big(\int_0^1 u(x,y)\phi_{(n_2,m_2)}(y) w_{n_2}(y) dy \Big) dx. \\ &= \int_0^1 \phi_{(n_1,m_1)}(x) w_{n_1}(x) \Big(\int_{(n_2-1)/2^{k_2-1}}^{n_2/2^{k_2-1}} 2^{k_2/2} u(x,y) \\ &\qquad \hat{T}_{m_2}(2^{k_2} y - 2n_2 + 1) \\ &\qquad w_{n_2}(2^{k_2} y - 2n_2 + 1)\, dy \Big) dx. \end{aligned} \tag{22}$$

If $m_1, m_2 > 1$, by substituting $2^{k_2} y - 2n_2 + 1 = \cos\theta$ in (22), it yields

$$\begin{aligned}u_{n_1m_1n_2m_2} &= \frac{1}{2^{k_2/2}}\int_0^1 \phi_{(n_1,m_1)}(x)w_{n_1}(x)\Big[\int_0^\pi u(x,\frac{\cos\theta_2 n_2-1}{2^{k_2}})\sqrt{\frac{2}{\pi}}\cos m_2\theta\,d\theta\Big]\\ &= \frac{\sqrt{2}}{2^{k_2/2}\sqrt{\pi}}\int_0^1 \phi_{(n_1,m_1)}(x)w_{n_1}(x)\Big[u(x,\frac{\cos\theta+2n_2-1}{2^{k_2}})(\frac{\sin m_2\theta}{m_2})\Big|_0^\pi\\ &\quad+\frac{\sqrt{2}}{2^{3k_2/2}m_2\sqrt{\pi}}\int_0^\pi \frac{\partial u(x,(\cos\theta_2 n_2-1)/2^{k_2})}{\partial y}\sin m_2\theta\sin\theta\,d\theta\Big]\,dx\\ &= \frac{1}{2^{3k_2/2}m_2\sqrt{2\pi}}\int_0^1 \phi_{(n_1,m_1)}(x)w_{k_1}(x)\Big[\int_0^\pi \frac{\partial u(x,(\cos\theta_2 n_2-1)/2^{k_2})}{\partial y}\\ &\qquad\Big(\frac{\sin(m_2-1)\theta}{m_2-1}-\frac{\sin(m_2+1)\theta}{m_2+1}\Big)\Big|_0^\pi\\ &\quad+\frac{1}{2^{5k_2/2}m_2\sqrt{2\pi}}\int_0^\pi \frac{\partial^2 u(x,(\cos\theta+2n_2-1)/2^{k_2})}{\partial^2 y}h_{m_2}(\theta)\,d\theta\Big]\,dx,\end{aligned}\tag{23}$$

where

$$h_m(\theta)=\sin\theta\Big(\frac{\sin(m_2-1)\theta}{m_2-1}-\frac{\sin(m_2+1)\theta}{m_2+1}\Big).$$

Then, we obtain

$$u_{n_1m_1n_2m_2}=\frac{1}{2^{5k_2/2}m_2\sqrt{2\pi}}\int_0^\pi\Big[\int_0^1\frac{\partial^2 u(x,(\cos\theta+2n_2-1)/2^{k_2})}{\partial^2 y}\phi_{(n_1,m_1)}(x)w_{k_1}(x)\,dx\Big]h_{m_2}(\theta)\,d\theta.\tag{24}$$

Now, similar to the discussion in (23), by substituting $2^{k_1}y-2n_1+1=\cos\alpha$ in (24), it yields

$$\begin{aligned}&u_{n_1m_1n_2m_2}\\ &\quad=\frac{1}{2^{5(k_1+k_2)/2}m_1m_2(2\pi)}\\ &\quad\times\int_0^\pi\int_0^\pi\frac{\partial^4 u((\cos\alpha+2n_1-1)/2^{k_1},(\cos\theta+2n_2-1)/2^{k_2})}{\partial^2 x\partial^2 y}h_{m_1}(\alpha)h_{m_2}(\theta)\,d\alpha\,d\theta.\end{aligned}$$

Thus we have

$$\begin{aligned}&|u_{n_1m_1n_2m_2}|\\ &\quad=\frac{1}{2^{5(k_1+k_2)/2}m_1m_2(2\pi)}\\ &\quad\Big|\int_0^\pi\int_0^\pi\frac{\partial^4 u((\cos\alpha+2n_1-1)/2^{k_1},(\cos\theta+2n_2-1)/2^{k_2})}{\partial^2 x\partial^2 y}h_{m_1}(\alpha)h_{m_2}(\theta)\,d\alpha\,d\theta\Big|\\ &\quad\le\frac{B}{2^{5(k_1+k_2)/2}m_1m_2(2\pi)}\int_0^\pi|h_{m_1}(\alpha)|\,d\alpha\int_0^\pi|h_{m_2}(\theta)|\,d\theta.\end{aligned}\tag{25}$$

However

$$\begin{aligned}\int_0^\pi|h_{m_2}(\theta)|\,d\theta&=\int_0^\pi\Big|\sin\theta\Big(\frac{\sin(m_2-1)\theta}{m_2-1}-\frac{\sin(m_2+1)\theta}{m_2+1}\Big)\Big|d\theta\\ &\le\int_0^\pi\Big|\frac{\sin\theta\sin(m_2-1)\theta}{m_2-1}\Big|+\Big|\frac{\sin\theta\sin(m_2+1)\theta}{m_2+1}\Big|\\ &\le\frac{2m_2\pi}{(m_2^2-1)},\end{aligned}\tag{26}$$

and similarly, it is obtained

$$\int_0^\pi |h_{m_1}(\alpha)|\, d\alpha \le \frac{2m_1\pi}{(m_1^2-1)}. \tag{27}$$

Since $n_1 \le 2^{k_1-1}$ and $n_2 \le 2^{k_2-1}$, by substituting (26) and (27) in (25), the desired result is obtained as follows:

$$|u_{n_1m_1n_2m_2}| \le \frac{2\pi B}{2^{5(k_1+k_2)/2}(m_1^2-1)(m_2^2-1)}.$$

□

Theorem 4. *Let $u(x,y) \in L^2([0,1]\times[0,1])$, with bounded forth partial derivations, $|\frac{\partial^4 u(x,y)}{\partial^2 x \partial^2 y}| < B$; then the error bound would be obtained as follows:*

$$\sigma_{k_1,M_1,k_2,M_2} = O\Big(2^{-\frac{5}{2}(k_1+k_2)}\Big), \tag{28}$$

where

$$\sigma_{k_1,M_1,k_2,M_2} = \Big(\int_0^1\int_0^1 \Big[u(x,y) - \sum_{n_1=1}^{2^{k_1-1}}\sum_{m_1=0}^{M_1-1}\sum_{n_2=1}^{2^{k_2-1}}\sum_{M_2=0}^{M_2-1} u_{n_1m_1n_2m_2}\phi_{(n_1,m_1)}(x)\phi_{(n_2,m_2)}(y)\Big]^2 w_{n_1}(x)w_{n_2}(y)\,dx\,dy\Big)^{\frac{1}{2}}.$$

Proof. From the error statement, we have

$$\begin{aligned}
\sigma^2_{k_1,M_1,k_2,M_2}
&= \int_0^1\int_0^1 \Big[u(x,y) - \sum_{n_1=1}^{2^{k_1-1}}\sum_{m_1=0}^{M_1-1}\sum_{n_2=1}^{2^{k_2-1}}\sum_{M_2=0}^{M_2-1} u_{n_1m_1n_2m_2}\phi_{(n_1,m_1)}(x)\phi_{(n_2,m_2)}(y)\Big]^2 \\
&\qquad w_{n_1}(x)w_{n_2}(y)\,dx\,dy \\
&= \int_0^1\int_0^1 \Big[\sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{M_2=0}^{\infty} u_{n_1m_1n_2m_2}\phi_{(n_1,m_1)}(x)\phi_{(n_2,m_2)}(y) \\
&\quad - \sum_{n_1=1}^{2^{k_1-1}}\sum_{m_1=0}^{M_1-1}\sum_{n_2=1}^{2^{k_2-1}}\sum_{m_2=0}^{M_2-1} u_{n_1m_1n_2m_2}\phi_{(n_1,m_1)}(x)\phi_{(n_2,m_2)}(y)\Big]^2 w_{n_1}(x)w_{n_2}(y)\,dx\,dy \\
&= \int_0^1\int_0^1 \sum_{n_1=2^{k_1}}^{\infty}\sum_{m_1=M_1}^{\infty}\sum_{n_2=2^{k_2}}^{\infty}\sum_{m_2=M_2}^{\infty} u^2_{n_1m_1n_2m_2}(\phi_{(n_1,m_1)}(x))^2(\phi_{(n_2,m_2)}(y))^2 \\
&\qquad w_{n_1}(x)w_{n_2}(y)\,dx\,dy
\end{aligned}$$

$$= \sum_{n_1=2^{k_1}}^{\infty}\sum_{m_1=M_1}^{\infty}\sum_{n_2=2^{k_2}}^{\infty}\sum_{m_2=M_2}^{\infty} u^2_{n_1m_1n_2m_2}\int_0^1\int_0^1 (\phi_{(n_1,m_1)}(x))^2(\phi_{(n_2,m_2)}(y))^2$$

$$w_{n_1}(x)w_{n_2}(y)\,dx\,dy$$

$$= \sum_{n_1=2^{k_1}}^{\infty} \sum_{m_1=M_1}^{\infty} \sum_{n_2=2^{k_2}}^{\infty} \sum_{m_2=M_2}^{\infty} u^2_{n_1m_1n_2m_2} \int_{\frac{n_1-1}{2^{k_1-1}}}^{\frac{n_1}{2^{k_1-1}}} \frac{\left[2^{\frac{k_1}{2}}\hat{T}_{m_1}(2^{k_1}x-2n_1+1)\right]^2}{\sqrt{1-(2^{k_1}x-2n_1+1)^2}}\,dx$$
$$\times \int_{\frac{n_2-1}{2^{k_2-1}}}^{\frac{n_2}{2^{k_2-1}}} \frac{\left[2^{\frac{k_2}{2}}\hat{T}_{m_2}(2^{k_2}y-2n_2+1)\right]^2}{\sqrt{1-(2^{k_2}x-2n_2+1)^2}}\,dy$$

$$= \sum_{n_1=2^{k_1}}^{\infty} \sum_{m_1=M_1}^{\infty} \sum_{n_2=2^{k_2}}^{\infty} \sum_{m_2=M_2}^{\infty} u^2_{n_1m_1n_2m_2} 2^{k_1+k_2} \int_{\frac{n_1-1}{2^{k_1-1}}}^{\frac{n_1}{2^{k_1-1}}} \frac{\left[\hat{T}_{m_1}(2^{k_1}x-2n_1+1)\right]^2}{\sqrt{1-(2^{k_1}x-2n_1+1)^2}}\,dx$$
$$\times \int_{\frac{n_2-1}{2^{k_2-1}}}^{\frac{n_2}{2^{k_2-1}}} \frac{\left[\hat{T}_{m_2}(2^{k_2}y-2n_2+1)\right]^2}{\sqrt{1-(2^{k_2}x-2n_2+1)^2}}\,dy.$$

Now, let $2^{k_1}y-2n_1+1=t_1$ and $2^{k_2}y-2n_2+1=t_2$, then it is obtained

$$sigma^2_{k_1,M_1,k_2,M_2} = \sum_{n_1=2^{k_1}}^{\infty} \sum_{m_1=M_1}^{\infty} \sum_{n_2=2^{k_2}}^{\infty} \sum_{m_2=M_2}^{\infty} u^2_{n_1m_1n_2m_2} \int_{-1}^{1} \frac{\hat{T}_{m_1}(t_1)}{\sqrt{1-t_1^2}}\,dt_1$$
$$\int_{-1}^{1} \frac{\hat{T}_{m_2}(t_2)}{\sqrt{1-t_2^2}}\,dt_2. \tag{29}$$

For $m_1 \geq 1$, $m_2 \geq 1$, we have

$$\int_{-1}^{1} \frac{\hat{T}_{m_1}(t_1)}{\sqrt{1-t_1^2}}\,dt_1 = \frac{\pi}{2}, \qquad \int_{-1}^{1} \frac{\hat{T}_{m_2}(t_2)}{\sqrt{1-t_2^2}}\,dt_2 = \frac{\pi}{2};$$

then (29) simplifies as follows:

$$\sigma^2_{k_1,M_1,k_2,M_2} = \frac{\pi^2}{4} \sum_{n_1=2^{k_1}}^{\infty} \sum_{m_1=M_1}^{\infty} \sum_{n_2=2^{k_2}}^{\infty} \sum_{m_2=M_2}^{\infty} u^2_{n_1m_1n_2m_2}. \tag{30}$$

Therefore from (21) and (30), we can conclude the desired result as follows:

$$\sigma^2_{k_1,M_1,k_2,M_2} \leq \frac{\pi^4 B}{2^{10}} \sum_{n_1=2^{k_1}}^{\infty} \sum_{m_1=M_1}^{\infty} \sum_{n_2=2^{k_2}}^{\infty} \sum_{m_2=M_2}^{\infty} \frac{1}{(n_1n_2)^5(m_1^2-1)^2(m_2^2-1)^2}. \tag{31}$$

Also

$$\sum_{n_1=2^{k_1}}^{\infty} \frac{1}{(n_1)^5} = \frac{1}{(2^{k_1})^5} + \frac{1}{(2^{k_1}+1)^5} \cdots = \frac{1}{(2^{k_1})^5} \sum_{n=0}^{\infty} \frac{1}{(1+\frac{n}{2^{k_1}})^5}. \tag{32}$$

From (31) and (32), we conclude that

$$\sigma_{k_1,M_1,k_2,M_2} = O\Big(2^{-\frac{5}{2}(k_1+k_2)}\Big).$$

□

Theorem 5. *Let us consider* $M = \max \|\mathbf{K}(t,\eta,x,s)\|$, *for* $\eta, t \in I = [0,1], s, x \in \Omega = [0,1]$. *Assume that* $U_\varepsilon(t,x)$ *is the exact solution of system (3) and that* $\hat{U}_\varepsilon(t,x)$ *denotes the Chebyshev wavelet approximation for the exact solution* U *which is given by (15) and (16) . Then* $\hat{U}_\varepsilon(t,x)$ *converges to* $U_\varepsilon(t,x)$ *and the following result can be obtained*

$$\|U_\varepsilon - \hat{U}_\varepsilon\| = O\Big(2^{-\frac{5}{2}(k_1+k_2)}\Big).$$

Proof. According to the proposed method in previous section, we consider (15), (16), and (17) and insert $\hat{U}_\varepsilon(t,x)$ and $\hat{\mathbf{K}}(t,\eta,x,s)$ as approximations of the exact solution U and kernel $\mathbf{K}$ into system (3)

$$F(t,x) = EU_\varepsilon(t,x) + \int_0^t \int_0^1 \hat{\mathbf{K}}(t,\eta,x,s)U_\varepsilon(\eta,s)dsd\eta. \tag{33}$$

Subtracting (3) from (33) and some manipulations, we get

$$\begin{aligned}\|E\|\|U_\varepsilon - \hat{U}_\varepsilon\| \leqslant \| &\int_0^t \int_0^1 (\mathbf{K}(t,\eta,x,s) - \hat{\mathbf{K}}(t,\eta,x,s))U_\varepsilon(\eta,s)dsd\eta\| \\ &+ \|\int_0^t \int_0^1 \hat{\mathbf{K}}(t,\eta,x,s)(U_\varepsilon(\eta,s) - \hat{U}_\varepsilon(\eta,s))dsd\eta\|.\end{aligned}$$

From $\|E\| \geq 1$ it is obtained

$$\|U_\varepsilon - \hat{U}_\varepsilon\| \leqslant \|\mathbf{K} - \hat{\mathbf{K}}\|\|U_\varepsilon\| + M\|U_\varepsilon - \hat{U}_\varepsilon\|. \tag{34}$$

Relation (34) together with Lemma 2, shows the convergence of the exact solution to the approximate solution. Considering Theorem 4, we have

$$\|\mathbf{K} - \hat{\mathbf{K}}\| = O\Big(2^{-\frac{5}{2}(k_1+k_2)}\Big). \tag{35}$$

From (34) and (35), we conclude that

$$\|U_\varepsilon - \hat{U}_\varepsilon\| = O\Big(2^{-\frac{5}{2}(k_1+k_2)}\Big).$$

□

6 Numerical examples

To demonstrate the efficiency and the practicability of the proposed method, we consider the following two examples. All results are computed by using a program written in the Mathematica®.

Example 1. Consider the following TIAEs:

$$\int_0^t \int_0^1 100t(t+x)u(\eta,s)dsd\eta = \frac{25}{3}t^2(3+4t)(t+x), \qquad t \in [0,1], \tag{36}$$

where the exact solution is $u(t,x) = t + x$.

Considering regularization techniques and (3) for (36), we have

$$\varepsilon u_\varepsilon(t,x) + \int_0^t \int_0^1 100t(t+x)u(\eta,s)dsd\eta = \frac{25}{3}t^2(3+4t)(t+x). \tag{37}$$

We assume that $\hat{u}_\varepsilon(t,x)$ is the approximation of the exact solution $u_\varepsilon(t,x)$ which is defined by (15). For analyzing the behavior of the error representations, we consider absolute error as

$$Error = |u(t_i,x_j) - \hat{u}_\varepsilon(t_i,x_j)|.$$

The Chebyshev wavelet method described in Section 4 has been implemented for problem (37) with $M_1 = M_2 = 3, k_1 = k_2 = 2$ and the error for different values of ε has been reported in Table 1, which confirms the theoretical results of Lemma 1. Figure 1 presents the plots of exact and approximate solution with $\varepsilon = 0.000001$, which are found to be in good agreement.

From subsection 4.2, we consider the normal form of equation (36) as follows:

$$\begin{aligned}\varepsilon u_\varepsilon(p,q) &+ \int_p^1 \int_0^1 \int_0^t \int_0^1 10000pt(p+q)(t+x)u_\varepsilon(\eta,s)dsd\eta dxdt \\ &= \int_p^1 \int_0^1 \frac{2500}{3}p(p+q)t^2(3+4t)(t+x)dxdt, \qquad p \in [0,1],\end{aligned} \tag{38}$$

and then solve this equation by the Chebyshev wavelet method with $M_1 = M_2 = 3$ and $k_1 = k_2 = 2$. We report the error for different values of ε in Table 2. From numerical results in Tables 1 and 2, we observe that the results in Table 2 are more accurate than the reported results in Table 1.

Example 2. Consider the following:

$$Au(t,x) + \int_0^t \int_0^1 K(t,\eta,x,s)u(\eta,s)dsd\eta = f(t,x), \qquad t \in [0,1], \tag{39}$$

Table 1: Absolute errors of u_ε for different values of ε in Example 1

(t_i, x_j)	Error $\varepsilon = 0.01$	Error $\varepsilon = 0.001$	Error $\varepsilon = 0.000001$
$(0.1, 0.1)$	3.62×10^{-3}	3.64×10^{-4}	3.65×10^{-7}
$(0.2, 0.2)$	1.62×10^{-4}	1.45×10^{-5}	1.40×10^{-8}
$(0.3, 0.3)$	1.37×10^{-3}	1.39×10^{-4}	1.39×10^{-7}
$(0.4, 0.4)$	1.32×10^{-3}	1.35×10^{-4}	1.37×10^{-7}
$(0.5, 0.5)$	9.09×10^{-3}	9.29×10^{-4}	9.32×10^{-7}
$(0.6, 0.6)$	1.71×10^{-3}	1.74×10^{-4}	1.72×10^{-7}
$(0.7, 0.7)$	1.69×10^{-3}	1.73×10^{-4}	1.70×10^{-7}
$(0.8, 0.8)$	4.19×10^{-5}	9.27×10^{-7}	1.87×10^{-9}
$(0.9, 0.9)$	7.42×10^{-3}	8.20×10^{-4}	8.20×10^{-7}

Table 2: Absolute errors of u_ε for different values of ε for normal equation in Example 1

(t_i, x_j)	Error $\varepsilon = 0.01$	Error $\varepsilon = 0.001$	Error $\varepsilon = 0.000001$
$(0.1, 0.1)$	6.18×10^{-6}	6.11×10^{-6}	1.95×10^{-7}
$(0.2, 0.2)$	6.84×10^{-6}	6.26×10^{-6}	5.22×10^{-8}
$(0.3, 0.3)$	3.79×10^{-6}	3.83×10^{-7}	8.93×10^{-8}
$(0.4, 0.4)$	2.43×10^{-5}	2.43×10^{-6}	5.70×10^{-8}
$(0.5, 0.5)$	2.06×10^{-5}	2.06×10^{-6}	6.66×10^{-7}
$(0.6, 0.6)$	2.14×10^{-5}	2.14×10^{-6}	2.42×10^{-7}
$(0.7, 0.7)$	7.78×10^{-6}	7.78×10^{-7}	1.67×10^{-7}
$(0.8, 0.8)$	2.25×10^{-5}	2.25×10^{-6}	2.41×10^{-8}
$(0.9, 0.9)$	7.18×10^{-5}	7.18×10^{-6}	3.13×10^{-7}

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad K(t,\eta,x,s) = \begin{bmatrix} \eta^2(s^2+t^2+1) & (s+t) \\ \eta s^2+\eta+\eta t^2 & 200s+200t \end{bmatrix},$$

$$u(t,x) = \big(u_1(t,x), u_2(t,x)\big)^T, \qquad f(t,x) = \big(f_1(t,x), f_2(t,x)\big)^T,$$

and f_1 and f_2 are such that the exact solution is

$$u_1(t,x) = t^2 + x^2 + 1, \qquad u_2(t,x) = t + x.$$

We apply the regularization method to convert the mixed system (39) to the system of the second kind integral equations. Then, the resulting second kind integral equation will be solved by the proposed numerical scheme in section 4 with $M_1 = M_2 = 3$ and $k_1 = k_2 = 2$. Let $(\hat{u}_1, \hat{u}_{2,\varepsilon})$ be the approximation of the exact solution $(u_1, u_{2,\varepsilon})$ which are defined by (15) and (16). Numerical errors with several values of ε are displayed in Tables 3 and 4.

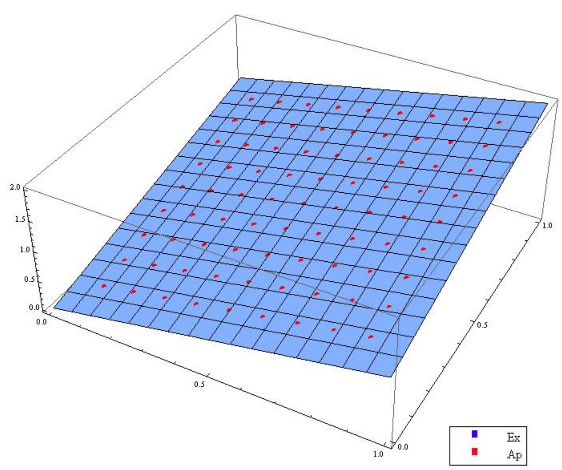

Figure 1: The plots of exact solution u and approximate solution of u with $\varepsilon = 0.000001$ in Example 1.

Similar to Example 2, we transform system (39) into the normal form of system (20) and then solve the normal system by Chebyshev wavelet method with $M_1 = M_2 = 3$ and $k_1 = k_2 = 2$. The error for different values of ε is

Table 3: Absolute errors of u_1 for different values of ε in Example 2

(x_i, y_j)	Error $\varepsilon = 0.01$	Error $\varepsilon = 0.001$	Error $\varepsilon = 0.000001$
$(0.1, 0.1)$	1.00×10^{-5}	1.00×10^{-6}	1.00×10^{-9}
$(0.2, 0.2)$	2.00×10^{-5}	2.00×10^{-6}	2.00×10^{-9}
$(0.3, 0.3)$	3.03×10^{-5}	3.03×10^{-6}	3.03×10^{-9}
$(0.4, 0.4)$	4.11×10^{-5}	4.11×10^{-6}	4.11×10^{-9}
$(0.5, 0.5)$	5.44×10^{-5}	5.44×10^{-6}	5.44×10^{-9}
$(0.6, 0.6)$	6.58×10^{-5}	6.58×10^{-6}	6.58×10^{-9}
$(0.7, 0.7)$	8.25×10^{-5}	8.25×10^{-6}	8.25×10^{-9}
$(0.8, 0.8)$	1.05×10^{-4}	1.05×10^{-5}	1.05×10^{-8}
$(0.9, 0.9)$	1.34×10^{-4}	1.34×10^{-5}	1.34×10^{-8}

Table 4: Absolute errors of $u_{2,\varepsilon}$ for different values of ε in Example 2

(x_i, y_j)	Error $\varepsilon = 0.01$	Error $\varepsilon = 0.001$	Error $\varepsilon = 0.000001$
$(0.1, 0.1)$	1.07×10^{-3}	1.83×10^{-4}	8.51×10^{-6}
$(0.2, 0.2)$	1.58×10^{-4}	2.84×10^{-5}	1.86×10^{-5}
$(0.3, 0.3)$	2.39×10^{-4}	2.36×10^{-5}	1.78×10^{-5}
$(0.4, 0.4)$	4.63×10^{-4}	2.48×10^{-5}	2.37×10^{-6}
$(0.5, 0.5)$	3.84×10^{-3}	8.20×10^{-4}	4.82×10^{-5}
$(0.6, 0.6)$	1.66×10^{-3}	1.12×10^{-4}	1.06×10^{-4}
$(0.7, 0.7)$	1.22×10^{-3}	1.80×10^{-4}	1.77×10^{-4}
$(0.8, 0.8)$	2.85×10^{-3}	2.81×10^{-4}	1.80×10^{-4}
$(0.9, 0.9)$	6.69×10^{-3}	3.88×10^{-4}	3.56×10^{-4}

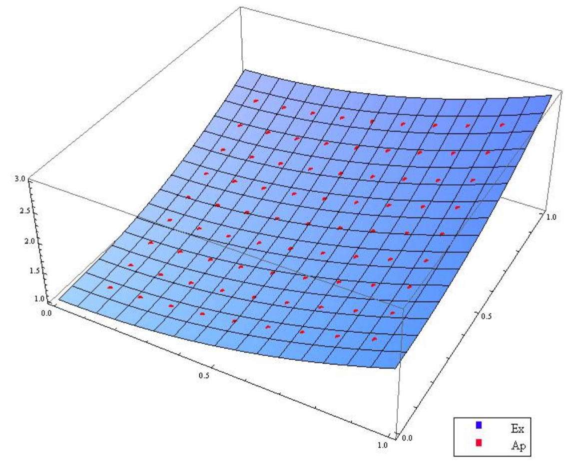

Figure 2: The plots of exact solution u_1 and approximate solution of u_1 with $\varepsilon = 0.000001$ in Example 2.

Table 5: Absolute errors of u_1 for different values of ε for normal equation in Example 2

	Error	Error	Error
(x_i, y_j)	$\varepsilon = 0.01$	$\varepsilon = 0.001$	$\varepsilon = 0.000001$
$(0.1, 0.1)$	8.54×10^{-8}	8.54×10^{-9}	8.54×10^{-12}
$(0.2, 0.2)$	2.56×10^{-8}	2.56×10^{-9}	2.56×10^{-12}
$(0.3, 0.3)$	4.19×10^{-8}	4.19×10^{-9}	4.19×10^{-12}
$(0.4, 0.4)$	1.10×10^{-7}	1.10×10^{-8}	1.10×10^{-11}
$(0.5, 0.5)$	5.46×10^{-7}	5.46×10^{-8}	5.46×10^{-11}
$(0.6, 0.6)$	1.73×10^{-6}	1.73×10^{-7}	1.73×10^{-10}
$(0.7, 0.7)$	2.54×10^{-6}	2.54×10^{-7}	2.54×10^{-10}
$(0.8, 0.8)$	2.86×10^{-6}	2.86×10^{-7}	2.86×10^{-10}
$(0.9, 0.9)$	2.58×10^{-6}	2.58×10^{-7}	2.58×10^{-10}

reported in Tables 5 and 6. Comparing displayed errors in the Tables 3, 4

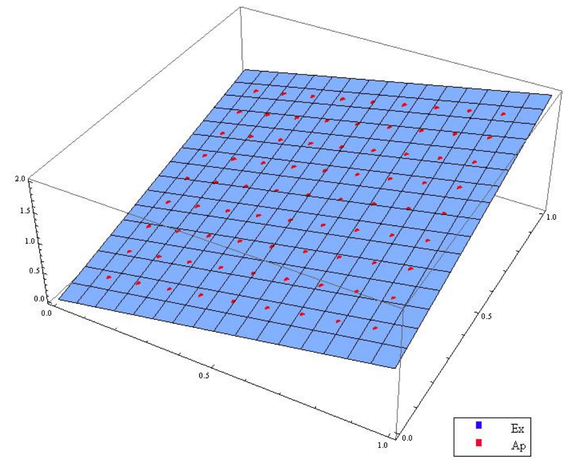

Figure 3: The plots of exact solution u_2 and approximate solution of u_2 with $\varepsilon = 0.000001$ in Example 2.

and 5, 6, we observe that the results in Tables 5 and 6 are more accurate than the reported results in Tables 3 and 4. This is predicted by the theory, in particular by Theorem 2. Figures 2 and 3 present the plots of exact and approximate solution with $\varepsilon = 0.000001$.

6.1 Haar wavelet method

In literature, several wavelets with different properties have been derived and depending upon the applications, different wavelet families are used. The Cheybyshev wavelets method is found to be simple, efficient, accurate, and computationally attractive for solving linear and non-linear problems. The properties of Chebyshev wavelets make the wavelet coefficient matrices sparse which eventually leads to the sparsity of the coefficients matrix of the obtained system. The Haar wavelet is also the simplest possible wavelet. The technical disadvantage of the Haar wavelet is that it is not continuous, and therefore not differentiable. This property can, however, be an advantage for the analysis of signals with sudden transitions, such as monitoring of tool failure in machines. For comparison, we consider the Haar wavelet method to solve Example 1. The Haar wavelet family is

Table 6: Absolute errors of $u_{2,\varepsilon}$ for different values of ε for normal equation in Example 2

(x_i, y_j)	Error $\varepsilon = 0.01$	Error $\varepsilon = 0.001$	Error $\varepsilon = 0.000001$
$(0.1, 0.1)$	1.25×10^{-7}	1.00×10^{-8}	2.19×10^{-7}
$(0.2, 0.2)$	9.92×10^{-7}	9.83×10^{-8}	8.59×10^{-8}
$(0.3, 0.3)$	9.58×10^{-8}	1.05×10^{-8}	9.59×10^{-8}
$(0.4, 0.4)$	3.64×10^{-6}	3.64×10^{-7}	5.67×10^{-9}
$(0.5, 0.5)$	5.75×10^{-6}	5.67×10^{-7}	2.19×10^{-7}
$(0.6, 0.6)$	4.75×10^{-6}	4.73×10^{-7}	4.37×10^{-7}
$(0.7, 0.7)$	1.18×10^{-6}	1.17×10^{-7}	1.52×10^{-7}
$(0.8, 0.8)$	5.24×10^{-6}	5.24×10^{-7}	7.34×10^{-8}
$(0.9, 0.9)$	1.48×10^{-5}	1.48×10^{-6}	1.39×10^{-7}

$$h_i(t) = \begin{cases} 1, & t \in [\tau_1, \tau_2], \\ -1, & t \in [\tau_2, \tau_3], \\ 0, & elsewhere \end{cases}$$

where

$$\tau_1 = \frac{k}{m}, \qquad \tau_2 = \frac{k + \frac{1}{2}}{m}, \qquad \tau_3 = \frac{k+1}{m}.$$

The integer $m = 2^j, j = 0, 1, \ldots, p$ and $k = 0, 1, \ldots, m-1$ are the level of the wavelet and translation parameter, respectively. The index i is calculated from the formula $i = m + k + 1$ and the maximal value is $i = 2M(M = 2^p)$. The index $i = 1$ corresponds to the scaling function of the Haar wavelet $h_1(t) = 1$. We can consider an approximate solution of equation (37) as

$$\hat{u}_\epsilon(t, x) = \sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a_{ij} h_i(t) h_j(x),$$

where the coefficients a_{ij} can be obtained by inserting approximate solution $\hat{u}_\epsilon(t, x)$ and the following collocation points in equation (37)

$$t_l = \frac{l - (\frac{1}{2})}{2M_1}, l = 1, 2, \ldots, 2M_1, \quad x_r = \frac{r - (\frac{1}{2})}{2M_2}, r = 1, 2, \ldots, 2M_2.$$

By assuming $M_1 = M_2 = 4$, the absolute error for different values of ε have been reported in Table 7. From Tables 1 and 7, we observe that the results obtained by the Chebyshev wavelet method are more accurate than the Haar wavelet method in this case.

Table 7: Absolute errors of u_ε for different values of ε by Haar wavelet method in Example 1

(t_i, x_j)	Error $\varepsilon = 0.01$	Error $\varepsilon = 0.001$	Error $\varepsilon = 0.000001$
$(0.1, 0.1)$	8.99×10^{-2}	8.07×10^{-2}	7.99×10^{-2}
$(0.2, 0.2)$	8.83×10^{-2}	2.11×10^{-2}	1.41×10^{-2}
$(0.3, 0.3)$	4.04×10^{-2}	1.72×10^{-2}	1.29×10^{-2}
$(0.4, 0.4)$	9.33×10^{-2}	7.99×10^{-2}	7.82×10^{-2}
$(0.5, 0.5)$	4.99×10^{-1}	5.00×10^{-1}	4.99×10^{-1}
$(0.6, 0.6)$	9.58×10^{-2}	8.25×10^{-2}	8.08×10^{-2}
$(0.7, 0.7)$	6.79×10^{-2}	1.98×10^{-2}	1.15×10^{-2}
$(0.8, 0.8)$	5.09×10^{-2}	1.78×10^{-2}	1.77×10^{-2}
$(0.9, 0.9)$	9.26×10^{-2}	8.01×10^{-2}	7.84×10^{-2}

7 Conclusion and future work

This work has been concerned with the regularization method to convert the mixed systems of the first and second-kind Volterra–Fredholm integral equations to the system of the second-kind Volterra–Fredholm integral equations. We presented the numerical method based on Chebyshev wavelets for solving the obtained second-kind problem. Convergence of the method was proved. These results were confirmed by some numerical examples.

In the present work, we considered the mixed system (1) in special case with $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The mixed system (1) in general form with $A(t)$ will be investigated as our future work.

Acknowledgements

The authors would like to express their gratitude for the referees of the paper for their valuable suggestions that improved the final form of the paper.

References

1. M.A. Abdou, F.A. Salama, *Volterra–Fredholm integral equation of the first kind and spectral relationships*, Appl. Math. Comput. 153 (2004), 141–153.

2. J. Ahmadi Shali, P. Darania, A.A. Jodayree, *Collocation method for nonlinear Volterra–Fredholm integral equations*, Open J. Appl. Sci. 2 (2012),115–121.

3. H. Brunner, *Collocation methods for Volterra integral and related functional equations*, Cambridge Monographs on Applied and Computational Mathematics, vol. 15, Cambridge University Press, Cambridge, 2004.

4. O. Diekman, T*hresholds and traveling waves for the geographical spread of infection*, J. Math. Biol. 6 (1978), 109–130.

5. M. Hamina, J. Saranen , *On the spline collocation method for the single-layer heat operator equation*, Math. Comp. 62 (1994), 41–64.

6. L. Hocia, *On Approximate Solution for Integral Equation of the Mixed Type*, ZAMM J. Appl. Math. Mech. 76 (1996), 415–416.

7. Y, Iso, K. Onishi, *On the stability of the boundary element collocation method applied to the linear heat equation*, J. Comput. Appl. Math. 38 (1991), 201–209

8. J.P. Kauthen, *Continuous time collocation for Volterra–Fredholm integral equations*, Numer. Math. 56 (1989), 409–424.

9. A. Kirsch, *An introduction to the mathematical theory of inverse problem*, Second edition. Applied Mathematical Sciences, 120. Springer, New York, 2011.

10. K. Maleknejad, H. Almasieh, M. Roodaki, *Triangular functions (TF) method for the solution of nonlinear Volterra–Fredholm integral equations*, Commun. Nonlinear Sci. Numer. Simul. 15 (2010), 3293–3298

11. K. Maleknejad, M. Hadizadeh, *A New computational method for Volterra–Fredholm integral equations*, Comput. Math. Appl., 37 (1999), 1–8.

12. D.L. Phillips, A *technique for the numerical solution of certain integral equations of the first kind*, J. Assoc. Comput. Mach. 9 (1962), 84–97.

13. H. R. Thieme, *A model for the spatio spread of an epidemic*, J. Math. Biol. 4, (1977), 337–351.

14. A. N. Tikhonov, *On the solution of incorrectly posed problem and the method of regularization*, Soviet Math. 4 (1963), 1035–1038.

15. S. Yousefi, M. Razzaghi, *Legendre Wavelets method for the nonlinear Volterra–Fredholm integral equations*, Math. Comput. Simul. 70 (2005), 1–8.

16. A.M. Wazwaz, *A reliable treatment for mixed Volterra–Fredholm integral equations*, Appl. Math. Comput. 127 (2002), 405–414.

منظم سازی و تحلیل عددی دستگاه مرکب معادلات انتگرالی ولترا-فردهلم نوع اول و دوم

سعید پیش بین و جواد شکری

دانشگاه ارومیه، دانشکده علوم، گروه ریاضی

دریافت مقاله ۱۸ اسفند ۱۳۹۵، دریافت مقاله اصلاح شده ۷ مهر ۱۳۹۷، پذیرش مقاله ۲۳ مهر ۱۳۹۷

چکیده : توجه به این نکته ضرروی است که دستگاه معادلات انتگرالی ولترا-فردهلم نوع اول و دوم جز معادلات بد وضع می باشند بنابراین حل دستگاه‌های گسسسته شده مربوط به این معادلات دارای مشکلات فراوانی است. ما در این مقاله ابتدا یک روش منظم سازی را در نظر گرفته و مسئله نوع اول بد وضع را به یک مسئله نوع دوم خوش وضع تبدیل می کنیم. در ادامه روش عددی براساس موجک چبیشف را برای حل دستگاه خوش وضع بکار برده و همگرایی روش مربوطه را تحلیل می‌کنیم. در پایان چند مثال عددی با جوابهای معلوم را برای نشان دادن کارایی روش عددی پیشنهاد شده در نظر می‌گیریم.

کلمات کلیدی : دستگاه مرکب معادلات انتگرالی ولترا-فردهلم نوع اول و دوم؛ روش منظم سازی؛ موجک چبیشف؛ آنالیز همگرایی.