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Research Article

Differential-integral Euler–Lagrange equations

M. Shehata*

Abstract

We study the calculus of variations problem in the presence of a system of differential-integral (D-I) equations. In order to identify the necessary optimality conditions for this problem, we derive the so-called D-I Euler–Lagrange equations. We also generalize this problem to other cases, such as the case of higher orders, the problem of optimal control, and we derive the so-called D-I Pontryagin equations. In special cases, these formulations lead to classical Euler–Lagrange equations. To illustrate our results, we provide simple examples and applications such as obtaining the minimum power for an RLC circuit.

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*Corresponding author

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Mohammed Shehata
Department of Basic Science, Bilbeis Higher Institute for Engineering, Sharqia, Egypt.
e-mail: mashehata_ math@yahoo.com, mashehata_ math@bhie.edu.eg

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1 Introduction

The calculus of variations began with Johann Bernoulli's Brachistochrone problem at the end of the 17th century. As a result of their work, Euler and Lagrange were able to develop a systematic way of dealing with this kind of problem by introducing what is now known as the Euler–Lagrange equation in the 18th century. This work was then extended in many ways by Bliss, Bolza, Caratheodory, Clebsch, Hahn, Hamilton, Hilbert, Kneser, Jacobi, Legendre, Mayer, Weierstrass, just to quote a few; see [4, 5, 11]. For an interesting historical book on one-dimensional problems of the calculus of variations, see [8].

The classical variational calculus has one major shortcoming despite its great success, it only deals with functionals containing derivatives. Many phenomena in nature can be modeled more accurately using differential integral equations. The application of these equations is found in science, biology, engineering, and economics; see [1, 2, 3, 6, 7, 9, 10, 12, 15, 16]. It is not worthwhile in applications to convert integrals into differentials, especially if there are many integrals of higher orders. In [13], an algorithm has been constructed to compute the exact solutions for the quadratic optimal control problem with integral constraints, and this algorithm has been used to find the optimal solution for single and coupled RC electrical circuits. In this paper, we identify differential-integral (D-I) Euler–Lagrange equations necessary conditions for a new class of variational problems in which a cost functional involves differential and integral operators.

2 Definitions and notations

Definition 1 (Lower and upper integrals)**.** For a given time horizon $[t_0, t_f]$, we define lower and upper integration of a continuous function $x : [t_0, t_f] \to \Re$ by

$$\underline{I}_K x = \int_{t_0}^{t} K(t,\tau)\, x(\tau)\, d\tau, \quad \overline{I}_K x = \int_{t}^{t_f} K(t,\tau)\, x(\tau)\, d\tau$$

with continuous kernel $K(t, \tau)$. We can define lower and upper higher order integrals as follows:

$$\underline{I}^2_{K_1K_2}x = \underline{I}_{K_1}\left(\underline{I}_{K_2}x\right), \qquad \overline{I}^2_{K_1K_2}x = \overline{I}_{K_1}\left(\overline{I}_{K_2}x\right),$$
$$\underline{I}^2_{K}x = \underline{I}_{K}\left(\underline{I}_{K}x\right), \qquad \overline{I}^2_{K}x = \overline{I}_{K}\left(\overline{I}_{K}x\right),$$

and so on.

Definition 2 (Complementary integral)**.** For a given time horizon $[t_0, t_f]$ and continuous function $x : [t_0, t_f] \to \Re$, we define the complement of the integral

$$\underline{I}_K x = \int_{t_0}^{t} K(t,\tau)\, x(\tau)\, d\tau, \text{ by } \overline{I}_{\overline{K}} x = \int_{t}^{t_f} \overline{K}(t,\tau)\, x(\tau)\, d\tau,$$

where $\overline{K}(t,s) := K(s,t)$.

For $K = 1$ we denote it by $\underline{I}_1 x = \underline{I}x, \qquad \overline{I}_1 x = \overline{I}x$.

Applying the Leibniz integral rule $n+1$ times to $\int_{t_0}^{t}(t-\tau)^n$ and $\int_{t_0}^{t_f}(\tau-t)^n$, respectively, we obtain the Cauchy formulas for repeated integration.

Theorem 1 (Cauchy formulas)**.** If $x(t)$ is a continuous function over $[t_0, t_f]$, then

1. $$\int_{t_0}^{t}(t-\tau)^n\, x(\tau)d\tau = n! \underbrace{\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{\tau_1}\cdots\int_{t_0}^{\tau_{n-1}}}_{n+1\; times} x(\tau_n)d\tau_n d\tau_{n-1}\dots d\tau_1 d\tau,$$

2. $$\int_{t}^{t_f}(\tau-t)^n\, x(\tau)d\tau = n! \underbrace{\int_{t}^{t_f}\int_{\tau}^{t_f}\int_{\tau_1}^{t_f}\cdots\int_{\tau_{n-1}}^{t_f}}_{n+1\; times} x(\tau_n)d\tau_n\, d\tau_{n-1}\dots d\tau_1 d\tau.$$

From this theorem, we can define lower and upper higher integrals $\underline{I}^n\, x$, $\overline{I}^n x$ by

$$\underline{I}^n\, x = \frac{1}{(n-1)!}\int_{t_0}^{t}(t-\tau)^{n-1}x(\tau)\, d\tau,$$
$$\overline{I}^n x = \frac{1}{(n-1)!}\int_{t}^{t_f}(\tau-t)^{n-1}x(\tau)\, d\tau.$$

Through this paper, $D = \dfrac{d}{dt}$, and in general $D^n = \dfrac{d^n}{dt^n}$.

Note that

(i) $D^n\,(\underline{I}^n\, x(t)) = x(t)$ and $D^n\left(\overline{I}^n\, x(t)\right) = (-1)^n x(t)$.

(ii) $\underline{I}\,(D\, x(t)) = x(t) - x(t_0)$ and $\overline{I}\,(D\, x(t)) = x(t_f) - x(t)$.

3 D-I Euler–Lagrange equations

The first simplest D-I variations problem with fixed ends can be defined as follows: Among all functions $x(t)$ that satisfy the fixed end conditions

$$x(t_0) = x_0, \quad x(t_f) = x_f, \tag{1}$$

find the function for which the functional

$$J(x) = \int_{t_0}^{t_f} f\left(t, x(t), D\,x(t), \underline{I}_{K_1}x(t), \overline{I}_{K_2}x(t)\right) dt, \tag{2}$$

is an extremum. We assume that $f : [t_0, t_f] \times \Re^4 \to \Re$ has continuous first and second partial derivatives with respect to all of its arguments.

To derive the necessary conditions for the extremum, assume that $x = x(t)$ is the desired curve, and take some admissible curve $x = \bar{x}(t)$ close to $x = x(t)$ and include the curves $x = x(t)$ in one parameter family of curves

$$x(t, \epsilon) = x(t) + \epsilon\eta, \quad \eta = \bar{x}(t) - x(t), where t belongs to [t_0, t_f].$$

If one considers the values of the functional (2) only on curves of the family $x(t, \epsilon)$, then the functional becomes a function of ϵ:

$$J(y(\epsilon)) = \varphi(\epsilon).$$

This function $\varphi(\epsilon)$ is extremized for $\epsilon = 0$ since for $\epsilon = 0$ we have $x = x(t)$. The necessary conditions for the extremum of the function $\varphi(\epsilon)$ for $\epsilon = 0$ is as we know that $\varphi'(0) = 0$. Therefore we have proved the following lemma.

Lemma 1 (First variation condition)**.** If $x = x(t)$ is a solution to problem (1)–(2), then $\frac{\partial}{\partial \epsilon}\left(J(x + \epsilon\eta)\right|_{\epsilon=0} = 0$, for some functions $\eta(t)$ satisfies $\eta(t_0) = \eta(t_f) = 0$.

We also know from the calculus of variations, the following fundamental lemma.

Lemma 2 (The fundamental lemma)**.** If for every continuous function $\eta(t)$

$$\int_{t_0}^{t_f} \Psi(t)\eta(t)dt = 0,$$

where the function $\Psi(t)$ is continuous on the interval $[t_0, t_f]$, then $\Psi(t) \equiv 0$ on that interval.

From the above two lemmas, we will prove the following theorem.

Theorem 2 (D-I Euler Lagrange conditions)**.** If $x = x(t)$ is a solution of problem (1)–(2), then

$$\frac{\partial f}{\partial x} - D\left(\frac{\partial f}{\partial Dx}\right) + \overline{I}_{\overline{K_1}}\left(\frac{\partial f}{\partial \underline{I}_{K_1}x}\right) + \underline{I}_{\overline{K_2}}\left(\frac{\partial f}{\partial \overline{I}_{K_2}x}\right) = 0. \tag{3}$$

Proof. By Lemma 1, if $x = x(t)$ is a solution of Problem 1, then $\frac{\partial}{\partial \epsilon}\left(J(x+\epsilon\eta)\right|_{\epsilon=0} = 0$, for some functions $\eta(t)$ satisfying $\eta(a) = \eta(b) = 0$, and it follows that

$$\int_{t_0}^{t_f}\left[\frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial Dx}D\eta + \frac{\partial f}{\partial \underline{I}_{K_1}x}\underline{I}_{K_1}\eta + \frac{\partial f}{\partial \overline{I}_{K_2}x}\overline{I}_{K_2}\eta\right]dt = 0. \tag{4}$$

We integrate the second term by parts, and we get

$$\begin{aligned}\int_{t_0}^{t_f}\frac{\partial f}{\partial Dx}D\eta dt &= \left[\frac{\partial f}{\partial x}(t_f)\eta(t_f) - \frac{\partial f}{\partial Dx}(t_0)\eta(t_0)\right] - \int_{t_0}^{t_f} D\left(\frac{\partial f}{\partial Dx}\right)\eta dt\\ &= -\int_{t_0}^{t_f} D\left(\frac{\partial f}{\partial Dx}\right)\eta dt. \end{aligned}\tag{5}$$

By changing the order of the integrations in the third and fourth term in (4), we get

$$\int_{t_0}^{t_f}\frac{\partial f}{\partial \underline{I}_{K_1}x}\underline{I}_{K_1}\eta dt = \int_{t_0}^{t_f}\overline{I}_{\overline{K_1}}\left(\frac{\partial f}{\partial \underline{I}_{K_1}x}\right)\eta dt, \tag{6}$$

$$\int_{t_0}^{t_f}\frac{\partial f}{\partial \overline{I}_{K_2}x}\overline{I}_{K_2}\eta dt = \int_{t_0}^{t_f}\underline{I}_{\overline{K_2}}\left(\frac{\partial f}{\partial \overline{I}_{K_2}x}\right)\eta dt. \tag{7}$$

Thus, substituting (5), (6), and (7) back into (4), gives us

$$\begin{aligned}&\left[\frac{\partial f}{\partial Dx}(t_f)\eta(t_f) - \frac{\partial f}{\partial Dx}(t_0)\eta(t_0)\right]\\ &\quad + \int_{t_0}^{t_f}\left[\frac{\partial f}{\partial x} - D\left(\frac{\partial f}{\partial Dx}\right) + \overline{I}_{\overline{K_1}}\left(\frac{\partial f}{\partial \underline{I}_{K_1}x}\right) + \underline{I}_{\overline{K_2}}\left(\frac{\partial f}{\partial \overline{I}_{K_2}x}\right)\right]\eta dt = 0. \end{aligned}\tag{8}$$

Finally, from Lemma 2 and $\eta(t_0) = \eta(t_f) = 0$, we obtain the desired D-I Euler–Lagrange equation (3). □

Remark 1. By substituting $t = t_0$ in (3), we obtain the natural condition

$$\left[\frac{\partial f}{\partial x} - D\left(\frac{\partial f}{\partial Dx}\right) + \int_{t_0}^{t_f} \overline{K_1}\left(\frac{\partial f}{\partial \underline{I}_{K_1} x}\right) d\tau\right]_{\tau=t_0} = 0, \tag{9}$$

and by substituting $t = t_f$ in (3), we obtain the natural condition

$$\left[\frac{\partial f}{\partial x} - D\left(\frac{\partial f}{\partial Dx}\right) + \int_{t_0}^{t_f} \overline{K_2}\left(\frac{\partial f}{\partial \overline{I}_{K_2} x}\right) d\tau\right]_{\tau=t_f} = 0. \tag{10}$$

Special cases. There are some special cases of D-I Euler–Lagrange, which are important in many applications:

case 1. If f is independent of $\underline{I}_K x$, then D-I Euler–Lagrange conditions are reduced to so called $(D - \overline{I})$ Euler–Lagrange equation:

$$\frac{\partial f}{\partial x} - D\left(\frac{\partial f}{\partial Dx}\right) + \underline{I}_{\overline{K}}\left(\frac{\partial f}{\partial \overline{I}_K x}\right) = 0,$$

and if $K = (\tau - t)^n$, then $(D - \overline{I})$ Euler–Lagrange conditions become

$$\frac{\partial f}{\partial x} - D\left(\frac{\partial f}{\partial Dx}\right) + \underline{I}^n\left(\frac{\partial f}{\partial \overline{I}^n x}\right) = 0.$$

case 2. If f is independent of $\underline{I}_K x$, then D-I Euler–Lagrange conditions are reduced to so called $(D - \underline{I})$ Euler–Lagrange equation:

$$\frac{\partial f}{\partial x} - D\left(\frac{\partial f}{\partial Dx}\right) + \overline{I}_{\overline{K}}\left(\frac{\partial f}{\partial \underline{I}_K x}\right) = 0,$$

and if $K = (t - \tau)^n$, then $(D - \underline{I})$ Euler–Lagrange conditions become

$$\frac{\partial f}{\partial x} - D\left(\frac{\partial f}{\partial Dx}\right) + \overline{I}^n\left(\frac{\partial f}{\partial \underline{I}^n x}\right) = 0.$$

case 3. If f is independent of both $\underline{I}_{K_1} x$, $\overline{I}_{K_2} x$, then D-I Euler–Lagrange conditions are reduced to the usual Euler–Lagrange equation:

$$\frac{\partial f}{\partial x} - D\left(\frac{\partial f}{\partial Dx}\right) = 0.$$

4 Generalizations

In this section, we generalized the fixed boundaries problem to the cases of integral with deferent kernels, moving boundaries, higher order, and several independent variables.

Integral with different kernel

Consider the functional

$$J(x) = \int_{t_0}^{t_f} f\left(t, x, Dx, \underline{I}_{K_{11}}\, x, \underline{I}_{K_{12}}\, x, \dots, \underline{I}_{K_{1\ell}} x, \overline{I}_{K_{21}}\, x, \overline{I}_{K_{22}}\, x, \dots, \overline{I}_{K_{2k}} x\right) dt, \tag{11}$$

where $f : [t_0, t_f] \times \Re^{2+m+\ell} \to \Re$ has continuous partial derivatives up to the order two with respect to all its arguments. Moreover, t_0 and t_f are specified, and the boundary conditions are

$$x(t_0) = x_0, \quad x(t_f) = x_f.$$

For this case, following the above approach, we obtain the following necessary conditions

$$\frac{\partial f}{\partial Dx} - D\left(\frac{\partial f}{\partial Dx}\right) + \sum_{j=1}^{\ell} \overline{I}_{\overline{K_{1j}}} \left(\frac{\partial f}{\partial\, \underline{I}_{K_{1j}} x}\right) + \sum_{j=1}^{k} \underline{I}_{\overline{K_{2j}}} \left(\frac{\partial f}{\partial\, \overline{I}_{K_{2j}} x}\right) = 0. \tag{12}$$

Moving boundaries

Let the terminal conditions at $t = t_0$ and/or at $t = t_f$ not be specified. For this case, following the above approach, we obtain the D-I Euler–Lagrange equation given by (3), and the following transversally conditions:

$$\left.\frac{\partial f}{\partial Dx}\right]_{t=t_0} = 0, \quad \text{if}\, x(t_0)\, \text{is not satisfied and}$$

$$\left.\frac{\partial f}{\partial Dx}\right]_{t=t_f} = 0, \quad \text{if}\, x(t_f)\, \text{is not satisfied.}$$

Higher order

Consider the functional

$$J(x)=\int_{t_0}^{t_f} f\left(t,x,Dx,\dots,D^m x,\underline{I}_{K_1}x,\underline{I}_{K_1}^2 x,\dots,\underline{I}_{K_1}^{\ell}x,\overline{I}_{K_2}x,\overline{I}_{K_2}^2 x,\dots,\overline{I}_{K_2}^{k}x\right)dt, \tag{13}$$

where $f:[t_0,t_f]\times\Re^{1+m+\ell+k}\to\Re$ has continuous partial derivatives up to the order $m+1$ with respect to all its arguments. Moreover, t_0 and t_f are specified, and the boundary conditions are

$$\begin{aligned} x(t_0)&=x_0, & x(t_f)&=x_f,\\ &\vdots & &\vdots\\ D^m x(t_0)&=x_{m0}, & D^m x(t_f)&=x_{mf}. \end{aligned}$$

For this case, following the above approach, we obtain the following necessary conditions:

$$\sum_{i=0}^{m}(-1)^i D^i\left(\frac{\partial f}{\partial D^i x}\right)+\sum_{j=1}^{\ell}\overline{I}_{\underline{K_1}}^{j}\left(\frac{\partial f}{\partial\,\underline{I}_{K_1}^{j}x}\right)+\sum_{j=1}^{k}\underline{I}_{\overline{K_2}}^{j}\left(\frac{\partial f}{\partial\,\overline{I}_{K_2}^{j}x}\right)=0. \tag{14}$$

Several independent variables

Consider the functional

$$\begin{aligned} &J(x_1,\dots,x_n) \\ &=\int_{t_0}^{t_f} f\left(t,x_1,\dots,x_n,x_1',\dots,x_n',\dots,\underline{I}_{K_1}x_1,\dots,\underline{I}_{K_1}x_n,\overline{I}_{K_2}x_1,\dots,\overline{I}_{K_2}x_n\right)dt, \end{aligned} \tag{15}$$

where $x_1,x_2,\dots,x_n$ are independent functions with continuous first derivatives and $f:[t_0,t_f]\times\Re^{4n}\to\Re$ has continuous first and second partial derivatives with respect to all of its arguments. Moreover, t_0 and t_f are specified, and the boundary conditions are

$$\begin{aligned} x_1(t_0)&=x_{10}, & x_1(t_f)&=x_{1f},\\ &\vdots & &\vdots\\ x_n(t_0)&=x_{n0}, & x_n(t_f)&=x_{nf}. \end{aligned}$$

For this case, following the above approach, we obtain the following necessary conditions:

$$\frac{\partial f}{\partial x_i} - D\left(\frac{\partial f}{\partial Dx_i}\right) + \overline{I}_{\overline{K_1}}\left(\frac{\partial f}{\partial\, \underline{I}_{K_1} x_i}\right) + \underline{I}_{\overline{K_2}}\left(\frac{\partial f}{\partial\, \overline{I}_{K_2} x_i}\right) = 0, \quad i = 1, 2, \ldots, n. \tag{16}$$

5 D-I optimal control problem

We shall consider the class of control problems where the dynamical system is described by the following ordinary $D - \underline{I}$ equations:

$$Dx = f(t, x, \underline{I}_{K_1} x, \overline{I}_{K_2} x, u), \tag{17}$$

$$x(t_0) = x_0, \quad t_0 \text{ and } t_f \text{ are specified}, \tag{18}$$

where $x(t)$ an n-vector function is determined by $u(t)$ an m-vector function, with $x \in \Re^n$, $u \in \Re^m$.

The performance of the system is measured by the cost functional:

$$J(x) = S(t_f, x(t_f)) + \int_{t_0}^{t_f} L(t, x, \underline{I}_{K_1} x, \overline{I}_{K_2} x, u) dt. \tag{19}$$

The problem is to find the functions $u(t)$ that minimize (or maximize) $)J$. It is assumed that $f(t, x, \underline{I}_{K_1} x, \overline{I}_{K_2} x, u)$ and $L(t, x, \underline{I}_{K_1} x, \overline{I}_{K_2} x, u)$ are continuous for all $t \in [t_0, t_f]$, $x \in \Re^n$, $u \in \Re^m$, and have continuous derivative up to the second order.

Theorem 3 (D-I (Pontryagin))**.** If $u(t)$ is a solution to the problem (17)–(19), then the following equations are satisfied:

state equations

$$Dx = \frac{\partial H}{\partial \lambda} = f(t, x, \underline{I}_{K_1} x, \overline{I}_{K_2} x, u); \tag{20}$$

$$x(t_0) = x_0; \tag{21}$$

adjoint equations

$$-D\lambda = \left(\frac{\partial H}{\partial x}\right)^T + \overline{I}_{\overline{K_1}}\left(\frac{\partial F}{\partial\, \underline{I}_{K_1} x}\right)^T + \underline{I}_{\overline{K_2}}\left(\frac{\partial F}{\partial\, \overline{I}_{K_2} x}\right)^T; \tag{22}$$

optimality conditions

$$0 = \left(\frac{\partial H}{\partial u}\right)^T; \tag{23}$$

transversality condition

$$\lambda(t_f) = \left.\left(\frac{\partial S}{\partial x}\right)^T\right]_{t=t_f}; \tag{24}$$

where

$$H = L(t, x, \underline{I}_{K_1}x, \overline{I}_{K_2}x, u) + \lambda^T (t, x, \underline{I}_{K_1}x, \overline{I}_{K_2}x, u) \tag{25}$$

is the usual Hamiltonian.

Proof. First $S(t_f, x(t_f))$ can be written as

$$S(t_f, x(t_f)) = S(t_0, x(t_0)) + \int_{t_0}^{t_f} \frac{d}{dt} S(t, x(t))dt \tag{26}$$

$$= S(t_0, x(t_0)) + \int_{t_0}^{t_f} \left[\frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}x'\right] dt. \tag{27}$$

Equation (19) becomes

$$J(x) = S(t_0, x(t_0)) + \int_{t_0}^{t_f} L(t, x, \underline{I}_{K_1}x, \overline{I}_{K_2}x, u) + \left[\frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}x'\right] dt. \tag{28}$$

Adjoin the system differential equations (19) to J with multiplier functions $\lambda(t)$ and we have

$$\begin{aligned}\hat{J}(x) &= S(t_0, x(t_0)) + \int_{t_0}^{t_f} H - \lambda^T Dx + \left[\frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}x'\right] dt\\ &= S(t_0, x(t_0)) + \int_{t_0}^{t_f} F(t, x, Dx, \underline{I}_{K_1}x, \overline{I}_{K_2}x, u, \lambda),\end{aligned} \tag{29}$$

where $F(t, x, Dx, \underline{I}_{K_1}x, \overline{I}_{K_2}x, u, \lambda) = H - \lambda^T Dx + \left[\frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}x'\right]$.

Following the same approach in the calculus of variations ((8)) gives

$$\begin{aligned}&\left[\frac{\partial F}{\partial Dx}(t_f)\eta(t_f) - \frac{\partial F}{\partial Dx}(t_0)\eta(t_0)\right]\\ &\quad + \int_{t_0}^{t_f} \left[\frac{\partial F}{\partial x} - D\left(\frac{\partial F}{\partial Dx}\right) + \overline{I}_{\overline{K_1}}\left(\frac{\partial F}{\partial \underline{I}_{K_1}x}\right) + \underline{I}_{\overline{K_2}}\left(\frac{\partial F}{\partial \overline{I}_{K_2}x}\right)\right] \eta dt = 0\end{aligned}$$

for some $\eta(t_0) = 0$.

From the definition of F and the fact that the D-I Euler equation must be satisfied, we have

$$\begin{aligned}
&\frac{\partial F}{\partial x} - D\left(\frac{\partial F}{\partial Dx}\right) + \overline{I}_{\overline{K_1}}\left(\frac{\partial F}{\partial \underline{I}_{K_1} x}\right) + \underline{I}_{\overline{K_2}}\left(\frac{\partial F}{\partial \overline{I}_{K_2} x}\right) \\
&\quad = \frac{\partial H}{\partial x} + \overline{I}_{\overline{K_1}}\left(\frac{\partial H}{\partial \underline{I}_{K_1} x}\right) + \underline{I}_{\overline{K_2}}\left(\frac{\partial H}{\partial \overline{I}_{K_2} x}\right) + \frac{\partial}{\partial x}\left[S_t + S_x x'\right] \\
&\quad\quad + D\left(\lambda^T - S_x\right) \\
&\quad = \frac{\partial H}{\partial x} + \overline{I}_{\overline{K_1}}\left(\frac{\partial F}{\partial \underline{I}_{K_1} x}\right) + \underline{I}_{\overline{K_2}}\left(\frac{\partial F}{\partial \overline{I}_{K_2} x}\right) + D\left(\lambda^T\right) = 0. \qquad (30)
\end{aligned}$$

This gives (22). Similarly, λ and u being independent variables, then

$$\begin{aligned}
\frac{\partial F}{\partial \lambda} &= \frac{\partial F}{\partial \lambda} = \frac{\partial H}{\partial \lambda} - Dx = 0, \\
\frac{\partial F}{\partial u} &= \frac{\partial F}{\partial u} = \frac{\partial H}{\partial u} = 0.
\end{aligned}$$

This gives (20) and (23), respectively. Finally, the transversally or boundary conditions given by the remaining terms of (30) are

$$\frac{\partial F}{\partial x'}(t_f)\eta(t_f) = \left[\frac{\partial S}{\partial x} - \lambda^T\right]\eta(t_f) = 0. \qquad (31)$$

The fact that $\eta(t_f)$ does not vanish, yields (24). □

6 Examples

To illustrate our result, we give some examples.

Example 1. In this example, we want to find the unknown supplied voltage $u(t)$ for the RLC circuit in Figure 1, which minimizes the cost functional given by

$$J = \frac{1}{2}i^2(5) + \frac{1}{2}\int_0^5 u^2(t)dt. \qquad (32)$$

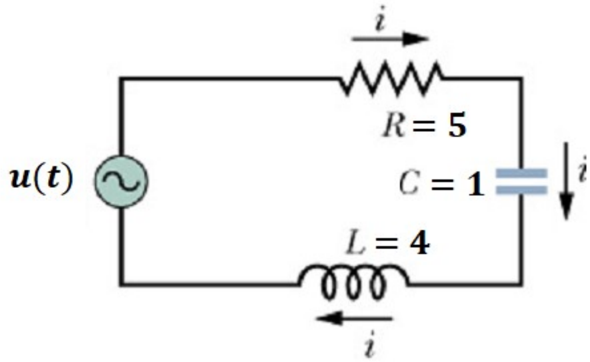

Figure 1: Series RLC circuit.

By applying the Kirchhoff's voltage law, we get

$$4\,\frac{d}{dt}\,i(t) + 5\,i(t) + \int_0^t i(\tau)d\tau = u(t). \tag{33}$$

By applying D-I Pontryagin necessary conditions to this problem with $x \equiv i$, $t_0 = 0$, $t_f = 5$ and

$$H = \frac{1}{2}u^2(t) + \lambda(t)\left[-\frac{5}{4}\,i(t) - \frac{1}{4}\int_0^t i(\tau)d\tau + \frac{1}{4}\,u(t)\right], \tag{34}$$

the optimal control for the problem (32)–(33) is characterized by

$$u(t) = -\frac{1}{4}\,\lambda(t),$$

where $i(t)$ and $\lambda(t)$ satisfy the following equations:

State equations

$$\frac{d\,i(t)}{dt} = -\frac{5}{4}\,i(t) - \frac{1}{4}\int_0^t i(\tau)d\tau - \frac{1}{16}\,\lambda(t), \tag{35}$$

$$i(0) = 1, \tag{36}$$

Adjoint equations

$$\frac{d\,\lambda(t)}{dt} = \frac{5}{4}\lambda(t) + \frac{1}{4}\int_t^5 \lambda(\tau)\,d\tau, \tag{37}$$

$$\lambda(5) = i(5). \tag{38}$$

Remark 2. Equations (35) and (37) provide the necessary conditions for the problem. They constitute two second order D-I equations whose solution contains four constants of integration. To evaluate these, we have 1-equation $i(0) = 1$, 1-equation $\lambda(5) = i(5)$, 1-equation $\underline{I}i(t) = 0$ at $t = 0$ and 1-equation $\overline{I}\lambda(t) = 0$ at $t = 5$.

To solve the adjoint equation (37), let

$$\lambda_1(t) = \int_t^5 \lambda(\tau)\,d\tau, \quad \lambda_2(t) = \frac{d\lambda_1(t)}{dt} = -\lambda(t).$$

Then (37)–(38) can be written in the following matrix form:

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ \frac{-1}{4} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix}$$

with final conditions:

$$\begin{bmatrix} \lambda_1(5) \\ \lambda_2(5) \end{bmatrix} = \begin{bmatrix} 0 \\ -i(5) \end{bmatrix},$$

which have the solution

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = e^{\begin{bmatrix} 0 & 1 \\ \frac{-1}{4} & \frac{5}{4} \end{bmatrix}(t-5)} \begin{bmatrix} 0 \\ -i(5) \end{bmatrix}.$$

Now (see, for example, [14]),

$$\begin{bmatrix} 0 & 1 \\ \frac{-1}{4} & \frac{5}{4} \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{-1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{bmatrix}^{-1}.$$

Then

$$\begin{aligned} e^{\begin{bmatrix} 0 & 1 \\ \frac{-1}{4} & \frac{5}{4} \end{bmatrix}(t-5)} &= \begin{bmatrix} \frac{-1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} e^{(t-5)} & 0 \\ 0 & e^{\frac{1}{4}(t-5)} \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -e^{5-t} + 4e^{\frac{1}{4}(5-t)} & 4e^{5-t} - 4e^{\frac{1}{4}(5-t)} \\ -e^{5-t} + e^{\frac{1}{4}(5-t)} & 4e^{5-t} - e^{\frac{1}{4}(5-t)} \end{bmatrix}. \end{aligned}$$

So,

$$\lambda(t) = -\lambda_2(t) = \frac{i(5)}{3}\left[4\,e^{5-t} - \,e^{\frac{1}{4}(5-t)}\right].$$

To solve the state equation (35), let

$$i_1(t) = \int_0^t i(\tau)\, d\tau, \quad i_2(t) = \frac{di_1(t)}{dt} = i(t).$$

Then (35)–(36) can be written in the following nonhomogeneous matrix form

$$\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ \frac{-1}{4} & \frac{-5}{4} \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + \psi(t, i(5))$$

with initial conditions:

$$\begin{bmatrix} i_1(0) \\ i_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $\psi(t, i(5)) = \begin{bmatrix} 0 \\ \frac{-i(5)}{16} \left[\frac{4}{3}\, e^{5-t} - \frac{1}{3}\, e^{\frac{1}{4}(5-t)} \right] \end{bmatrix}$, which have the solution

$$\begin{aligned}
\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} &= e^{\begin{bmatrix} 0 & 1 \\ \frac{-1}{4} & \frac{-5}{4} \end{bmatrix} t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t e^{\begin{bmatrix} 0 & 1 \\ \frac{-1}{4} & \frac{-5}{4} \end{bmatrix} (t-\tau)} \psi(\tau, i(5)) d\tau \\
&= \begin{bmatrix} \frac{-4}{3} e^{-t} + \frac{4}{3} e^{\frac{-t}{4}} \\ \frac{4}{3} e^{-t} - \frac{1}{3} e^{\frac{-t}{4}} \end{bmatrix} \\
&\quad - \frac{i(5)}{16} \int_0^t \begin{bmatrix} \left(\frac{-4}{3} e^{\tau - t} + \frac{4}{3} e^{\frac{1}{4}(\tau - t)} \right) \left(\frac{4}{3}\, e^{5-t} - \frac{1}{3}\, e^{\frac{1}{4}(5-t)} \right) \\ \left(\frac{4}{3} e^{\tau - t} - \frac{1}{3} e^{\frac{1}{4}(\tau - t)} \right) \left(\frac{4}{3}\, e^{5-t} - \frac{1}{3}\, e^{\frac{1}{4}(5-t)} \right) \end{bmatrix} d\tau.
\end{aligned}$$

So,

$$\begin{aligned}
i(t) = i_2(t) &= \frac{4}{3} e^{-t} - \frac{1}{3} e^{\frac{-t}{4}} + \frac{i(5)}{9} \left[e^{5-2t} - \frac{1}{4} e^{\frac{5}{4} - \frac{5}{4} t} - e^{5 - \frac{5}{4} t} + \frac{1}{4} e^{\frac{5}{4} - \frac{1}{2} t} \right] \\
\Rightarrow i(5) &= \frac{4 \left(4e^{-5} - e^{-\frac{5}{4}} \right)}{12 - e^{-5} + e^{\frac{-5}{4}}}.
\end{aligned}$$

Hence, we obtain the control

$$u(t) = -\frac{\left(4e^{-5} - e^{-\frac{5}{4}} \right) \left(e^{5-t} - e^{\frac{1}{4}(5-t)} \right)}{3 \left(12 - e^{-5} + e^{\frac{-5}{4}} \right)} \tag{39}$$

and the current (see Figure 2)

$$i(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{\frac{-t}{4}} + \frac{4\left(4e^{-5} - e^{-\frac{5}{4}}\right)\left(e^{5-2t} - \frac{1}{4}e^{\frac{5}{4}-\frac{5}{4}t} - e^{5-\frac{5}{4}t} + \frac{1}{4}e^{\frac{5}{4}-\frac{1}{2}t}\right)}{9\left(12 - e^{-5} + e^{\frac{-5}{4}}\right)}. \tag{40}$$

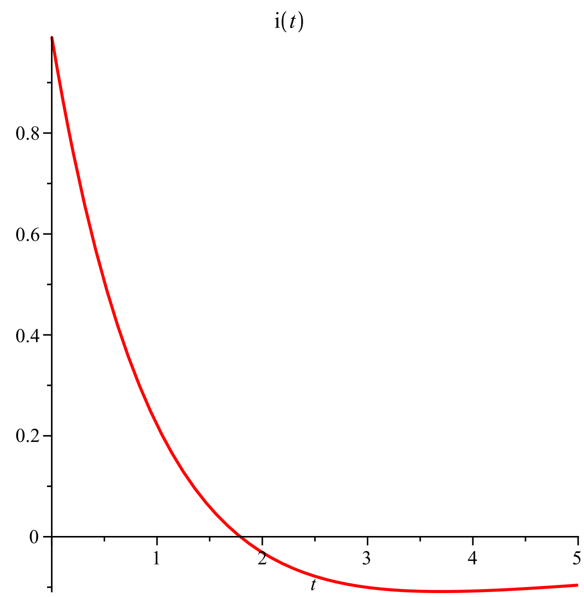

Figure 2: Optimal electrical current.

Example 2. In this example, we want to find $u(t)$ that minimizes the cost functional given by

$$J = \frac{1}{2}\int_0^2 u^2(t)dt + \frac{1}{2}\int_0^2 \left[\int_0^t (t-\tau)x(\tau)\, d\tau\right]^2 dt$$

with constraints

$$Dx(t) = u(t), \quad 0 < t \leq 2,$$
$$x(0) = 1.$$

By applying D-I Pontryagin necessary conditions to this problem, the optimal control is characterized by

$$u = -\lambda,$$

$$\begin{aligned}
Dx &= -\lambda,\\
-D\lambda &= \int_t^2 \left\{(\tau - t)\int_0^\tau (\tau - s)x(s)\,ds\right\} d\tau,\\
x(0) &= 1,\\
\lambda(2) &= 0.
\end{aligned}$$

The above system is simplified to the following equations:

$$Dx(t) = -\int_t^2 \left\{\int_\tau^2 (r - \tau_1)\int_0^r (r - s)x(s)\,dsdr\right\} d\tau_1\,d\tau, \tag{41}$$

$$x(0) = 1. \tag{42}$$

To solve (41)–(42). Let $x_1 = x(t)$, $x_2 = \int_0^t x_1(\tau)\,d\tau$, $x_3(t) = \int_0^t x_2(\tau)\,d\tau$, $x_4(t) = \int_t^2 x_3(\tau)\,d\tau$, $x_5(t) = \int_t^2 x_4(\tau)\,d\tau$ and $x_6(t) = \int_t^2 x_5(\tau)\,d\tau$. Then (41)–(42) is equivalent to the following system:

$$\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6 \end{bmatrix}' = \begin{bmatrix} 0&0&0&0&0&-1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0 \end{bmatrix} \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6 \end{bmatrix}$$

with

$$x_1(0) = 1,\quad x_2(0) = 0,\quad x_3(0) = 0,\quad x_4(2) = 0,\quad x_5(2) = 0,\quad x_6(2) = 0,$$

which leads to the graph of $x(t)$ as shown in Figure 3.

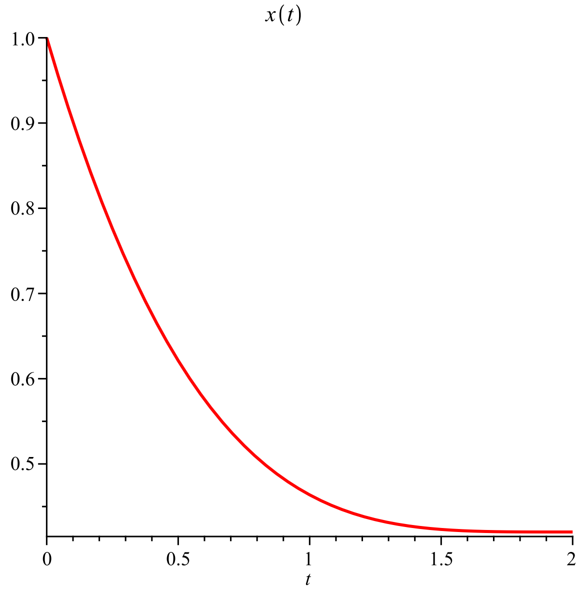

Figure 3: Optimal state solution $x(t)$.

7 Conclusion

In this paper, we have identified D-I Euler–Lagrange equations necessary conditions for a new class of variational problems in which a cost functional involving differential and integral operators. We concluded that if Euler–Lagrange equations contain an integral, then they must contain the complementary integral. We also generalized results to other problems.

Declarations

Conflict of interest: The author declare that he has no conflict of interest.

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