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Research Article

Toeplitz-like preconditioner for linear systems from spatial fractional diffusion equations

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Abstract

The article deals with constructing a Toeplitz-like preconditioner for linear systems arising from finite difference discretization of the spatial fractional diffusion equations. The coefficient matrices of these linear systems have an $S+L$ structure, where S is a symmetric positive definite (SPD) matrix and L satisfies $\text{rank}(L) \leq 2$. We introduce an approximation for the SPD part S, which is called P_S, and then we show that the preconditioner $P = P_S + L$ has the Toeplitz-like structure and its displacement rank is 6. The analysis shows that the eigenvalues of the corresponding preconditioned matrix are clustered around 1. Numerical experiments exhibit that the Toeplitz-like preconditioner can significantly improve the convergence properties of the applied iteration method.

AMS subject classifications (2020): 65F10, 35R05, 65F08, 65M06.

Keywords: Fractional diffusion equation; Toeplitz-like matrix; Krylov subspace methods; PGMRES.

1 Introduction

In this article, we aim to propose a Toeplitz-like preconditioner for solving the linear system resulting from a finite difference approximation of an initial-boundary value problem of the spatial fractional diffusion equations (FDEs) that were introduced in [4]. In this article, the authors show that applying the finite difference method to the FDEs leads to the following linear system:

$$Ax = b, \tag{1}$$

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where the coefficient matrix of the linear system (1) is a dense Toeplitz-like matrix. As we see in [4], the coefficient matrix of (1) can be rewritten as the sum of a symmetric positive definite matrix S and a low rank matrix L, that is, $A = S + L$ (for more details, see [4, 2]). The matrix S has the following structure:

$$S = \eta I_n + GD_1G^T + G^TD_2G, \tag{2}$$

where $\eta > 0$, G is a Toeplitz matrix, and D_1 and D_2 are diagonal matrices with positive diagonal elements (more details are available in Section 2). In [4], the authors proposed a preconditioner by replacing G with its Strang's circulant approximation and two matrices D_1 and D_2 by their scalar (identity matrix) approximation. They showed the superlinear convergence of their preconditioner, when D_1 and D_2 are scalar identity matrices. In [2], we introduced a preconditioner based on the approximation of S defined in (2), and we proved that the proposed preconditioner is strong even though D_1 and D_2 are not a multiple of the identity matrix. In the following, we give some definitions about Toeplitz-like matrices.

Toeplitz-like matrices are defined by means of a displacement operator. Given an $n \times n$ matrix A, we consider the down-shift matrix of order n, $Z_n = \left(e_2\; e_3\; \cdots\; e_n\; 0_n\right)$, and the displacement operator ∇ defined by

$$\nabla(A) = A - Z_nAZ_n^T. \tag{3}$$

We define the displacement rank of matrix A as $rank(\nabla(A))$. The matrix A is said to have the Toeplitz-like structure if its displacement rank is low compared with its order n, that is, $rank(\nabla(A)) = m \ll n$. If the displacement rank of A is m, then we can rewrite $\nabla(A)$ as $\nabla(A) = CD^T$, where $C, D \in \mathbb{R}^{n\times m}$. Two matrices C and D are called the generators of the matrix A.

Throughout this article, we use the following notations: Capital letters, boldface lowercase letters, and regular lowercase letters denote matrices, vectors, and scalars, respectively. Moreover, I_n denotes the identity matrix of order n, while J_n denotes the $n\times n$ exchange matrix $J_n = \text{antidiag}(1, 1, \ldots, 1)$. We denote by e_j the jth column of identity matrix, and $\mathbf{0}$ denotes the zero vector of proper dimensions.

The organization of this article is as follows. In Section 2, we present the new preconditioner for the discretized linear systems arising from FDEs. Some computational remarks are given in Section 3. Numerical experiments are provided in Section 4 to prove the performance of our preconditioner. Finally, in Section 5, some concluding remarks are provided.

2 Preconditioning technique

The finite difference discretization of FDEs was developed in [4]. By introducing

$$a_i^{(\alpha)} = (i+1)^{1-\alpha} - i^{1-\alpha}, \qquad i = 0, 1, 2, \dots,$$
$$g_0^{(\alpha)} = \frac{a_0^{(\alpha)}}{\Gamma(2-\alpha)}, \quad g_k^{(\alpha)} = \frac{a_k^{(\alpha)} - a_{k-1}^{(\alpha)}}{\Gamma(2-\alpha)}, \qquad k = 1, 2, 3, \dots,$$

and

$$\mathbf{a} = -\frac{1}{\Gamma(2-\alpha)} \begin{pmatrix} a_0^{(\alpha)} \\ a_1^{(\alpha)} \\ \vdots \\ a_{n-1}^{(\alpha)} \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} g_1^{(\alpha)} \\ g_2^{(\alpha)} \\ \vdots \\ g_n^{(\alpha)} \end{pmatrix}, \tag{4}$$

the authors showed that the coefficient matrix of resulting linear system by finite difference discretization of FDEs possesses $A = S+L \in \mathbb{R}^{n\times n}$ structure, where

$$S = \eta I_n + G^{(\alpha)} D_+ G^{(\alpha)T} + G^{(\alpha)T} D_- G^{(\alpha)}, \tag{5}$$
$$L = c_1 \mathbf{a}\mathbf{g}^T + c_2 J\mathbf{a}(J\mathbf{g})^T, \tag{6}$$

in which $\eta > 0$, $D_\pm = \operatorname{diag}(d_{\pm,1}, d_{\pm,2}, \dots, d_{\pm,n})$ with $d_{\pm,i} > 0$, $c_1, c_2 \geq 0$, and

$$G^{(\alpha)} = \begin{pmatrix} g_0^{(\alpha)} & 0 & \cdots & 0 \\ g_1^{(\alpha)} & g_0^{(\alpha)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ g_{n-1}^{(\alpha)} & \cdots & g_1^{(\alpha)} & g_0^{(\alpha)} \end{pmatrix} \in \mathbb{R}^{n\times n}. \tag{7}$$

For simplicity, in the rest of the the superscript, (α) will be ignored. We see that the matrix S is a symmetric positive definite and that L is a matrix with low rank, that is, $rank(L) \leq 2$. Denote by $d_{\pm,\min}$ and $d_{\pm,\max}$ the smallest and the largest elements of $D_\pm$, respectively. Letting $\bar{d}_\pm = \frac{d_{\pm,\max}+d_{\pm,\min}}{2}$, in [2], we used the following preconditioner :

$$P_S = \eta I_n + \bar{d}_+ GG^T + \bar{d}_- G^T G. \tag{8}$$

This preconditioner is an approximation of S. We can replace S in A by P_S, obtaining so-called TL preconditioner

$$P = P_S + L \tag{9}$$

for the matrix A.

Theorem 1. If P is defined as (9), then $P^{-1}A = M + N$, where M is similar to $P_S^{-1}S$ and $rank(N) \leq 2$.

Proof. We can extend $\mathcal{S} = \operatorname{span}\{\mathbf{g}, J\mathbf{g}\}$ to an orthonormal basis of $\mathbb{R}^n$. Let $V = \begin{pmatrix}\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n\end{pmatrix}$ be the matrix representation of such orthonormal basis, such that $\mathbf{v}_1 = \frac{\mathbf{g}}{\|\mathbf{g}\|_2}$ and $\mathbf{v}_2 = \frac{J\mathbf{g}+h\mathbf{g}}{\|J\mathbf{g}+h\mathbf{g}\|_2}$, where $h = -\frac{(J\mathbf{g})^T\mathbf{g}}{\|\mathbf{g}\|_2^2}$. Suppose that V

is partitioned as $V = \begin{pmatrix} V_1 \ V_2 \end{pmatrix}$, where $V_1 = \begin{pmatrix} \mathbf{v}_1 \ \mathbf{v}_2 \end{pmatrix}$ and $V_2 = \begin{pmatrix} \mathbf{v}_3 \ \mathbf{v}_4 \ \cdots \ \mathbf{v}_n \end{pmatrix}$. From (6), we see that $LV_2 = 0$. We have

$$V^T P_S^{-1} L V = \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} P_S^{-1} L \begin{pmatrix} V_1 \ V_2 \end{pmatrix} = \begin{pmatrix} V_1^T P_S^{-1} L V_1 & 0 \\ V_2^T P_S^{-1} L V_1 & 0 \end{pmatrix}. \tag{10}$$

If we define $K = I_2 + V_1^T P_S^{-1} L V_1$ and

$$Q = V \begin{pmatrix} I_2 - K^{-1} V_1^T & 0 \\ -V_2^T P_S^{-1} L V_1 K^{-1} V_1^T & 0 \end{pmatrix} V^T,$$

then

$$\begin{aligned}
(I + P_S^{-1} L)^{-1} &= V \left[I + V^T P_S^{-1} L V \right]^{-1} V^T \\
&= V \left[I + \begin{pmatrix} V_1^T P_S^{-1} L V_1 & 0 \\ V_2^T P_S^{-1} L V_1 & 0 \end{pmatrix} \right]^{-1} V^T \\
&= V \begin{pmatrix} K^{-1} & 0 \\ -V_2^T P_S^{-1} L V_1 K^{-1} & I_{n-2} \end{pmatrix} V^T \\
&= I + Q.
\end{aligned}$$

We note that $QV_2 = 0$. By this assumption, we can prove our assertion in the following

$$\begin{aligned}
P^{-1} A &= (P_S + L)^{-1} (S + L) = (I + P_S^{-1} L)^{-1} P_S^{-1} S (I + S^{-1} L) \\
&= (I + P_S^{-1} L)^{-1} P_S^{-1} S (I + P_S^{-1} L)(I + P_S^{-1} L)^{-1} (I + S^{-1} L).
\end{aligned} \tag{11}$$

In (11), if we define $M = (I + P_S^{-1} L)^{-1} P_S^{-1} S (I + P_S^{-1} L)$, then

$$\begin{aligned}
P^{-1} A &= M (I + Q)(I + S^{-1} L) \\
&= M + N,
\end{aligned}$$

where $N = M(S^{-1} L + Q + Q S^{-1} L)$. We see that M is similar to $P_S^{-1} S$. It can be easily verified that $NV_2 = 0$. Hence $rank(N) \leq 2$. □

Theorem 2. [2] If $\lambda \in \sigma(P_S^{-1} S)$, then $|1 - \lambda| < k < 1$, where $k = \frac{1}{\hat{\eta} + 1} < 1$ with $\hat{\eta} = \frac{\eta}{2\kappa \|\tilde{G}\|_2^2}$ and $\kappa = \max\{\frac{d_{+,\max} + d_{+,\min}}{2}, \frac{d_{-,\max} + d_{-,\min}}{2}\}$.

Based on Theorems 1 and 2, the eigenvalues of the preconditioned matrix $P^{-1} A$ are clustered around one, and thus Krylov subspace methods with the proposed preconditioner coverage very quickly.

Remark 1. The main differences between our proposed preconditioner P and the proposed preconditioner in [2] P_S are as follows:

1- P_S is an approximation of S in (5), while P consists of two parts. The first part P_S is an approximation of the matrix S and the second one is the matrix L (the rest of the matrix A). Hence, we expect that P is more accurate than P_S.

2- In the next section, in order to show the computational efficiency of preconditioner P, we prove that P has the Toeplitz-like structure, and its generators will be computed precisely. By using them, we can use fast algorithms to compute P^{-1}. Hence, in comparison with the circulant approximation of P_S in [2], the P^{-1} is computed straightforward.

In Section 4, numerical experiments show that the efficiency of the proposed preconditioner P.

3 Some computational results

In this section, we show that the matrix P defined in (9) has the Toeplitz-like structure, and we construct its generators. To this end, we define the following temporary vectors:

$$\bar{\mathbf{g}} = \begin{pmatrix} g_0 \ g_1 \ \cdots \ g_{n-1} \end{pmatrix}^T \qquad \bar{t} = \begin{pmatrix} g_1 \ g_2 \ \cdots \ g_{n-1} \end{pmatrix}^T. \tag{12}$$

The following relations can be easily deduced from (4) and (12)

$$\mathbf{a} - \mathbf{g} = \frac{1}{\Gamma(2-\alpha)} \begin{pmatrix} -a_0 - (a_1 - a_0) \\ -a_1 - (a_2 - a_1) \\ \vdots \\ -a_{n-1} - (a_n - a_{n-1}) \end{pmatrix} = Z^T \mathbf{a} + a_n e_n, \tag{13}$$

$$\bar{\mathbf{g}} = g_0 \mathbf{e}_1 + Z\mathbf{g}, \tag{14}$$

$$Z^T \bar{\mathbf{g}} = \mathbf{g} - g_n e_n = \begin{pmatrix} \bar{\mathbf{t}} \\ 0 \end{pmatrix}. \tag{15}$$

We use the following lemmas in our subsequent discussion.

Lemma 1. Let G be the $n \times n$ matrix defined in (7). Then the following relations hold:

(i) $\nabla(GG^T) = \bar{\mathbf{g}}\bar{\mathbf{g}}^T$,

(ii) $\nabla(G^TG) = \mathbf{e}_1\bar{\mathbf{g}}^T G + G^T\bar{\mathbf{g}}\mathbf{e}_1^T - \bar{\mathbf{g}}^T\bar{\mathbf{g}}\mathbf{e}_1\mathbf{e}_1^T - JZ^T\bar{\mathbf{g}}\bar{\mathbf{g}}^T ZJ$,

(iii) $\nabla(P_S) = \eta\mathbf{e}_1\mathbf{e}_1^T + \bar{d}_+\bar{\mathbf{g}}\bar{\mathbf{g}}^T + \bar{d}_-(\mathbf{e}_1\bar{\mathbf{g}}^T G + G^T\bar{\mathbf{g}}\mathbf{e}_1^T - \bar{\mathbf{g}}^T\bar{\mathbf{g}}\mathbf{e}_1\mathbf{e}_1^T - JZ^T\bar{\mathbf{g}}\bar{\mathbf{g}}^T ZJ)$.

Proof. The matrix G can be partitioned as

$$G = \begin{pmatrix} g_0 & \mathbf{0} \\ \bar{\mathbf{t}} & G_1 \end{pmatrix}, \tag{16}$$

where $G_1 = G(2:n, 2:n)$ is a lower triangular Toeplitz matrix, so the displacement structure of GG^T can be viewed as

$$\begin{aligned}\nabla(GG^T) &= GG^T - ZGG^TZ^T\\ &= \begin{pmatrix} g_0 & \mathbf{0}\\ \bar{\mathbf{t}} & G_1\end{pmatrix}\begin{pmatrix} g_0 & \bar{\mathbf{t}}^T\\ \mathbf{0} & G_1^T\end{pmatrix} - \begin{pmatrix} \mathbf{0} & 0\\ G_1 & \mathbf{0}\end{pmatrix}\begin{pmatrix} \mathbf{0} & G_1^T\\ 0 & \mathbf{0}\end{pmatrix}.\end{aligned} \tag{17}$$

Now, by a straightforward computation, part (i) of the lemma can be verified. For part (ii), first we know that

$$\bar{\mathbf{g}}^T G = \begin{pmatrix}\bar{\mathbf{g}}^T\bar{\mathbf{g}} & \bar{\mathbf{t}}^T G_1\end{pmatrix}. \tag{18}$$

Therefore,

$$\nabla(G^TG) = G^TG - ZG^TGZ^T \tag{19}$$

$$= \begin{pmatrix} g_0 & \bar{\mathbf{t}}^T\\ 0 & G_1^T\end{pmatrix}\begin{pmatrix} g_0 & 0\\ \bar{\mathbf{t}} & G_1\end{pmatrix} - \begin{pmatrix} 0 & 0\\ G_1^T & J\bar{\mathbf{t}}\end{pmatrix}\begin{pmatrix} 0 & G_1\\ 0 & (J\bar{\mathbf{t}})^T\end{pmatrix} \tag{20}$$

$$= \begin{pmatrix}\bar{\mathbf{g}}^T\bar{\mathbf{g}} & \bar{\mathbf{t}}^T G_1\\ G_1^T\bar{\mathbf{t}} & -J\bar{\mathbf{t}}\bar{\mathbf{t}}^T J\end{pmatrix}. \tag{21}$$

It gives that

$$\begin{aligned}\nabla(G^TG) &= \begin{pmatrix}\bar{\mathbf{g}}^T\bar{\mathbf{g}} & \bar{\mathbf{t}}^T G_1\\ \bar{\mathbf{0}} & \bar{\mathbf{0}}\end{pmatrix} + \begin{pmatrix}\bar{\mathbf{g}}^T\bar{\mathbf{g}} & \bar{\mathbf{0}}\\ G_1^T\bar{\mathbf{t}} & \bar{\mathbf{0}}\end{pmatrix} - \begin{pmatrix}\bar{\mathbf{g}}^T\bar{\mathbf{g}} & \bar{\mathbf{0}}\\ \bar{\mathbf{0}} & \bar{\mathbf{0}}\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & J\bar{\mathbf{t}}^T\bar{\mathbf{t}}J\end{pmatrix}\\ &= \mathbf{e}_1\bar{\mathbf{g}}^T G + G^T\bar{\mathbf{g}}\mathbf{e}_1^T - \bar{\mathbf{g}}^T\bar{\mathbf{g}}\mathbf{e}_1\mathbf{e}_1^T - JZ^T\bar{\mathbf{g}}\bar{\mathbf{g}}^T ZJ.\end{aligned} \tag{22}$$

We note that $\nabla(I) = \mathbf{e}_1\mathbf{e}_1^T$, so

$$\begin{aligned}\nabla(P_S) &= \eta\mathbf{e}_1\mathbf{e}_1^T + \bar{d}_+\nabla(GG^T) + \bar{d}_-\nabla(G^TG)\\ &= \eta\mathbf{e}_1\mathbf{e}_1^T + \bar{d}_+\bar{\mathbf{g}}\bar{\mathbf{g}}^T + \bar{d}_-(\mathbf{e}_1\bar{\mathbf{g}}^T G + G^T\bar{\mathbf{g}}\mathbf{e}_1^T - \bar{\mathbf{g}}^T\bar{\mathbf{g}}\mathbf{e}_1\mathbf{e}_1^T - JZ^T\bar{\mathbf{g}}\bar{\mathbf{g}}^T ZJ),\end{aligned}$$

which completes the proof of (iii). □

Lemma 2. If we define the auxiliary vectors $\mathbf{h}_1 = \bar{d}_-(J\mathbf{g}) - c_2 Z(J\mathbf{g})$ and $\mathbf{h}_2 = c_1\mathbf{g} - \bar{d}_+(Z\mathbf{g})$, then the following relations hold:

(i) $c_2\left[(J\mathbf{a})(J\mathbf{g})^T - Z(J\mathbf{a})(J\mathbf{g})^T Z^T\right] - \bar{d}_- JZ^T\bar{\mathbf{g}}\bar{\mathbf{g}}^T ZJ = (c_2 - \bar{d}_-)(J\mathbf{a})(J\mathbf{g})^T + Z(J\mathbf{a})\mathbf{h}_1^T + \mathbf{e}_1(\bar{d}_- a_n + \bar{d}_- g_n)(J\mathbf{g})^T + \bar{d}_-(g_n(J\mathbf{g}) - g_n^2\mathbf{e}_1)\mathbf{e}_1^T$,

(ii) $c_1\left[(\mathbf{a})(\mathbf{g})^T - Z(\mathbf{a})(\mathbf{g})^T Z^T\right] + \bar{d}_+\bar{\mathbf{g}}\bar{\mathbf{g}}^T = (\bar{d}_+ - c_1)(Z\mathbf{a})(Z\mathbf{g})^T + \mathbf{a}\mathbf{h}_2^T + \mathbf{e}_1(a_0\bar{d}_+ + \bar{d}_+ g_0)(Z\mathbf{g})^T + (\bar{d}_+ g_0(Z\mathbf{g}) + \bar{d}_+ g_0^2\mathbf{e}_1)\mathbf{e}_1^T$.

Proof. We see that

$$
\begin{aligned}
& c_2\left[(J\mathbf{a})(J\mathbf{g})^T - Z(J\mathbf{a})(J\mathbf{g})^T Z^T\right] - \bar{d}_- J Z^T \bar{\mathbf{g}}\bar{\mathbf{g}}^T Z J \\
&= c_2\left[(J\mathbf{a})(J\mathbf{g})^T - Z(J\mathbf{a})(J\mathbf{g})^T Z^T\right] - \bar{d}_-\left[J(\mathbf{g} - g_n\mathbf{e}_n)(\mathbf{g} - g_n\mathbf{e}_n)^T J\right] \qquad \text{By (15)} \\
&= \left[c_2(J\mathbf{a}) - \bar{d}_-(J\mathbf{g})\right](J\mathbf{g})^T - c_2 Z(J\mathbf{a})(J\mathbf{g})^T Z^T \\
&\quad + \bar{d}_- g_n \mathbf{e}_1 (J\mathbf{g})^T + \bar{d}_- g_n (J\mathbf{g})\mathbf{e}_1^T - \bar{d}_- g_n^2 \mathbf{e}_1\mathbf{e}_1^T \\
&= J\left[c_2\mathbf{a} - \bar{d}_-(\mathbf{a} - Z^T\mathbf{a} - a_n\mathbf{e}_n)\right](J\mathbf{g})^T - c_2 Z(J\mathbf{a})(J\mathbf{g})^T Z^T \qquad \text{By (13)} \\
&\quad + \bar{d}_- g_n \mathbf{e}_1 (J\mathbf{g})^T + \bar{d}_- g_n (J\mathbf{g})\mathbf{e}_1^T - \bar{d}_- g_n^2 \mathbf{e}_1\mathbf{e}_1^T \\
&= (c_2 - \bar{d}_-)(J\mathbf{a})(J\mathbf{g})^T + Z(J\mathbf{a})\mathbf{h}_1^T + \mathbf{e}_1(\bar{d}_- a_n + \bar{d}_- g_n)(J\mathbf{g})^T \\
&\quad + (\bar{d}_- g_n (J\mathbf{g}) - \bar{d}_- g_n^2 \mathbf{e}_1)\mathbf{e}_1^T.
\end{aligned}
$$

In the similar way, we can prove part (ii) as follows:

$$
\begin{aligned}
& c_1\left[(\mathbf{a})(\mathbf{g})^T - Z(\mathbf{a})(\mathbf{g})^T Z^T\right] + \bar{d}_+ \bar{\mathbf{g}}\bar{\mathbf{g}}^T \\
&= c_1\left[(\mathbf{a})(\mathbf{g})^T - Z(\mathbf{a})(\mathbf{g})^T Z^T\right] + \bar{d}_+(g_0\mathbf{e}_1 + Z\mathbf{g})(g_0\mathbf{e}_1 + Z\mathbf{g})^T \qquad \text{By (14)} \\
&= Z\left[\bar{d}_+\mathbf{g} - c_1\mathbf{a}\right](Z\mathbf{g})^T + c_1\mathbf{a}\mathbf{g}^T + \bar{d}_+\left[g_0^2\mathbf{e}_1\mathbf{e}_1^T + g_0\mathbf{e}_1(Z\mathbf{g})^T + g_0 Z\mathbf{g}\mathbf{e}_1^T\right] \\
&= Z\left[(\bar{d}_+ - c_1)\mathbf{a} - \bar{d}_+ Z^T\mathbf{a} - \bar{d}_+ a_n\mathbf{e}_n\right](Z\mathbf{g})^T + c_1\mathbf{a}\mathbf{g}^T \qquad \text{By (13)} \\
&\quad + \bar{d}_+\left[g_0^2\mathbf{e}_1\mathbf{e}_1^T + g_0\mathbf{e}_1(Z\mathbf{g})^T + g_0 Z\mathbf{g}\mathbf{e}_1^T\right] \\
&= (\bar{d}_+ - c_1)(Z\mathbf{a})(Z\mathbf{g})^T + \mathbf{a}\mathbf{h}_2^T + \mathbf{e}_1(a_0\bar{d}_+ + \bar{d}_+ g_0)(Z\mathbf{g})^T \\
&\quad + (\bar{d}_+ g_0 (Z\mathbf{g}) + \bar{d}_+ g_0^2\mathbf{e}_1)\mathbf{e}_1^T.
\end{aligned}
$$

□

Theorem 3 indicates that P in (9) has the Toeplitz-like structure, and we can use a superfast solver to compute P^{-1}.

Theorem 3. The matrix P defined in (8) is a Toeplitz-like matrix and its displacement operator is

$$
\begin{aligned}
\nabla(P) =& (c_2 - \bar{d}_-)(J\mathbf{a})(J\mathbf{g})^T + (\bar{d}_+ - c_1)(Z\mathbf{a})(Z\mathbf{g})^T + Z(J\mathbf{a})\mathbf{h}_1^T \\
& + \mathbf{a}\mathbf{h}_2^T + \mathbf{e}_1\mathbf{h}_3^T + \mathbf{h}_4\mathbf{e}_1^T,
\end{aligned} \tag{23}
$$

where $\mathbf{h}_1$ and $\mathbf{h}_2$ are defined in Lemma 2, and

$$
\begin{aligned}
\mathbf{h}_3 &= (\bar{d}_- a_n + \bar{d}_- g_n)(J\mathbf{g}) + \bar{d}_+(a_0 + g_0)Z\mathbf{g} + \eta\mathbf{e}_1 + \bar{d}_- G^T\bar{\mathbf{g}}, \\
\mathbf{h}_4 &= \bar{d}_- g_n (J\mathbf{g}) - g_n^2\mathbf{e}_1 + \bar{d}_+ g_0 (Z\mathbf{g}) + \bar{d}_+ g_0^2\mathbf{e}_1 + \bar{d}_- G^T\bar{\mathbf{g}} - \bar{d}_-\bar{\mathbf{g}}^T\bar{\mathbf{g}}\mathbf{e}_1.
\end{aligned}
$$

Proof. We have $P = P_S + L$. Hence

$$
\begin{aligned}
\nabla(P) =& \nabla(P_S) + \nabla(L) = \nabla(P_S) + c_1((a)(g)^T - Z(a)(Z\mathbf{g})^T) \\
& + c_2(J\mathbf{a}(J\mathbf{g}^T) - ZJ\mathbf{a}(ZJ\mathbf{g})^T).
\end{aligned} \tag{24}
$$

By part (iii) of Lemma 1, we can rewrite (24) as follows:

$$\begin{aligned}
\nabla(P) &= \eta \mathbf{e}_1\mathbf{e}_1^T + \bar{d}_+\bar{\mathbf{g}}\bar{\mathbf{g}}^T + \bar{d}_-(\mathbf{e}_1\bar{\mathbf{g}}^T G + G^T\bar{\mathbf{g}}\mathbf{e}_1^T - \bar{\mathbf{g}}^T\bar{\mathbf{g}}\mathbf{e}_1\mathbf{e}_1^T - JZ^T\bar{\mathbf{g}}\bar{\mathbf{g}}^T ZJ) \\
&\quad + c_1((a)(g)^T - Z(a)(Z\mathbf{g})^T) + c_2(J\mathbf{a}(J\mathbf{g}^T) - ZJ\mathbf{a}(ZJ\mathbf{g})^T) \\
&= c_2\left[(J\mathbf{a})(J\mathbf{g})^T - Z(J\mathbf{a})(J\mathbf{g})^T Z^T\right] - \bar{d}_- JZ^T\bar{\mathbf{g}}\bar{\mathbf{g}}^T ZJ \\
&\quad + c_1\left[(\mathbf{a})(\mathbf{g})^T - Z(\mathbf{a})(\mathbf{g})^T Z^T\right] + \bar{d}_+\bar{\mathbf{g}}\bar{\mathbf{g}}^T \\
&\quad + \eta\mathbf{e}_1\mathbf{e}_1^T + \bar{d}_-(\mathbf{e}_1\bar{\mathbf{g}}^T G + G^T\bar{\mathbf{g}}\mathbf{e}_1^T - \bar{\mathbf{g}}^T\bar{\mathbf{g}}\mathbf{e}_1\mathbf{e}_1^T)
\end{aligned}$$

Parts (i) and (ii) of Lemma 2 and the above equations imply that

$$\begin{aligned}
\nabla(P) =& (c_2 - \bar{d}_-)(J\mathbf{a})(J\mathbf{g})^T + Z(J\mathbf{a})\mathbf{h}_1^T + \mathbf{e}_1(\bar{d}_- a_n + \bar{d}_- g_n)(J\mathbf{g})^T \\
&+ (\bar{d}_- g_n(J\mathbf{g}) - g_n^2\mathbf{e}_1)\mathbf{e}_1^T + (\bar{d}_+ - c_1)(Z\mathbf{a})(Z\mathbf{g})^T + \mathbf{a}\mathbf{h}_2^T \\
&+ \mathbf{e}_1(a_0\bar{d}_+ + \bar{d}_+ g_0)(Z\mathbf{g})^T + (\bar{d}_+ g_0(Z\mathbf{g}) + \bar{d}_+ g_0^2\mathbf{e}_1)\mathbf{e}_1^T \\
&+ \eta\mathbf{e}_1\mathbf{e}_1^T + \bar{d}_-(\mathbf{e}_1\bar{\mathbf{g}}^T G + G^T\bar{\mathbf{g}}\mathbf{e}_1^T - \bar{\mathbf{g}}^T\bar{\mathbf{g}}\mathbf{e}_1\mathbf{e}_1^T) \\
=& (c_2 - \bar{d}_-)(J\mathbf{a})(J\mathbf{g})^T + Z(J\mathbf{a})\mathbf{h}_1^T + (\bar{d}_+ - c_1)(Z\mathbf{a})(Z\mathbf{g})^T + \mathbf{a}\mathbf{h}_2^T \\
&+ \mathbf{e}_1\left((\bar{d}_- a_n + \bar{d}_- g_n)(J\mathbf{g}) + \bar{d}_+(a_0 + g_0)Z\mathbf{g} + \eta\mathbf{e}_1 + \bar{d}_- G^T\bar{\mathbf{g}}\right)^T \\
&+ \left(\bar{d}_- g_n(J\mathbf{g}) - g_n^2\mathbf{e}_1 + \bar{d}_+ g_0(Z\mathbf{g}) + \bar{d}_+ g_0^2\mathbf{e}_1 + \bar{d}_- G^T\bar{\mathbf{g}} - \bar{d}_-\bar{\mathbf{g}}^T\bar{\mathbf{g}}\mathbf{e}_1\right)\mathbf{e}_1^T \\
=& (c_2 - \bar{d}_-)(J\mathbf{a})(J\mathbf{g})^T + (\bar{d}_+ - c_1)(Z\mathbf{a})(Z\mathbf{g})^T \\
&+ Z(J\mathbf{a})\mathbf{h}_1^T + \mathbf{a}\mathbf{h}_2^T + \mathbf{e}_1\mathbf{h}_3^T + \mathbf{h}_4\mathbf{e}_1^T.
\end{aligned}$$

□

We showed that our preconditioner P has the Toeplitz-like structure and its displacement rank of P ($rank(\nabla(P))$) is 6. So we can use fast and stable Levinson like algorithms [5] to compute $P^{-1}r$ in $O(n^2)$ operations. For Krylov subspace iteration methods such as preconditioned GMRES [10], high quality preconditioning plays a crucial role in accelerating the convergence speed of the Krylov subspace iteration methods. Instead of $O(n^2)$ operations to apply our preconditioner in each iteration, Circulant preconditioners require only $O(n\log(n))$ operations in each step. Significant reduction of total iterations (and total CPU time) by using our preconditioner in numerical experiments show the efficiency of our preconditioner.

4 Numerical experiments

The GMRES(20) (restarts every 20 inner iterations) with the proposed preconditioner is applied to solve the linear system (1). We choose the right-hand side such that $x^* = (1, 2 \ldots, n)^T$ is the exact solution, that is, $b = Ax^*$. The stopping criterion in the numerical experiments is $\|r_k\|_2/\|b\|_2 < 1e-7$, where r_k is the residual vector of the linear system after k iterations and b is the right-hand side. For all experiments, the initial guess is chosen as the zero

Table 1: Numerical results for Example 1

α	n	I		P_C		P_S		TL	
		Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
0.7	2^{10}	81	23.5113	6	0.6189	4	0.4012	1	0.3001
	2^{11}	†	-	7	2.1718	5	1.6312	1	0.9017
	2^{12}	†	-	7	7.9142	5	5.1305	1	2.7321
	2^{13}	†	-	8	9.8320	5	6.3483	1	5.9821
	2^{14}	†	-	11	14.8639	7	9.6407	1	8.9231
0.9	2^{10}	†	-	13	1.4921	8	1.2134	1	0.5743
	2^{11}	†	-	21	9.5901	13	5.8513	1	1.2401
	2^{12}	†	-	29	46.2163	17	26.2245	1	3.5421
	2^{13}	†	-	†	-	†	-	2	7.3142
	2^{14}	†	-	†	-	†	-	2	10.6893

vector. All the numerical experiments are run in MATLAB on a desktop with the configuration: Intel(R) Core(TM) CPU Q9450 2.66 GHz and 8.00 GB RAM. In the following tables, "I" represents the GMRES method without preconditioning technique. We use P_C to denote the circulant preconditioner [4], P_S to denote preconditioner in [2], and TL to present the proposed preconditioner defined in (9). Also the "Iter" denotes the number of iterations, "CPU" denotes the total CPU time in seconds for solving the problem, and n denotes the size of our linear system. In the tables, † indicates no convergence within 100 iterations.

Example 1. In this example, we consider

$$D_+ = \text{diag}\{0, (\frac{1}{n-1})^{1-\alpha}, (\frac{2}{n-1})^{1-\alpha}, \dots, 1\} \tag{25}$$

$$D_- = \text{diag}\{1, (\frac{n-2}{n-1})^{1-\alpha}, (\frac{n-3}{n-1})^{1-\alpha}, \dots, 0\}, \tag{26}$$

and $c_1 = c_2 = 2$.

Example 2. In this example, $D_\pm$ are random diagonal matrices with diagonal entries taken independently at random from $[0, 1]$ and $c_1 = c_2 = 2$.

The numerical results of Examples 1 and 2 are shown in Tables 1 and 2, respectively. Tables 1–2 show clearly the fast convergence of the TL preconditioner. We see that TL preconditioner is more effective in both CPU time and the number of iterations than P_S and P_C preconditioners.

In what follows, we compare the eigenvalues of A and $P^{-1}A$ for Example 1, the results are shown in Figures 1 and 2. As we see the eigenvalues of preconditioned matrix $P^{-1}A$ are clustered around 1.

Table 2: Numerical results for Example 2

α	n	I		P_C		P_S		TL	
		Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
0.7	2^{10}	94	0.5931	6	0.6907	4	0.4455	1	0.2517
	2^{11}	†	-	7	1.7012	5	1.3259	1	0.8993
	2^{12}	†	-	7	8.2302	5	6.4216	1	2.3263
	2^{13}	†	-	9	11.4134	6	8.1983	1	7.3129
	2^{14}	†	-	11	14.3921	7	11.0214	2	10.9386
0.9	2^{10}	†	-	30	2.3102	17	1.7492	1	0.4352
	2^{11}	†	-	41	14.0561	25	10.9318	1	1.3107
	2^{12}	†	-	75	47.5169	44	20.9312	1	4.3563
	2^{13}	†	-	†	-	†	-	2	9.8311
	2^{14}	†	-	†	-	†	-	2	14.0968

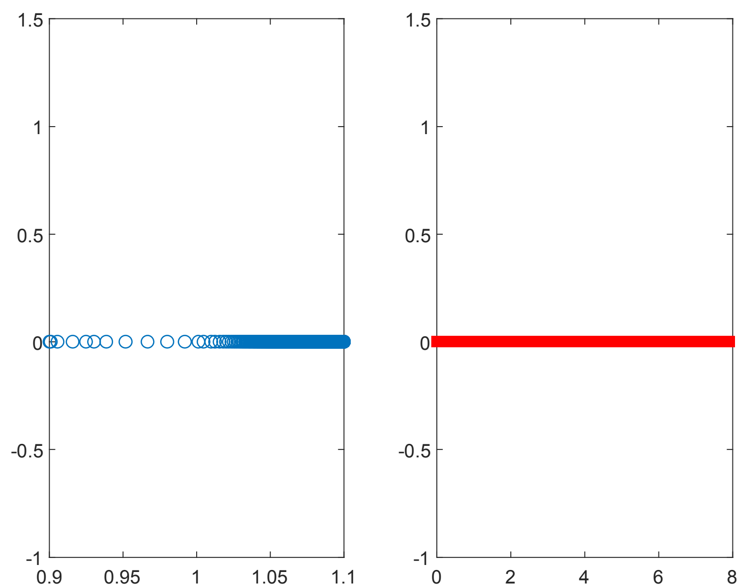

Figure 1: Eigenvalues of $P^{-1}A$ (left) and A (right) for Example 1, with $\alpha = .9$ and $n = 2^{10}$.

5 Concluding remarks

In this article, the preconditioned GMRES method with Toeplitz-like structure was employed to solve the discretized linear systems arising from FDEs. The displacement structure of the Toeplitz-like preconditioner was computed

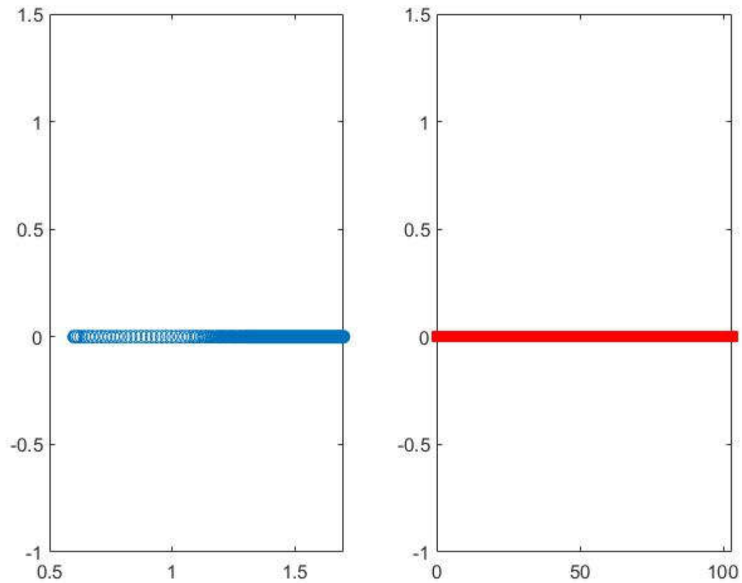

Figure 2: Eigenvalues of $P^{-1}A$ (left) and A (right) for Example 1, with $\alpha = .9$ and $n = 2^{12}$.

precisely. The efficiency of the proposed preconditioner was proved even though the diffusion coefficients are not constants. Numerical experiments have demonstrated the efficiency of the proposed preconditioner.

Nevertheless, it is interesting that to propose the possible Toeplitz-like preconditioner for two-dimensional FDEs, and then examine the efficiency of the proposed preconditioner.

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