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The variational iteration method for solving linear and nonlinear Schrodinger equations*

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Abstract

In this paper, the variational iteration method which proposed by Ji-Huan He is applied to solve both linear and nonlinear Schrodinger equations. The main property of the method is in its flexibility and ability to solve linear and nonlinear equations accurately and conveniently. In this method, general Lagrange multipliers are introduced to construct correction functionals to the problems. The multipliers in the functionals can be identified optimally via the variational theory. Numerical results show that this method can readily be implemented with excellent accuracy to linear and nonlinear Schrodinger equations. This technique can be extended to higher dimensions linear and nonlinear Schrodinger equations without a serious difficulties.

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1 Introduction

The variational iteration method (VIM) was first proposed by Ji-Huan He in 1998 [6,7] and systematically illustrated in 1999 [11]. Since then, it has been success-

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fully applied to various engineering problems [15,16]. This method is employed in [18] to solve the Klein-Gordon equation which is the relativistic version of the Schrodinger equation, which is used to describe spinless particles. Application of He's variational iteration technique to an inverse parabolic problem is described in [4]. In [2] the VIM is employed to solve the time dependent reaction-diffusion equation which has special importance in engineering and sciences and constitutes a good model for many systems in various fields. This technique is also employed in [5] to solve the Fokker-Planck equation and in [3] to solve a biological population model. For more application of the method, the interested reader is referred to [1,17,19,21]. The VIM [11,12] is a powerful tool to search for approximate solutions of linear and nonlinear equations without requirement of linearization or perturbation. Another important advantage is that the VIM is capable of greatly reducing the size of calculation while still maintaining high accuracy of the numerical solution. Moreover, the power of the method gives a wider applicability in handling a huge number of analytical and numerical applications. The convergence of He'variational iterative method is investigated in [20]. Here, we apply VIM to one and two dimensional linear and nonlinear Schrodinger equations. This paper is organized as follows: In Section 2, we introduce the model of the problems. In Section 3, first we describe VIM method and then we apply VIM in a direct manner to establish exact solutions for linear and nonlinear Schrodinger equations. In Section 4, we describe the numerical solution of linear and nonlinear Schrodinger equations to show the power of the method in a unified manner without requiring any additional restriction.

2 The model of the problem

In this paper, the linear Schrodinger equation is considered as follows:

$$\frac{\partial \psi}{\partial t}(x,t) + i\frac{\partial^2 \psi}{\partial x^2}(x,t) = 0, \quad \psi(x,0) = f(x), \;\; x \in \mathbb{R}, \;\; t \geq 0, \;\; i^2 = -1, \tag{1}$$

and we consider the nonlinear Schrodinger equation of the form

$$i\frac{\partial\psi}{\partial t}(X,t) = -\frac{1}{2}(\frac{\partial^2\psi}{\partial x^2}(X,t) + \frac{\partial^2\psi}{\partial y^2}(X,t)) + \beta|\psi|^2\psi, \;\; X \in \mathbb{R}^2, \;\; t \geq 0, \qquad (2)$$

where $X = (x, y)$, $|\psi|^2 = \psi\overline{\psi}$, and β is a real constant.

3 Basic ideas of He's variational iteration method

In this section, the application of the VIM is discussed for linear and nonlinear Schrodinger equations. Considering the following general differential equation:

$$L\psi(x,t) + R\psi(x,t) + N\psi(x,t) = g(x,t), \qquad (3)$$

where L is a first order partial differential operator, R is a linear operator, N is a nonlinear operator and $g(x,t)$ is a known analytical function. According to the VIM[8–10], we can construct the following correction functional:

$$\psi_{n+1}(x,t) = \psi_n(x,t) + \int_0^t \lambda(L\psi_n(\xi) + R\tilde{\psi}_n(\xi) + N\tilde{\psi}_n(\xi) - g(\xi))d\xi \quad ,n \geq 0, \; (4)$$

where λ is a general Lagrange multiplier [14], which should be identified optimally via the variational theory [14], the subscript n denotes the nth approximation, and $\tilde{\psi}_n$ is considered as a restricted variation [6,7,11] and [13] i.e $\delta\tilde{\psi}_n = 0$. We first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $\psi_{n+1}(x,t)$ $,n \geq 0$ of the solution $\psi(x,t)$ will be readily obtained using the derived Lagrange multiplier and by using any selective function ψ_0. The initial values $\psi(x,0)$ and $\psi_t(x,0)$ are usually used for selecting the zeroth approximation ψ_0 . With λ determined, several approximation $\psi_j(x,0)$, $j \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using (see [20])

$$\psi = \lim_{n\to\infty} \psi_n. \qquad (5)$$

According to the VIM, we consider linear Schrodinger equation (1) in the following form[8–10]:

$$\psi_{n+1}(x,t) = \psi_n(x,t) + \int_0^t \lambda(\xi)(\frac{\partial\psi_n}{\partial\xi}(x,\xi) + i\frac{\partial^2\widetilde{\psi}_n}{\partial x^2}(x,\xi))d\xi. \qquad (6)$$

To find the optimal value of λ, we have

$$\delta\psi_{n+1}(x,t) = \delta\psi_n(x,t) + \delta\int_0^t \lambda(\xi)(\frac{\partial\psi_n}{\partial\xi}(x,\xi) + i\frac{\partial^2\widetilde{\psi}_n}{\partial x^2}(x,\xi))d\xi = 0, \quad (7)$$

or

$$\delta\psi_{n+1}(x,t) = \delta\psi_n(x,t) + \delta\int_0^t \lambda(\xi)(\frac{\partial\psi_n}{\partial\xi}(x,\xi))d\xi = 0. \quad (8)$$

which follows

$$\delta\psi_{n+1}(x,t) = \delta\psi_n(x,t)(1+\lambda(t)) - \delta\int_0^t \lambda'(\xi)\psi_n(x,\xi)d\xi = 0, \quad (9)$$

The following stationary conditions

$$1 + \lambda(t) = 0, \quad (10)$$

$$\lambda'(\xi) = 0, \quad (11)$$

follow immediately. This in turn gives

$$\lambda(\xi) = -1. \quad (12)$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (6) gives in the following iteration formula

$$\psi_{n+1}(x,t) = \psi_n(x,t) - \int_0^t (\frac{\partial\psi_n}{\partial\xi}(x,\xi) + i\frac{\partial^2\widetilde{\psi}_n}{\partial x^2}(x,\xi))d\xi. \quad (13)$$

Similarly, we obtain the correction functional for (2). Hence we have

$$\begin{aligned}\psi_{n+1}(X,t) &= \psi_n(X,t) + \int_0^t \lambda(\xi)(i\frac{\partial\psi_n}{\partial\xi}(X,\xi) + \frac{1}{2}(\frac{\partial^2\widetilde{\psi}_n}{\partial x^2}(X,\xi) + \frac{\partial^2\widetilde{\psi}_n}{\partial y^2}(X,\xi)) \\ &- \beta\psi^2\overline{\psi})d\xi. \quad (14)\end{aligned}$$

The stationary conditions are of the following form

$$1 + i\lambda(t) = 0, \quad (15)$$

$$\lambda'(\xi) = 0, \quad (16)$$

and so we have

$$\lambda(\xi) = i. \quad (17)$$

Substituting this value of the Lagrange multiplier $\lambda = i$ into the functional (14) gives the following iteration formula

$$\begin{aligned} \psi_{n+1}(X,t) &= \psi_n(X,t) + i\int_0^t (i\frac{\partial \psi_n}{\partial \xi}(X,\xi) + \frac{1}{2}(\frac{\partial^2 \widetilde{\psi}_n}{\partial x^2}(X,\xi) + \frac{\partial^2 \widetilde{\psi}_n}{\partial y^2}(X,\xi)) \\ &- \beta\psi^2\overline{\psi})d\xi. \end{aligned} \tag{18}$$

Here, we will use this method to solve linear and nonlinear Schrodinger equations to establish exact solutions for these equations.

4 Examples

To illustrate the solution procedure and show the ability of the method, some examples are provided.

Example 4.1 Consider the following linear Schrodinger equation :

$$\frac{\partial \psi}{\partial t}(x,t) + i\frac{\partial^2 \psi}{\partial x^2}(x,t) = 0, \tag{19}$$

$$\psi_0(x) = \sinh 2x. \tag{20}$$

Using (13), we obtain the following successive approximations:

$$\begin{aligned} \psi_1(x,t) &= (1-4it)\sinh 2x, \\ \psi_2(x,t) &= (1-4it+\frac{(-4it)^2}{2!})\sinh 2x, \\ &\vdots \\ \psi_n(x,t) &= (1-4it+\frac{(-4it)^2}{2!}+\frac{(-4it)^3}{3!}+...+\frac{(-4it)^n}{n!})\sinh 2x. \end{aligned}$$

Consequently, the exact solution is

$$\psi_{exact}(x,t) = e^{-4it}\sinh 2x. \tag{21}$$

Example 4.2 Consider the nonlinear Schrodinger equation

$$i\frac{\partial \psi}{\partial t}(x,t) + \frac{\partial^2 \psi}{\partial x^2}(x,t) + 2|\psi|^2\psi = 0, \tag{22}$$

$$\psi_0(x) = e^{-ix}. \tag{23}$$

Using (18), we obtain the following successive approximations:

$$\begin{aligned}
\psi_1(x,t) &= (1+it)e^{-ix},\\
\psi_2(x,t) &= (1+(it)+\frac{(it)^2}{2!})e^{-ix},\\
\psi_3(x,t) &= (1+(it)+\frac{(it)^2}{2!}+\frac{(it)^3}{3!})e^{-ix},\\
&\vdots\\
\psi_n(x,t) &= (1+(it)+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}+\frac{(it)^4}{4!}+...+\frac{(it)^n}{n!})e^{-ix}.
\end{aligned}$$

Consequently, the exact solution is

$$\psi_{exact}(x,t) = e^{i(t-x)}. \tag{24}$$

Example 4.3 Consider the nonlinear Schrodinger equation

$$i\frac{\partial\psi}{\partial t}(X,t)+\frac{1}{2}(\frac{\partial^2\psi}{\partial x^2}(X,t)+\frac{\partial^2\psi}{\partial y^2}(X,t))+2|\psi|^2\psi = 0 \ , \tag{25}$$

$$\psi_0(x,y) = e^{i(x+y)}. \tag{26}$$

Using (9), we obtain the following successive approximations:

$$\begin{aligned}
\psi_1(X,t) &= (1+it)e^{i(x+y)},\\
\psi_2(X,t) &= (1+(it)+\frac{(it)^2}{2!})e^{i(x+y)},\\
\psi_3(X,t) &= (1+(it)+\frac{(it)^2}{2!}+\frac{(it)^3}{3!})e^{i(x+y)},\\
&\vdots\\
\psi_n(X,t) &= (1+(it)+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}+\frac{(it)^4}{4!}+...+\frac{(it)^n}{n!})e^{i(x+y)}.
\end{aligned}$$

Consequently, the exact solution is

$$\psi_{exact}(X,t) = e^{i(t+x+y)}.$$

Conclusions

In this paper, He's variational iteration method has been successfully applied to find the solution of the linear and nonlinear Schrodinger equations. The main advantage of the method is the fact that it provides an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed term. Analytical solutions enable researchers to study the effect of different variables or parameters on the function under study easily. A clear conclusion can be drawn from the numerical results which VIM provides with highly accurate numerical solution without spatial discretizations for linear and nonlinear Schrodinger equations.

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