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Research Article

Modified Runge–Kutta method with convergence analysis for nonlinear stochastic differential equations with Hölder continuous diffusion coefficient

A. Haghighi

Abstract

The main goal of this work is to develop and analyze an accurate truncated stochastic Runge–Kutta (TSRK2) method to obtain strong numerical solutions of nonlinear one-dimensional stochastic differential equations (SDEs) with continuous Hölder diffusion coefficients. We will establish the strong L^1-convergence theory to the TSRK2 method under the local Lipschitz condition plus the one-sided Lipschitz condition for the drift coefficient and the continuous Hölder condition for the diffusion coefficient at a time T and over a finite time interval $[0, T]$, respectively. We show that the new method can achieve the optimal convergence order at a finite time T compared to the classical Euler–Maruyama method. Finally, numerical examples are given to support the theoretical results and illustrate the validity of the method.

AMS subject classifications (2020): 65C30; 65L07; 65L04

Keywords: Stochastic differential equation; Strong convergence; Truncated methods; Hölder continuous coefficient.

1 Introduction

Consider the scalar stochastic differential equation of Itô type

$$\begin{aligned} dx(\xi) &= a(x(\xi))d\xi + b(x(\xi))dW(\xi), \qquad \xi \in [0, T], \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned} \tag{1}$$

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Amir Haghighi
Department of Mathematics, Faculty of Science, Razi University, Kermanshah 67149, Iran. e-mail: a.haghighi@razi.ac.ir

where $T > 0$ and $a, b : \mathbb{R} \to \mathbb{R}$ are Borel measurable functions and $W(\xi)$ is a one-dimensional Wiener process. We refer the reader to [5, 16, 24, 26] for an overview of stochastic differential equations (SDEs) and their applications. We assume that the drift coefficient a satisfies the one-sided Lipschitz and local Lipschitz conditions and that the diffusion coefficient b satisfies the Hölder continuity condition. Such applications exist in finance, for example, the Aït–Sahalia-type interest rate model [1, 4] and the Cox–Ingersoll–Ross model [6], and in biology, for example, the stochastic SIS epidemic model [7] and laser emission models in chemical physics [16]. Since these equations can be very complicated and analytical solutions are not always possible, numerical methods have become an efficient tool for computing approximate solutions for SDEs. Many numerical methods have been developed for SDEs under the global Lipschitz and linear growth conditions; see, for example, [24, 16, 29]. In particular, derivative-free stochastic Runge–Kutta (SRK) methods for strong approximations have been proposed [3, 27, 8]. However, Hutzenthaler, Jentzen, and Kloeden [13] have shown that the classical Euler–Maruyama (EM) and Milstein methods do not converge strongly to the solution of (1) when the global Lipschitz and linear growth conditions of the drift or diffusion coefficients are perturbed.

In recent years, an increasing number of numerical methods have been developed for solving nonlinear SDEs without global Lipschitz conditions. These methods include implicit methods [25, 23, 9], the tamed numerical methods, the first of which was presented in Hutzenthaler et al. [14, 28, 12], and the stopped EM method [19]. However, the use of implicit or drift-implicit numerical methods requires the solution of a nonlinear algebraic equation at each time step and thus can be very inefficient. Moreover, the tamed methods may lead to inaccurate results due to the perturbation of the flow by changing the drift and diffusion coefficients even at moderately small step sizes [30]. Recently, Mao [21] proposed the truncated EM method for a strong approximation of the nonlinear SDEs under the local Lipschitz condition and the Khasminskii-type condition. After that, the L^q-convergence rates and stability properties of the truncated EM method have been studied by some researchers [22, 11]. Yang et al. [32] investigated the strong convergence of the truncated EM method for one-dimensional SDEs with superlinearly growing drifts and the Hölder continuous diffusion conditions. Then, several new techniques of the partially truncated EM method were proposed in [33] to determine the optimal convergence rate. The authors also investigated the stability of these methods.

Despite the strength of explicit methods in terms of computational cost, there is still a drawback. As mentioned in [16, 2], explicit methods may have to use very small step sizes if the SDEs to be solved are stiff. Although under the classical global Lipschitz condition, there are some classes of explicit methods with extended stability regions that are well suited for solving stiff problems, especially those whose eigenvalues are close to the negative real axis. For example, Komori and Burrage [17] have developed strong first-order SROCK methods for Itô and Stratonovich SDEs. However, they are still far from accurate numerical algorithms suitable for highly nonlinear stiff SDEs with explicit methods.

In this paper, we will bring all these ideas together. Based on Mao's truncation strategies [21, 22, 18], we derive a stochastic two-stage truncated Runge–Kutta method for nonlinear SDEs with the superlinearly growing drift coefficient and the continuous Hölder diffusion coefficient. The proposed scheme is explicit and includes some free parameters that can extend the accuracy of the results and stability regions. To the best of the author's knowledge, truncated SRK is the only truncated Runge–Kutta method with the relevant results for a strong approximation of solutions of SDEs with Hölder diffusion coefficients. We will study the strong convergence of the proposed method under the local Lipschitz condition plus the one-sided Lipschitz condition for the drift term and the continuous Hölder condition for the diffusion term at a time T and over a finite time interval $[0, T]$, respectively. We show that the new method can achieve the optimal order of convergence compared to the classical EM method at a finite time T without any restriction on the step size. To show the effectiveness of our methods, we simulate some stiff SDEs with several Hölder parameters $\alpha \in [0, \frac{1}{2})$.

The rest of the paper is organized as follows. In Section 2, we describe some relevant assumptions that must be satisfied for the drift and diffusion coefficients, as well as results for solving the original SDEs. In Section 3, we first present mathematical notations and preliminary results for truncated methods. We then develop the two-stage truncated stochastic Runge–Kutta (TSRK2) method, which is the main goal of the paper. In this section, we present convergence results of the new method at time T and give several technical lemmas. In Section 4, we study the convergence rate over a finite interval $[0, T]$. Numerical results and conclusions are given in Sections 5 and 6, respectively.

2 Mathematical preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with right continuous and increasing filtration $\{\mathcal{F}_\xi\}_{0\leq\xi\leq T}$, where $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\{W(\xi)\}_{0\leq\xi\leq T}$ be a one-dimensional standard $\{\mathcal{F}_\xi\}_{0\leq\xi\leq T}$-adapted Wiener process on the probability space. For $z_1, z_2 \in \mathbb{R}$, we use $z_1 \vee z_2 = \max\{z_1, z_2\}$ and $z_1 \wedge z_2 = \min\{z_1, z_2\}$. If B is a set, then its indicator function is denoted by I_B, namely, $I_B(z) = 1$ if $z \in B$ and 0 otherwise. For any fixed $p \in [1, \infty)$, we frequently make use of the Young inequality

$$z_1^p z_2 \leq \frac{p\delta}{q} z_1^q + \frac{q-p}{q\delta^{p/q-p}} z_2^{p/q-p}, \qquad \text{for all } z_1, z_2 \in \mathbb{R}^+, \tag{2}$$

for any $\delta > 0$ and $q \in [p, \infty)$. To construct the new method, we now make assumptions about a and b.

Assumption 1. Suppose that there exist real positive constants K_1 and ρ such that

$$|a(z_1) - a(z_2)|^2 \leq K_1(1 + |z_1|^\rho + |z_2|^\rho)|z_1 - z_2|^2, \tag{3}$$

for all $z_1, z_2 \in \mathbb{R}^d$.

From (3) we can conclude that the drift coefficient a satisfies the local Lipschitz condition: For any $u > 0$, there exists $K_u > 0$ such that

$$|a(z_1) - a(z_2)| \leq K_u|z_1 - z_2|, \tag{4}$$

for all $z_1, z_2 \in \mathbb{R}$ with $|z_1| \vee |z_2| \leq u$.

Assumption 2. We assume that the drift coefficient a satisfies the one-sided Lipschitz condition and the diffusion coefficient b satisfies the Hölder continuity condition: Real constants $H_1, H_2 \in \mathbb{R}^+$ and $0 \leq \alpha < 0.5$ exist such that

$$(z_1 - z_2)(a(z_1) - a(z_2)) \leq H_1|z_1 - z_2|^2, \tag{5}$$

$$|b(z_1) - b(z_2)| \leq H_2|z_1 - z_2|^{\frac{1}{2}+\alpha}, \tag{6}$$

for all $z_1, z_2 \in \mathbb{R}$.

Clearly, from Assumption 2, it is easy to verify that

$$za(z) \leq H_1|z|^2 + |z||a(0)| \leq M_1(1 + |z|^2), \tag{7}$$

$$|b(z)| \leq H_2|z|^{0.5+\alpha} + |b(0)| \leq M_2(1 + |z|), \tag{8}$$

for all $z \in \mathbb{R}$, where $M_1 = 0.5\big(|a(0)|^2 \vee [2H_1 + 1]\big)$ and $M_2 = H_2(0.5 + \alpha) \vee \big[H_2(0.5 - \alpha) + |b(0)|\big]$. The relation (8) shows that the coefficient function b satisfies the linear growth condition.

In this work, we use C for the generic positive real constant that depends on p, T, α, x_0, and so on but are independent of the time step size Δ and R, and whose values can change between occurrences.

Remark 1. Under Assumption 2, for all $p \in (2, \infty)$, there is a positive constant C such that

$$za(z) + \frac{p-1}{2}|b(z)|^2 \leq C(1 + |z|^2), \qquad \text{for all } z \in \mathbb{R}. \tag{9}$$

To construct the numerical method, we first estimate the growth rate of a as follows. We choose a strictly increasing continuous function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\nu(r) \to \infty$ as $r \to \infty$ and

$$\sup_{0<|z_1|\vee|z_2|\leq r}\frac{|a(z_1)-a(z_2)|}{|z_1-z_2|}\vee\sup_{|z|\leq r}|a(z)|\leq\nu(r),\qquad\text{for all } r\geq 1.\tag{10}$$

We see that $\nu^{-1}:[\nu(0),\infty)\to(0,\infty)$ as the inverse function of ν is also a strictly increasing continuous function. From Assumption 1, we observe that

$$\sup_{0<|z_1|\vee|z_2|\leq r}\frac{|a(z_1)-a(z_2)|}{|z_1-z_2|}\leq\sqrt{K_1}(1+\sqrt{2}r^{\rho/2}),$$
$$\sup_{|z|\leq r}|a(z)|\leq|a(0)|+\sqrt{K_1}r(1+r^{\rho/2}).\tag{11}$$

Therefore, we can set $\nu(r)=\eta_\nu r^{1+\rho/2}$ for all $r\geq 1$ with $\eta_\nu=\sqrt{2K_1}+|a(0)|$. Yang et al. [32] proved the existence and uniqueness of the strong solution of the scalar SDE (1) with Hölder continuous diffusion coefficients presented in the following theorem. They proved the theorem based on [31, Yamada–Watanabe theorem] and [10, Lemma 3.2].

Theorem 1. Let Assumptions 1 and 2 be satisfied. Then the SDE (1) with initial value $x(0)=x_0\in\mathbb{R}$ has a unique global solution $x(t)$. Moreover, for all $p>0$, there is a positive constant C that depends on T, p, and x_0 such that

$$\mathbb{E}\Big(\sup_{0\leq\xi\leq T}|x(\xi)|^p\Big)\leq C,\tag{12}$$

where $\mathbb{E}$ denotes the probability expectation under the probability measure $\mathbb{P}$.

For any real number $R>|x(0)|$, we define a stopping time

$$\tau_R=\inf\{t\geq 0:|x(t)|\geq R\}.\tag{13}$$

Lemma 1. Let Assumptions 1 and 2 hold. Fix any $p\in(0,\infty)$. Then, for any real number $R>|x(0)|$, we have

$$\mathbb{P}(\tau_R\leq T)\leq\frac{C}{R^p},\tag{14}$$

where C stands for the generic positive real constant here in independent of R.

Proof. By replacing ξ by $\tau_R\wedge T$ in (12), we see

$$\mathbb{E}|x(\tau_R\wedge T)|^p\leq C.\tag{15}$$

Then, by Markov's inequality, we have

$$R^p\mathbb{P}(\tau_R\leq T)\leq\mathbb{E}(|x(\tau_R)|^pI_{\tau_R\leq T})\leq\mathbb{E}(|x(\tau_R\wedge T)|^p)\leq C,\tag{16}$$

which completes the proof. □

In the following sections, we consider numerical methods on a uniform mesh $t_n = n\Delta$ for $n = 1, \ldots, N$ with $\Delta = T/N$ for some $N \in \mathbb{N}$.

3 An explicit two-stage truncated Runge–Kutta method

In this section, we develop an explicit two-stage truncated Runge–Kutta scheme for the nonlinear SDE (1) with a superlinearly growing drift coefficient and a continuous Hölder diffusion coefficient. It is worth mentioning that in this regard, we adopt the idea of constructing the truncating functions of Mao [22] and Li, Mao, and Yin [18] to construct the new method. First, we outline some notations and preliminary results of the truncated methods, which will be used in the following sections. Further details can be found in the literature [21, 22, 11]. Let $h : (0, 1] \to (0, \infty)$ be a strictly decreasing function such that for a constant $\hat{h} \geq 1$

$$h(1) \geq \nu(1), \quad \Delta^{1/4} h(\Delta) \leq \hat{h}, \quad \lim_{\Delta \to 0^+} h(\Delta) = \infty, \quad \text{for all } \Delta \in (0, 1]. \tag{17}$$

For example, we can consider $h(\Delta) = \eta_h \Delta^{-\epsilon\omega}$ with $\eta_h \geq \eta_\nu$ for any $\epsilon \in (0, 1/4\omega)$, where $\omega > 0$. For a given step size $\Delta \in (0, 1]$, let $\kappa_\Delta : \mathbb{R} \to \mathbb{R}$ denote the truncation mapping defined by $\kappa_\Delta(z) := (\nu^{-1}(h(\Delta)) \wedge |z|)\frac{z}{|z|}$, for all $z \in \mathbb{R}$. We set $\frac{z}{|z|} = 0$ if $z = 0$. It can be easily deduced

$$|\kappa_\Delta(z_1)| \leq |z_1|, \qquad |\kappa_\Delta(z_1) - \kappa_\Delta(z_2)| \leq 2|z_1 - z_2|, \qquad \text{for all } z_1, z_2 \in \mathbb{R}^d. \tag{18}$$

Accordingly, we define the truncated coefficient by

$$a_\Delta(z) := a(\kappa_\Delta(z)), \tag{19}$$

for all $z \in \mathbb{R}$. It is obvious from (3), (10) and (18) that

$$|a_\Delta(z)| \leq \nu\big(\nu^{-1}(h(\Delta))\big) = h(\Delta), \qquad \text{for all } z \in \mathbb{R}^d, \tag{20}$$

and

$$|a_\Delta(z_1) - a_\Delta(z_2)|^2 \leq 4K_1\big(1 + |z_1|^\rho + |z_2|^\rho\big)|z_1 - z_2|^2, \tag{21}$$

for all $z_1, z_2 \in \mathbb{R}$.

Remark 2. From (10) and (18) it can be deduced for all $z \in \mathbb{R}$ and $\Delta \in (0, 1]$ that

$$\begin{aligned} |a_\Delta(z)| &\leq |a_\Delta(z) - a(0)| + |a(0)| \leq \nu(\nu^{-1}(h(\Delta))|\kappa_\Delta(z)| + |a(0)| \\ &\leq Ch(\Delta)(1 + |z|). \end{aligned} \tag{22}$$

Also, from (8) we can easily write

$$|b(z)|^2 \le C(1+|z|^2). \tag{23}$$

In fact, relations (22) and (23) are the same inequalities (2.5) and (2.6) presented in Remark 2.1 in [33], which play a fundamental role in determining the optimal convergence rate of the partially truncated EM methods.

The truncated function a_Δ preserves the relation (9) in Remark 1 for all $\Delta \in (0,1]$. We describe this in the following lemma.

Lemma 2. Let Assumption 2 holds. Then for all $p > 2$, there is a constant K_2 such that for all $\Delta \in (0,1]$

$$za_\Delta(z) + \frac{p-1}{2}|b(z)|^2 \le K_2(1+|z|^2), \qquad \text{for all } z \in \mathbb{R}, \tag{24}$$

where $K_2 = \frac{5}{2}\Big(M_1 \vee \big[\frac{M_1}{\nu^{-1}(h(1))}\big] \vee \big[(p-1)M_2^2\big]\Big)$.

Proof. For any $z \in \mathbb{R}$ with $|z| \le \nu^{-1}(h(\Delta))$ from (7) and (8), we have

$$\begin{aligned} za_\Delta(z) + \frac{p-1}{2}|b(z)|^2 &= za(z) + \frac{p-1}{2}|b(z)|^2 \\ &\le \big(M_1 \vee [(p-1)M_2^2]\big)(1+|z|^2). \end{aligned} \tag{25}$$

Furthermore, for any $z \in \mathbb{R}$ with $|z| > \nu^{-1}(h(\Delta))$, we can write

$$\begin{aligned} za_\Delta(z) + \frac{p-1}{2}|b(z)|^2 \le& \nu^{-1}(h(\Delta))\frac{z}{|z|}a\Big(\nu^{-1}(h(\Delta))\frac{z}{|z|}\Big) + (p-1)M_2^2(1+|z|^2) \\ &+ \big(\frac{|z|}{\nu^{-1}(h(\Delta))} - 1\big)\nu^{-1}(h(\Delta))\frac{z}{|z|}a\Big(\nu^{-1}(h(\Delta))\frac{z}{|z|}\Big) \\ \le& M_1\frac{|z|}{\nu^{-1}(h(\Delta))}\Big(1 + [\nu^{-1}(h(\Delta))]^2\Big) + (p-1)M_2^2(1+|z|^2) \\ \le& M_1|z|\Big(\frac{1}{\nu^{-1}(h(1))} + |z|\Big)\Big) + (p-1)M_2^2(1+|z|^2) \\ \le& \frac{5}{2}\Big(M_1 \vee \big[\frac{M_1}{\nu^{-1}(h(1))}\big] \vee \big[(p-1)M_2^2\big]\Big)(1+|z|^2) \end{aligned} \tag{26}$$

The inequalities (25) and (26) imply the required assertion (24) easily. □

Next, we construct our numerical algorithm TSRK2 to approximate the exact solution (1). For any given step size $\Delta \in (0,1]$, define

$$\begin{cases} Y(t_0) = x_0 \\ Z_\Delta(t_n) = Y_\Delta(t_n) + \Delta\theta a_\Delta(Y_\Delta(t_n)), \\ Y_\Delta(t_{n+1}) = Y_\Delta(t_n) + \Delta\Big(\alpha_1 a_\Delta(Y_\Delta(t_n)) + \alpha_2 a_\Delta(Z_\Delta(t_n))\Big) + b(Z_\Delta(t_n))\Delta W_n, \end{cases} \tag{27}$$

for $n = 0, 1, \ldots, N$, where $t_n = n\Delta$ and $\Delta W_n := W(t_{n+1}) - W(t_n)$. Here $\theta \in \mathbb{R}$ and $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 + \alpha_2 = 1$ are free parameters of the TSRK2 procedure. As an example, for $\theta = 0$, we obtain the explicit truncated EM method [21]. Thus, the presented class of TSRK2 methods turns out to be a generalization of the truncated EM method. It is worth noting that when the truncated function a_Δ is replaced by a, the TSRK2 method with $\alpha_1 = 0$, $\alpha_2 = 1$, and $\theta = 0.5$ is the SRK scheme DDIRDI1 [3] and reduces to the midpoint rule when applied to an ordinary differential equation.

We now form a continuous-time version of the TSRK2 procedure (27). To do this, we first for any fixed step size $\Delta \in (0, 1]$ set

$$Y_\Delta(\xi) = \sum_{n=0}^{\infty} Y_\Delta(t_n) I_{[t_n, t_{n+1})}(\xi), \quad Z_\Delta(\xi) = \sum_{n=0}^{\infty} Z_\Delta(t_n) I_{[t_n, t_{n+1})}(\xi), \qquad \text{for all } \xi \geq 0.$$

$$y_\Delta(\xi) = Y_\Delta(\xi) + \int_{t_n}^{\xi} \Big(\alpha_1 a_\Delta(Y_\Delta(\zeta)) + \alpha_2 a_\Delta(Z_\Delta(\zeta))\Big) d\zeta + \int_{t_n}^{\xi} b(Z_\Delta(\zeta)) dW(\zeta), \tag{28}$$

or equivalently

$$y_\Delta(\xi) = x_0 + \int_{0}^{\xi} \Big(\alpha_1 a_\Delta(Y_\Delta(\zeta)) + \alpha_2 a_\Delta(Z_\Delta(\zeta))\Big) d\zeta + \int_{0}^{\xi} b(Z_\Delta(\zeta)) dW(\zeta). \tag{29}$$

3.1 Moment bound of the TSRK2 method

In this subsection, we will state a new result showing the uniform boundedness of the solutions of the TSRK2 method (27). First, the following lemma shows how to conclude that the values $Y_\Delta(\xi)$ and $Z_\Delta(\xi)$ are close to $y_\Delta(\xi)$ with respect to the L^p-norm.

Lemma 3. For any $\tilde{p} > 0$ and any $T > 0$, there is a positive constant number $C_{\tilde{p}}$ such that for every step size $\Delta \in (0, 1]$

$$\mathbb{E}|y_\Delta(\xi) - Y_\Delta(\xi)|^{\tilde{p}} \vee \mathbb{E}|y_\Delta(\xi) - Z_\Delta(\xi)|^{\tilde{p}} \leq C_{\tilde{p}} \Delta^{\tilde{p}/2} (h(\Delta))^{\tilde{p}}, \qquad \text{for all } \xi \in [0, T]. \tag{30}$$

Proof. We first prove the lemma for any $\tilde{p} \geq 2$. In this case, for a given $\xi \in [0, T]$, there is an unique $n \geq 0$ such that $t_n \leq \xi < t_{n+1}$. By (28) and from the Hölder inequality, [20, Theorem 7.1], (8), and (20), one can conclude

$$\mathbb{E}|y_\Delta(\xi) - Y_\Delta(\xi)|^{\tilde{p}} \leq 3^{\tilde{p}-1} \Big(\alpha_1 \Delta^{\tilde{p}-1} \mathbb{E} \int_{t_n}^{\xi} \big|a_\Delta(Y_\Delta(\zeta))\big|^{\tilde{p}} d\zeta \tag{31}$$

$$
\begin{aligned}
&+\alpha_2\Delta^{\tilde{p}-1}\mathbb{E}\int_{t_n}^{\xi}\left|a_\Delta(Z_\Delta(\zeta))\right|^{\tilde{p}}d\zeta+\Delta^{\frac{\tilde{p}-2}{2}}\mathbb{E}\int_{t_n}^{\xi}\left|b(Z_\Delta(\zeta))\right|^{\tilde{p}}d\zeta\Big)\\
\leq&3^{\tilde{p}-1}\Big(\alpha_1\Delta^{\tilde{p}}(h(\Delta))^{\tilde{p}}+\alpha_2\Delta^{\tilde{p}}(h(\Delta))^{\tilde{p}}\\
&+2^{\tilde{p}-1}M_2\Delta^{\tilde{p}/2}\big(1+\mathbb{E}|Z_\Delta(t_n)|^{\tilde{p}}\big)\Big)
\end{aligned}
$$

Moreover, for every $\zeta\in[0,T]$, we can find a unique positive integer k such that $t_k\leq\xi<t_{k+1}$. So, by (20) and (27), we have

$$
\mathbb{E}|Z_\Delta(\zeta)-Y_\Delta(\zeta)|^{\tilde{p}}=\mathbb{E}|Z_\Delta(t_k)-Y_\Delta(t_k)|^{\tilde{p}}=|\theta|^{\tilde{p}}\Delta^{\tilde{p}}\mathbb{E}|a_\Delta(Y_\Delta(t_k))|^{\tilde{p}}\leq|\theta|^{\tilde{p}}\Delta^{\tilde{p}}(h(\Delta))^{\tilde{p}}. \tag{32}
$$

On the other hand, by relations (8), (29), and (32) and the Doob martingale inequality, we have

$$
\begin{aligned}
\mathbb{E}|y_\Delta(\xi)|^{\tilde{p}}\leq&3^{\tilde{p}-1}\Big(|x_0|^{\tilde{p}}+\mathbb{E}\int_0^{\xi}\left|a_\Delta(Y_\Delta(\zeta))\right|^{\tilde{p}}d\zeta+\alpha_2\mathbb{E}\int_0^{\xi}\left|a_\Delta(Z_\Delta(\zeta))\right|^{\tilde{p}}d\zeta\\
&+T^{\frac{\tilde{p}-2}{2}}\mathbb{E}\int_0^{\xi}\left|b(Z_\Delta(\zeta))\right|^{\tilde{p}}d\zeta\Big)\leq C\Big(1+(h(\Delta))^{\tilde{p}}+\int_0^{\xi}\mathbb{E}|Z_\Delta(\zeta)|^{\tilde{p}}d\zeta\Big)\\
\leq&C\Big(1+(h(\Delta))^{\tilde{p}}+\int_0^{\xi}\mathbb{E}|Z_\Delta(\zeta)-Y_\Delta(\zeta)|^{\tilde{p}}d\zeta+\int_0^{\xi}\mathbb{E}|Y_\Delta(\zeta)|^{\tilde{p}}d\zeta\Big).
\end{aligned} \tag{33}
$$

From (32) and (33), we can conclude

$$
\sup_{0\leq u\leq\xi}\mathbb{E}|Y_\Delta(u)|^{\tilde{p}}\leq\sup_{0\leq u\leq\xi}\mathbb{E}|y_\Delta(u)|^{\tilde{p}}\leq C\Big(1+(h(\Delta))^{\tilde{p}}+\int_0^{\xi}\sup_{0\leq u\leq\zeta}\mathbb{E}|Y_\Delta(u)|^{\tilde{p}}d\zeta\Big).
$$

By the Gronwall's inequality, we can deduce

$$
\sup_{0\leq\xi\leq T}\mathbb{E}|Y_\Delta(\xi)|^{\tilde{p}}\leq C\Big(1+(h(\Delta))^{\tilde{p}}\Big). \tag{34}
$$

So, from (32), (34), and (17), we can write

$$
\sup_{0\leq\xi\leq T}\mathbb{E}|Z_\Delta(\xi)|^{\tilde{p}}\leq2^{\tilde{p}-1}|\theta|^{\tilde{p}}\Delta^{\tilde{p}}(h(\Delta))^{\tilde{p}}+2^{\tilde{p}-1}\sup_{0\leq\xi\leq T}\mathbb{E}|Y_\Delta(\xi)|^{\tilde{p}}\leq C\Big(1+(h(\Delta))^{\tilde{p}}\Big). \tag{35}
$$

By inserting (35) in (31), we have

$$
\mathbb{E}|y_\Delta(\xi)-Y_\Delta(\xi)|^{\tilde{p}}\leq C\Big(\Delta^{\tilde{p}}(h(\Delta))^{\tilde{p}}+\Delta^{\tilde{p}/2}+\Delta^{\tilde{p}/2}(h(\Delta))^{\tilde{p}}\Big)\leq C\Delta^{\tilde{p}/2}(h(\Delta))^{\tilde{p}}. \tag{36}
$$

To prove the lemma for any $\bar{p}\in(0,2)$, fix a number $\tilde{p}\geq2$. Then by the Hölder inequality, we can write

$$\mathbb{E}|y_\Delta(\xi)-Y_\Delta(\xi)|^{\bar{p}} \leq \Big(\mathbb{E}|y_\Delta(\xi)-Y_\Delta(\xi)|^{p}\Big)^{\bar{p}/\tilde{p}} \leq \Big(C\Delta^{\tilde{p}/2}(h(\Delta))^{\tilde{p}}\Big)^{\bar{p}/\tilde{p}}$$
$$= C_{\bar{p}}\Delta^{\bar{p}/2}(h(\Delta))^{\bar{p}}. \qquad (37)$$

So, from (32), (36), and (37) the proof of theorem is complete. □

Lemma 4. Let Assumption 1 and 2 hold. Then, we have

$$\sup_{0\leq\Delta\leq 1}\sup_{0\leq\xi\leq T}\mathbb{E}|y_\Delta(\xi)|^p \leq C, \qquad \text{for all } p>0. \qquad (38)$$

Proof. Let us fix $\Delta\in(0,1]$ and $\xi\in[0,T]$. Using the Itô formula from (29) for any $p>2$, we can write

$$\begin{aligned}|y_\Delta(\xi)|^p \leq& |x_0|^p + \int_0^\xi p|y_\Delta(\zeta)|^{p-2}y_\Delta(\zeta)b(Z_\Delta(\zeta))dW(\zeta)\\ &+\int_0^\xi p|y_\Delta(\zeta)|^{p-2}\Bigg(y_\Delta(\zeta)\Big(\alpha_1 a_\Delta(Y(\zeta))+\alpha_2 a_\Delta(Z_\Delta(\zeta))\Big)\\ &+\frac{p-1}{2}|b(Z_\Delta(\zeta)|^2\Bigg)d\zeta,\end{aligned}$$

for all $\xi\in[0,T]$. Since $\alpha_1+\alpha_2=1$, we have

$$\begin{aligned}|y_\Delta(\xi)|^p \leq& |x_0|^p+\Gamma_1(\xi)+\Gamma_2(\xi)+\Gamma_3(\xi)\\ &+\int_0^\xi p|y_\Delta(\zeta)|^{p-2}y_\Delta(\zeta)b(Z_\Delta(\zeta))dW(\zeta),\end{aligned} \qquad (39)$$

where

$$\begin{aligned}\Gamma_1(\xi) &:= \int_0^\xi p|y_\Delta(\zeta)|^{p-2}\Big(Z_\Delta(\zeta)a_\Delta(Z_\Delta(\zeta))+\frac{p-1}{2}|b(Z_\Delta(\zeta))|^2\Big)d\zeta,\\ \Gamma_2(\xi) &:= \int_0^\xi p|y_\Delta(\zeta)|^{p-2}\big(y_\Delta(\zeta)-Z_\Delta(\zeta)\big)a_\Delta(Z_\Delta(\zeta))d\zeta,\\ \Gamma_3(\xi) &:= \alpha_1\int_0^\xi p|y_\Delta(\zeta)|^{p-2}y_\Delta(\zeta)\Big(a_\Delta(Y_\Delta(\zeta))-a_\Delta(Z_\Delta(\zeta))\Big)d\zeta.\end{aligned} \qquad (40)$$

By the facts that $p|y_\Delta(\zeta)|^{p-2}y_\Delta(\zeta)b(Z_\Delta(\zeta))$ is $\mathcal{F}_\zeta-$measurable, we have

$$\mathbb{E}|y_\Delta(\xi)|^p \leq \mathbb{E}|x_0|^p+\mathbb{E}(\Gamma_1(\xi))+\mathbb{E}(\Gamma_2(\xi))+\mathbb{E}(\Gamma_3(\xi)). \qquad (41)$$

Next, we try to estimate the values $\mathbb{E}(\Gamma_1(\xi))$, $\mathbb{E}(\Gamma_2(\xi))$, and $\mathbb{E}(\Gamma_3(\xi))$ in (41). By the relations (20), (24), and (27) and the special form of Young's inequality (2), which reads

$$z_1^{p-2}z_2 \leq \frac{p-2}{p}z_1^p + \frac{2}{p}z_2^{p/2}, \qquad \text{for all } z_1, z_2 \geq 0, \tag{42}$$

we can approximate $\Gamma_1(\xi)$ as follows:

$$\begin{aligned}\mathbb{E}(\Gamma_1(\xi)) \leq &\mathbb{E}\int_0^\xi \big((p-2)|y(\zeta)|^p + 2K_2^{p/2}(1+|Z_\Delta(\zeta)|^2)^{p/2}\big)d\zeta \\ \leq &(2K_2)^{p/2}T + 2^{3p/2-1}K_2^{p/2}T|\theta|^p\Delta^p(h(\Delta))^p \\ &+ (p-2)\int_0^\xi \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + 2^{3p/2-1}K_2^{p/2}\int_0^\xi \mathbb{E}|Y_\Delta(\zeta)|^p d\zeta. \end{aligned} \tag{43}$$

For $\Gamma_2(\xi)$, with the help of Young's inequality (42), relations (10) and (20) and Lemma 3, we obtain

$$\begin{aligned}\mathbb{E}(\Gamma_2(\xi)) &\leq (p-2)\int_0^\xi \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + 2(h(\Delta))^{p/2}\int_0^T \mathbb{E}|y_\Delta(\zeta) - Z_\Delta(\zeta)|^{p/2}d\zeta \\ &\leq (p-2)\int_0^\xi \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + 2TC_{p/2}(h(\Delta))^p\Delta^{p/4}. \end{aligned} \tag{44}$$

For $\Gamma_3(\xi)$, the Young inequality as well as (10), (18), and (20) yields

$$\begin{aligned}\mathbb{E}(\Gamma_3(\xi)) &\leq \alpha_1\mathbb{E}\int_0^\xi \Big((p-2)|y_\Delta(\zeta)|^p + 2|y_\Delta(\zeta)|^{p/2}\big|a_\Delta(Y(\zeta)) - a_\Delta(Z_\Delta(\zeta))\big|^{p/2}\Big)d\zeta \\ &\leq \alpha_1(p-1)\int_0^\xi \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + \alpha_1(h(\Delta))^p\int_0^T \big|\kappa_\Delta(Y(\zeta)) - \kappa_\Delta(Z_\Delta(\zeta))\big|^p d\zeta \\ &\leq \alpha_1(p-1)\int_0^\xi \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + 2\alpha_1|\theta|^pT(h(\Delta))^{2p}\Delta^p \\ &\leq (p-1)\alpha_1\int_0^\xi \sup_{0\leq u\leq\zeta}\mathbb{E}|y_\Delta(u)|^p du + 2|\theta|^p\alpha_1T\hat{h}^2. \end{aligned} \tag{45}$$

Inserting (43)–(45) into (41), we have

$$\mathbb{E}|y_\Delta(\xi)|^p \leq C\Big(1 + \int_0^\xi \sup_{0\leq u\leq\zeta}\mathbb{E}|y_\Delta(u)|^p d\zeta\Big).$$

Therefore, we can write

$$\sup_{0\leq u\leq\xi}\mathbb{E}|y(u)|^p \leq C\Big(1 + \int_0^\xi \sup_{0\leq u\leq\zeta}\mathbb{E}|y(u)|^p d\zeta\Big).$$

It can be deduced from the Gronwall's inequality that

$$\sup_{0\leq u\leq\xi}\mathbb{E}|y(u)|^p \leq C,$$

for a positive constant number C independent of Δ. For $0 < \hat{p} \le p$, according to the Lyapunov inequality, (38) still holds, which completes the proof of the lemma. $\square$

Remark 3. For every $\xi \in [0, T]$, there is a single integer $n \ge 0$ such that $t_n \le \xi < t_{n+1}$. From (17) and (27) we can deduce that

$$\mathbb{E}|Z_\Delta(\xi)|^p = \mathbb{E}|Z_\Delta(t_n)|^p \le 2^{p-1}\Big(\mathbb{E}|Y_\Delta(t_n))|^p + \Delta^p|\theta|^p\mathbb{E}\big|a_\Delta(Y_\Delta(t_n))\big|^p\Big)$$
$$\le 2^{p-1}\Big(\mathbb{E}|Y_\Delta(t_n)|^p + \Delta^{3p/4}|\theta|^p\hat{h}^p\Big).$$

Therefore, by Lemma 3 we can write

$$\sup_{0\le\Delta\le 1}\sup_{0\le\xi\le T}\mathbb{E}|Z_\Delta(\xi)|^p \le C. \tag{46}$$

In addition to (13), for any real number $R > |x(0)|$, we define two other stopping times

$$\theta_R^{(1)} := \inf\{t \ge 0 : |Y_\Delta(t)| \ge R\}, \quad \theta_R^{(2)} := \inf\{t \ge 0 : |Z_\Delta(t)| \ge R\}. \tag{47}$$

Lemma 5. Let Assumption 1 and 2 hold. Fix any $p \in (0, \infty)$. Then, for any real number $R > |x(0)|$, we have

$$\mathbb{P}(\theta_R^{(1)} \le T) \vee \mathbb{P}(\theta_R^{(2)} \le T) \le \frac{C}{R^p},$$

where C stands for the generic positive real constant here in independent of R.

Proof. The proof of this lemma is similar to that of Lemma 1. Namely, replacing ξ by $\theta_R^{(1)} \wedge T$ in (38), we see

$$\mathbb{E}|y_\Delta(\theta_R^{(1)} \wedge T)|^p \le C.$$

Then, by Markov's inequality we have

$$R^p\mathbb{P}(\theta_R^{(1)} \le T) \le \mathbb{E}(|y_\Delta(\theta_R^{(1)})|^p I_{\theta_R^{(1)}\le T}) \le \mathbb{E}(|y_\Delta(\theta_R^{(1)} \wedge T)|^p) \le C.$$

Moreover, it follows from Remark 3 that $\mathbb{E}|Z_\Delta(\theta_R^{(2)} \wedge T)|^p \le C$. Therefore, as above

$$R^p\mathbb{P}(\theta_R^{(2)} \le T) \le \mathbb{E}(|Z_\Delta(\theta_R^{(2)})|^p I_{\theta_R^{(2)}\le T}) \le \mathbb{E}(|Z_\Delta(\theta_R^{(2)} \wedge T)|^p) \le C,$$

which completes the proof. $\square$

3.2 Convergence of the new method at time T

In this section, we study the effectiveness of the method TASK2 in solving problem (1) and obtain the corresponding convergence results at finite time T. Using the method of Yamada and Watanabe, for each $\delta \in [1, \infty)$ and $\epsilon \in (0, \infty)$, we choose a nonnegative continuous function $\chi_{\delta\epsilon} : [0, \infty) \to \mathbb{R}$ such that [15, 31]:

- $\chi_{\delta\epsilon}(\xi) \leq \frac{2}{x \ln(\delta)}$, if $\xi \in [\epsilon/\delta, \epsilon]$, $\quad \chi_{\delta\epsilon}(\xi) = 0$ otherwise;
- $\int_{\epsilon/\delta}^{\epsilon} \chi_{\delta\epsilon}(\xi) d\xi = 1$.

Moreover, we approximate the function $\xi :\to |\xi|$ by the function $\varpi_{\delta\epsilon}$ defined by

$$\varpi_{\delta\epsilon}(\xi) := \int_0^{|\xi|} \int_0^{\zeta_1} \chi_{\delta\epsilon}(\zeta_2) d\zeta_2 d\zeta_1, \qquad \text{for all } \xi \in \mathbb{R}. \tag{48}$$

Below we outline some properties of the $\varpi_{\delta\epsilon}$ function that will be used in the following sections. Further details can be found in the literature [32, 33].

Lemma 6. Let $\delta \in [1, \infty)$ and let $\epsilon \in (0, \infty)$. Then for all $\xi \in \mathbb{R}$

1. $|\xi| \leq \varpi_{\delta\epsilon}(\xi) + \epsilon$,
2. $0 \leq |\varpi'_{\delta\epsilon}(\xi)| \leq 1$,
3. $\varpi''_{\delta\epsilon}(\xi) = \chi_{\delta\epsilon}(|\xi|) \leq \frac{2}{|\xi| \ln \delta} I_{\{\epsilon/\delta \leq |\xi| \leq \epsilon\}}$,
4. $\frac{\varpi'_{\delta\epsilon}(\xi)}{\xi} > 0$, for all $\xi \in \mathbb{R} \setminus \{0\}$.

Remark 4. Since $\sup_{\xi \in \mathbb{R}} |\varpi'_{\delta\epsilon}(\xi)| \leq 1$, under Assumption 2, we can easily conclude

$$\varpi'_{\delta\epsilon}(z_1 - z_2)(a(z_1) - a(z_2)) \leq H_1 |z_1 - z_2|, \tag{49}$$

for all $z_1, z_2 \in \mathbb{R}$; see [32].

Define the stopping time $\beta_{\Delta,R} := \tau_R \wedge \theta_R^{(1)} \wedge \theta_R^{(2)}$, where τ_R and $\theta_R^{(i)}$ for $i = 1, 2$, are defined by (13) and (47), respectively. Then, the moment deviation between $x(\xi \wedge \beta_{\Delta,R})$ and $y_\Delta(\xi \wedge \beta_{\Delta,R})$ is estimated as follows.

Lemma 7. Consider the initial problem (1), which satisfies Assumptions 1 and 2. Suppose that $R > |x_0|$ is a real number and that $\Delta \in (0, 1]$ is sufficiently small such that $\nu^{(-1)}(h(\Delta)) \geq R$. Then there exists a constant C such that

$$\mathbb{E}|e_\Delta(\xi \wedge \beta_{\Delta,R})| \leq \begin{cases} C\Big(-\frac{1}{\ln(\Delta)} + \Delta^{\frac{1}{4}} (h(\Delta))^{\frac{1}{2}}\Big), & \text{if } \alpha = 0, \\ C\big(\Delta^{\frac{1}{2}} h(\Delta)\big)^{2\alpha}, & \text{if } 0 < \alpha < 0.5, \end{cases} \tag{50}$$

where

$$e_\Delta(\xi) := x(\xi) - y_\Delta(\xi), \qquad \text{for } 0 \leq \xi \leq T.$$

Proof. We note for $0 \leq \zeta \leq \xi \wedge \beta_{\Delta,R}$ that $|Y_\Delta(\zeta)| \leq R$ and $|Z_\Delta(\zeta)| \leq R$. Therefore, we conclude by the condition $\nu^{(-1)}(h(\Delta)) \geq R$ and the definition of the truncation function a_Δ in (19) that

$$a_\Delta(Y_\Delta(\zeta)) = a(Y_\Delta(\zeta)) \qquad \text{and} \qquad a_\Delta(Z_\Delta(\zeta)) = a(Z_\Delta(\zeta)).$$

It therefore follows from the Itô formula that

$$\begin{aligned}\mathbb{E}|e_\Delta(\xi \wedge \beta_{\Delta,R})| \leq & \epsilon + \mathbb{E}\big(\varpi_{\delta\epsilon}(e_\Delta(\xi \wedge \beta_{\Delta,R}))\big) \\ = & \epsilon + \mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}} \varpi'_{\delta\epsilon}(e_\Delta(\zeta))\Big[a(x(\zeta)) - \alpha_1 a(Y_\Delta(\zeta)) - \alpha_2 a(Z_\Delta(\zeta))\Big]d\zeta \\ & + \frac{1}{2}\mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}} \varpi''_{\delta\epsilon}(e_\Delta(\zeta))\big[b(x(\zeta)) - b(Z_\Delta(\zeta))\big]^2 d\zeta := \epsilon + \Pi_1 + \Pi2.\end{aligned} \tag{51}$$

In what follows, we attempt to estimate the values Π_1 and Π_2 in (51). Since $\alpha_1 + \alpha_2 = 1$, by re-arranging we get that

$$\begin{aligned}\Pi_1 \leq & \mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}} \varpi'_{\delta\epsilon}(e_\Delta(\zeta))\big[a(x(\zeta)) - a(y_\Delta(\zeta))\big]d\zeta \\ & + \mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}} \big|\varpi'_{\delta\epsilon}(e_\Delta(\zeta))\big|\big|a(y_\Delta(\zeta)) - a(Y_\Delta(\zeta))\big|d\zeta \\ & + \alpha_2\mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}} \big|\varpi'_{\delta\epsilon}(e_\Delta(\zeta))\big|\big|a(Y_\Delta(\zeta)) - a(Z_\Delta(\zeta))\big|d\zeta.\end{aligned}$$

By Assumption 1, Lemma 6, and Remark 4 we can write

$$\begin{aligned}\Pi_1 \leq & H_1\mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}} |e_\Delta(\zeta))|d\zeta \\ & + \sqrt{K_1}\mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}} \big(1 + |y_\Delta(\zeta)|^\rho + |Y_\Delta(\zeta)|^\rho\big)^{\frac{1}{2}}\big|y_\Delta(\zeta)) - Y_\Delta(\zeta)\big|d\zeta \\ & + \alpha_2\sqrt{K_1}\mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}} \big(1 + |Y_\Delta(\zeta)|^\rho + |Z_\Delta(\zeta)|^\rho\big)^{\frac{1}{2}}\big|Y_\Delta(\zeta)) - Z_\Delta(\zeta)\big|d\zeta.\end{aligned}$$

So, by the Hölder inequality, Lemmas 3 and 4, we have

$$\Pi_1 \leq H_1\mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}} |e_\Delta(\zeta))|d\zeta + C\Delta^{\frac{1}{2}}h(\Delta). \tag{52}$$

As for Π_2, it follows from Assumption 2 and Lemmas 3 Lemma 6 that

$$\begin{aligned}\Pi_2 \leq & H_2^2\mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}} \big(x(\zeta) - Z_\Delta(\zeta)\big)^{1+2\alpha}\frac{1}{|e_\Delta(\zeta)|\ln\delta} I_{\{\epsilon/\delta \leq |e_\Delta(\zeta)| \leq \epsilon\}} d\zeta \\ \leq & \frac{2^{2\alpha}H_2^2\epsilon^{2\alpha}T}{\ln(\delta)} + \frac{2^{2\alpha}H_2^2\delta}{\epsilon\ln(\delta)}\mathbb{E}\int_0^{\xi} \big|y_\Delta(\zeta) - Z_\Delta(\zeta)\big|^{1+2\alpha}d\zeta\end{aligned}$$

$$\leq 2^{2\alpha} H_2^2 T\Big[\frac{\epsilon^{2\alpha}}{\ln(\delta)} + \frac{\delta}{\epsilon \ln(\delta)}\Big]\big(\Delta^{\frac{1}{2}} h(\Delta)\big)^{1+2\alpha}. \tag{53}$$

Substituting (52) and (53) in (51) and by the Gronwall inequality, we have

$$\mathbb{E}|e_\Delta(\xi \wedge \beta_{\Delta,R})| \leq C\Big(\epsilon + \Delta^{\frac{1}{2}} h(\Delta) + \frac{\epsilon^{2\alpha}}{\ln(\delta)} + \frac{\delta}{\epsilon \ln(\delta)}\big(\Delta^{\frac{1}{2}} h(\Delta)\big)^{1+2\alpha}\Big).$$

In the case when $\alpha = 0$, we set $\delta = \Delta^{-\frac{1}{8}}$ and $\epsilon = -\frac{1}{\ln(\Delta)}$, which implies

$$\mathbb{E}|e_\Delta(\xi \wedge \beta_{\Delta,R})| \leq C\Big(-\frac{1}{\ln(\Delta)} + \Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\Big). \tag{54}$$

If we take $\delta = 2$ and $\epsilon = \Delta^{\frac{1}{2}} h(\Delta)$ for the case where $\alpha \in (0, 0.5)$, then we have

$$\mathbb{E}|e_\Delta(\xi \wedge \beta_{\Delta,R})| \leq C\big(\Delta^{\frac{1}{2}} h(\Delta)\big)^{2\alpha}. \tag{55}$$

The inequalities (54) and (55) prove the desired. □

Theorem 2. Let conditions in Assumptions 1 and 2 be fulfilled and let $p > 1$. Let $R_\Delta := \big(\Delta^{\frac{1}{2}} h(\Delta)\big)^{-1/(p-1)}$ for any $\Delta \in (0, 1]$. If there is a positive real number $\Delta^* \in (0, 1]$ such that

$$\nu^{-1}(h(\Delta)) \geq R_\Delta, \qquad \text{for all } \Delta \in (0, \Delta^*]. \tag{56}$$

Then, there exists a positive constant C independent of Δ such that

$$\mathbb{E}|e_\Delta(T)| \leq \begin{cases} C\Big(-\frac{1}{\ln(\Delta)} + \Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\Big), & \text{if } \alpha = 0, \\ C\big(\Delta^{\frac{1}{2}} h(\Delta)\big)^{2\alpha}, & \text{if } \alpha \in (0, \frac{1}{2}). \end{cases} \tag{57}$$

Proof. We first divide the left side of (57) as below:

$$\mathbb{E}|e_\Delta(T)| = \mathbb{E}|e_\Delta(T) I_{\{\beta_{\Delta,R} > T\}}| + \mathbb{E}|e_\Delta(T) I_{\{\beta_{\Delta,R} \leq T\}}| \tag{58}$$

By Young's inequality (2), Theorem 1, and Lemmas 4 and 5, we obtain

$$\begin{aligned} \mathbb{E}|e_\Delta(T) I_{\{\beta_{\Delta,R} \leq T\}}| &\leq \frac{\Delta^{\frac{1}{2}} h(\Delta)}{p} \mathbb{E}|e_\Delta(T)|^p + \frac{p-1}{p\big(\Delta^{\frac{1}{2}} h(\Delta)\big)^{1/(p-1)}} \mathbb{P}(\beta_{\Delta,R} \leq T) \\ &\leq C \Delta^{\frac{1}{2}} h(\Delta). \end{aligned} \tag{59}$$

Since $\nu^{-1}(h(\Delta)) \geq R_\Delta$, substituting (59) in (58) by Lemma 7, we obtain the result of the theorem. □

Remark 5. This theorem shows that the TSRK2 method has an order of convergence close to α for $\alpha \in (0, 0.5)$. This is theoretically almost optimal if we recall that the classical EM method has a convergence order of

α. However, the condition (56) could sometimes make the TSRK2 method impractical; see [33] for more details.

In what follows, we use the mathematical techniques developed by Yang and Huang [33] to remove the imposed condition (56). In this context, we replace the condition $\Delta^{1/4}h(\Delta) \leq \hat{h}$ by the more general condition $\Delta^{1/2}h(\Delta) \leq \hat{h}$ for all $\Delta \in [0,1]$. Without limiting generality, for simplicity, we use here $h(\Delta) = \eta_h \Delta^{-\frac{1}{2}}$ with $\eta_h > 0$.

Lemma 8. Let Assumptions 1 and 2 be satisfied. If $h(\Delta) = \eta_h \Delta^{-\frac{1}{2}}$ with $\eta_h > 0$, then for every $\Delta \in (0,1]$ and every $p > 0$, it holds that

$$\sup_{0<\Delta\leq 1} \mathbb{E}\Big[\sup_{0\leq\xi\leq T} |y_\Delta(\xi)|^p\Big] \leq C, \tag{60}$$

and

$$\Big(\sup_{0\leq\xi\leq T} \mathbb{E}|y_\Delta(\xi) - Y_\Delta(\xi)|^p\Big) \vee \Big(\sup_{0\leq\xi\leq T} \mathbb{E}|y_\Delta(\xi) - Z_\Delta(\xi)|^p\Big) \leq C\Delta^{\frac{p}{2}}, \tag{61}$$

for all real positive number T.

Proof. We fix $\Delta \in (0,1]$ and use the same notation as in the proof of Lemma 4. By using the Itô formula, we can write that for any $p > 2$,

$$\begin{aligned}\sup_{0\leq\xi\leq T} |y_\Delta(\xi)|^p \leq& |x_0|^p + \Gamma_1(T) + \Gamma_2(T) + \Gamma_3(T)\\ &+ \sup_{0\leq\xi\leq T}\Big|\int_0^\xi p|y_\Delta(\zeta)|^{p-2} y_\Delta(\zeta) b(Z_\Delta(\zeta)) dW(\zeta)\Big|.\end{aligned} \tag{62}$$

So, by the Burkholder–Davis-Gundy inequality and the linear growth condition property of b in (8), we have

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\leq\xi\leq T} |y_\Delta(\xi)|^p\Big) \leq& |x_0|^p + \Gamma_1(T) + \Gamma_2(T) + \Gamma_3(T)\\ &+ 8pM_2 \mathbb{E}\Big(\Big[\sup_{0\leq\xi\leq T} |y_\Delta(\xi)|^p \int_0^T |y_\Delta(\xi)|^{p-2}\big(1 + |Z_\Delta(\xi)|^2\big) d\xi\Big]^{1/2}\Big).\end{aligned} \tag{63}$$

In what follows, we estimate the values $\mathbb{E}(\Gamma_1(T))$, $\mathbb{E}(\Gamma_2(T))$, and $\mathbb{E}(\Gamma_3(T))$ in (63) with the more general condition $h(\Delta) = \eta_h \Delta^{-\frac{1}{2}}$. For $\mathbb{E}(\Gamma_1(\xi))$, we obtain from (43)

$$\mathbb{E}(\Gamma_1(\xi)) \leq C\Big(1 + \int_0^T \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + \int_0^T \mathbb{E}|Y_\Delta(\zeta)|^p d\zeta\Big). \tag{64}$$

As for $\Gamma_2(\xi)$, we obtain by Young's inequality (42) that

$$\begin{aligned}\mathbb{E}(\Gamma_2(\xi)) \leq&(p-2)\int_0^T \mathbb{E}|y_\Delta(\zeta)|^p d\zeta + 2\mathbb{E}\int_0^T |y_\Delta(\zeta)-Z_\Delta(\zeta)|^{\frac{p}{2}}|a_\Delta(Y_\Delta(\zeta))|^{\frac{p}{2}} d\zeta\\ \leq& C\Delta^{-\frac{p}{4}}\int_0^T (\mathbb{E}|y_\Delta(\zeta)-Z_\Delta(\zeta)|^p)^{\frac{1}{2}}(1+\mathbb{E}|Y_\Delta(\zeta)|^p)^{\frac{1}{2}} d\zeta\\ &+(p-2)\int_0^T \mathbb{E}|y_\Delta(\zeta)|^p d\zeta. \end{aligned} \tag{65}$$

Since $(h(\Delta))^{2p}\Delta^p \leq \eta_h^{2p}\hat{h}^p$, following an approach very similar to that used in (45), we can show that

$$\mathbb{E}(\Gamma_3(\xi)) \leq C\big(1+\int_0^T \mathbb{E}|y_\Delta(\zeta)|^p du\big). \tag{66}$$

For any $\zeta \in [0,T]$, there is a unique integer $n \geq 0$ such that $t_n \leq \zeta \leq t_{n+1}$. Then we can write

$$\begin{aligned}\mathbb{E}|Z_\Delta(\zeta)-Y_\Delta(\zeta)|^p &= \mathbb{E}|Z_\Delta(t_n)-Y_\Delta(t_n)|^p = \Delta^p|\theta|^p\mathbb{E}|a_\Delta(Y_\Delta(t_n))|^p\\ &\leq C\Delta^{\frac{p}{2}}(1+\mathbb{E}|Y_\Delta(t_n)|^p) = C\Delta^{\frac{p}{2}}(1+\mathbb{E}|Y_\Delta(\zeta)|^p), \end{aligned} \tag{67}$$

and

$$\begin{aligned}\mathbb{E}|y_\Delta(\zeta)-Y_\Delta(\zeta)|^p =&\mathbb{E}|y_\Delta(\zeta)-Y_\Delta(t_n)|^p \leq C\Delta^{\frac{p}{2}}\Big(\alpha_1\Delta^{\frac{p}{2}}\mathbb{E}\big[|a_\Delta(Y_\Delta(t_n))|^p\big]\\ &+\alpha_2\Delta^{\frac{p}{2}}\mathbb{E}\big[|a_\Delta(Z_\Delta(t_n))|^p\big]+\mathbb{E}\big[|b(Z_\Delta(t_n))|^p\big]\Big). \end{aligned} \tag{68}$$

Therefore, by Remark 2 and relations (67) and (68) we can obtain

$$\begin{aligned}\mathbb{E}|y_\Delta(\zeta)-Y_\Delta(\zeta)|^p &\leq C\Delta^{\frac{p}{2}}\big(1+\mathbb{E}|Y_\Delta(t_n)|^p+\mathbb{E}|Z_\Delta(t_n)|^p\big)\\ &\leq C\Delta^{\frac{p}{2}}(1+\mathbb{E}|Y_\Delta(\zeta)|^p). \end{aligned} \tag{69}$$

From (67) and (69) we can simply conclude

$$\mathbb{E}|y_\Delta(\zeta)-Z_\Delta(\zeta)|^p \leq C\Delta^{\frac{p}{2}}(1+\mathbb{E}|Y_\Delta(\zeta)|^p). \tag{70}$$

Substituting (64)–(66) and (70) into (63) and applying Young's inequality that is

$$z_1z_2 \leq \frac{z_1^2}{2\delta}+\frac{\delta z_2^2}{2}, \qquad \text{for all } z_1,z_2\in\mathbb{R}, \quad \text{for all } \delta>0, \tag{71}$$

with $\delta = 8pM_2$, we can write

$$\mathbb{E}\Big(\sup_{0\leq\xi\leq T}|y_\Delta(\xi)|^p\Big) \leq C\Big(1+\int_0^T \mathbb{E}|y_\Delta(\zeta)|^p d\zeta+\int_0^T \mathbb{E}|Y_\Delta(\zeta)|^p d\zeta\Big)$$

$$
\begin{aligned}
&+\frac{1}{2}\mathbb{E}\Big(\sup_{0\leq\xi\leq T}|y_\Delta(\xi)|^p\Big)\\
&+32p^2M_2^2\mathbb{E}\Big(\int_0^T|y_\Delta(\xi)|^{p-2}\big(1+|Z_\Delta(\xi)|^2\big)d\xi\Big)\\
\leq&\frac{1}{2}\mathbb{E}\Big(\sup_{0\leq\xi\leq T}|y_\Delta(\xi)|^p\Big)\\
&+C\Big(1+\int_0^T\mathbb{E}|y_\Delta(\zeta)|^pd\zeta+\int_0^T\mathbb{E}|Y_\Delta(\zeta)|^pd\zeta\Big),
\end{aligned}
$$

which implies

$$
\mathbb{E}\Big(\sup_{0\leq\xi\leq T}|y_\Delta(\xi)|^p\Big)\leq C\Big(1+\int_0^T\mathbb{E}\big[\sup_{0\leq s\leq\xi}|y_\Delta(s)|^p\big]ds\Big).
$$

Then, the Gronwall inequality implies the relation (60) for $p>2$. For $\bar{p}\in(0,2]$, (60) still holds as desired due to the Lyapunov inequality. Moreover, (61) is obtained directly by substituting (60) into (69) and (70) , which completes the proof. □

Theorem 3. Consider the initial problem (1) that satisfies Assumptions 1 and 2. Let $\nu(r)=\eta_h r^{1+\bar{\rho}}$ and $h(\Delta)=\eta_h\Delta^{-\frac{1}{2}}$ in which $\bar{\rho}\geq\frac{\rho}{2}$. Then there exists a constant C such that

$$
\mathbb{E}|e_\Delta(T)|\leq\begin{cases}C\Big(-\frac{1}{\ln(\Delta)}+\Delta^{\frac{1}{4}}\Big), & \text{if } \alpha=0,\\ C\Delta^\alpha, & \text{if } 0<\alpha<0.5.\end{cases}\tag{72}
$$

Proof. We split the left side of (72) into two parts

$$
\mathbb{E}|e_\Delta(T)|=\mathbb{E}|e_\Delta(T)I_{\{\beta_{\Delta,R}\leq T\}}|+\mathbb{E}|e_\Delta(T)I_{\{\beta_{\Delta,R}>T\}}|.\tag{73}
$$

Using the Young inequality, we have

$$
\mathbb{E}|e_\Delta(T)I_{\{\beta_{\Delta,R}\leq T\}}|\leq\frac{\Delta^{\frac{1}{2}}}{\bar{\rho}+2}\mathbb{E}|e_\Delta(T)|^{\bar{\rho}+2}+\frac{\bar{\rho}+1}{\bar{\rho}+2}\Delta^{-\frac{1}{2(\bar{\rho}+1)}}\mathbb{P}(\beta_{\Delta,R}\leq T).\tag{74}
$$

For any $R>|x_0|$, by Lemmas 1 and 5, we can write

$$
R^{\bar{\rho}+2}\mathbb{P}(\beta_{\Delta,R}\leq T)\leq C,
$$

which follows that

$$
\mathbb{P}(\beta_{\Delta,R}\leq T)\leq\frac{C}{R^{\bar{\rho}+2}}.\tag{75}
$$

If we substitute (75) into (74) and use Theorem 1 and Lemma 8, we get

$$
\mathbb{E}|e_\Delta(T)I_{\{\beta_{\Delta,R}\leq T\}}|\leq C\Big(\Delta^{\frac{1}{2}}+\frac{\Delta^{-\frac{1}{2(\bar{\rho}+1)}}}{R^{\bar{\rho}+2}}\Big).
$$

By choosing $R = \nu^{-1}(h(\Delta)) = (\frac{\eta_h}{\eta_\nu}\Delta^{-\frac{1}{2}})^{\frac{1}{1+\bar{\rho}}}$, we have

$$\mathbb{E}|e_\Delta(T)I_{\{\beta_{\Delta,R}\leq T\}}| \leq C\Delta^{\frac{1}{2}}. \tag{76}$$

In what follows, we try to estimate the second terms in (73). Since $R = \nu^{-1}(h(\Delta))$, if $0 \leq \zeta \leq \beta_{\Delta,R}$, then $|Y_\Delta(\zeta)| \vee |Z_\Delta(\zeta)| \leq \nu^{-1}(h(\Delta))$ for all $\Delta \in [0,1]$. By the definition of the truncated function (19), $a_\Delta(|Y_\Delta(\zeta)|) = a(|Y_\Delta(\zeta)|)$ and $a_\Delta(|Z_\Delta(\zeta)|) = a(|Z_\Delta(\zeta)|)$. Using the same notation as in the proof of Lemma 7, according to the Itô formula we get

$$\mathbb{E}|e_\Delta(\xi \wedge \beta_{\Delta,R})| \leq \epsilon + \Pi_1 + \Pi 2, \tag{77}$$

where Π_1 and $\Pi 2$ are defined in (51). According to Assumption 1, Lemma 6, and Remark 4 as well as the Hölder inequality the following equation holds for Π_1

$$\begin{aligned}\Pi_1 \leq& H_1\int_0^\xi \mathbb{E}|e_\Delta(\zeta\wedge\beta_{\Delta,R}))|d\zeta\\ &+\sqrt{K_1}\int_0^{\xi\wedge\beta_{\Delta,R}}\big(1+\mathbb{E}|y_\Delta(\zeta)|^\rho+\mathbb{E}|Y_\Delta(\zeta)|^\rho\big)^{\frac{1}{2}}(\mathbb{E}\big|y_\Delta(\zeta))-Y_\Delta(\zeta)\big|^2)^{\frac{1}{2}}d\zeta\\ &+\alpha_2\sqrt{K_1}\mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}}\big(1+\mathbb{E}|Y_\Delta(\zeta)|^\rho+\mathbb{E}|Z_\Delta(\zeta)|^\rho\big)^{\frac{1}{2}}(\mathbb{E}\big|Y_\Delta(\zeta))-Z_\Delta(\zeta)\big|^2)^{\frac{1}{2}}d\zeta.\end{aligned}$$

From Theorem 1 and Lemma 8 we can thus conclude

$$\Pi_1 \leq C\int_0^\xi \mathbb{E}|e_\Delta(\zeta\wedge\beta_{\Delta,R}))|d\zeta + C\Delta^{\frac{1}{2}}. \tag{78}$$

As for Π_2, it follows from (6) and Lemmas 6 and 8 that

$$\begin{aligned}\Pi_2 &\leq H_2^2\mathbb{E}\int_0^{\xi\wedge\beta_{\Delta,R}}\big(x(\zeta)-Z_\Delta(\zeta)\big)^{1+2\alpha}\frac{1}{|e_\Delta(\zeta)|\ln\delta}I_{\{\epsilon/\delta\leq|e_\Delta(\zeta)|\leq\epsilon\}}d\zeta\\ &\leq \frac{2^{2\alpha}H_2^2\epsilon^{2\alpha}T}{\ln(\delta)}+\frac{2^{2\alpha}H_2^2\delta}{\epsilon\ln(\delta)}\mathbb{E}\int_0^\xi\big|y_\Delta(\zeta)-Z_\Delta(\zeta)\big|^{1+2\alpha}d\zeta\\ &\leq C\big(\frac{\epsilon^{2\alpha}}{\ln(\delta)}+\frac{\delta\Delta^{1/2+\alpha}}{\epsilon\ln(\delta)}\big).\end{aligned} \tag{79}$$

Substituting (78) and (79) into (77) and applying Gronwall's inequality yield

$$\mathbb{E}|e_\Delta(\xi\wedge\beta_{\Delta,R})| \leq C\big(\frac{\epsilon+\epsilon^{2\alpha}}{\ln(\delta)}+\frac{\delta\Delta^{1/2+\alpha}}{\epsilon\ln(\delta)}+\Delta^{1/2}\big).$$

In what follows we divide the proof into two cases $\alpha = 0$ and $\alpha \in (0, \frac{1}{2})$. In the case when $\alpha = 0$, we set $\delta = \Delta^{-\frac{1}{4}}$ and $\epsilon = \frac{1}{4}$, implying

$$\mathbb{E}|e_\Delta(\xi\wedge\beta_{\Delta,R})| \leq C\big(-\frac{1}{\ln(\Delta)}+\Delta^{\frac{1}{4}}\big). \tag{80}$$

In the case where $\alpha \in (0, \frac{1}{2})$, we set $\Delta = 2$ and $\epsilon = \Delta^{\frac{1}{2}}$, which implies

$$\mathbb{E}|e_\Delta(\xi \wedge \beta_{\Delta,R})| \leq C\Delta^\alpha. \tag{81}$$

From (77), (80), and (81) the proof follows directly. □

4 Convergence rate over a finite interval

Sometimes we need to approximate path-dependent quantities, for example, the value of the European barrier option value. In these cases, we need a stronger convergence result such as

$$\lim_{\Delta\to 0} \mathbb{E}\Big(\sup_{0\leq\xi\leq T} |x(\xi) - y_\Delta(\xi)|\Big) = 0.$$

Lemma 9. Consider the initial problem (1), which satisfies Assumptions 1 and 2. Suppose that $R > |x_0|$ is a real number and that $\Delta \in (0, 1]$ is sufficiently small such that $\nu^{(-1)}(h(\Delta)) \geq R$. Then for all $0 \leq \xi \leq T$, a constant C exists such that

$$\mathbb{E}\Big(\sup_{0\leq\zeta\leq\xi} |x(\zeta) - y_\Delta(\zeta)|\Big) \leq \begin{cases} C\Big(\frac{1}{\ln(\Delta^{-1})} + \Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\Big)^{\frac{1}{2}}, & \text{if} \quad \alpha = 0, \\ C\big(\Delta^{\frac{1}{2}}h(\Delta)\big)^{4\alpha^2}, & \text{if} \quad \alpha \in (0, \frac{1}{2}). \end{cases} \tag{82}$$

Proof. By the Itô formula, we can write

$$\begin{aligned} |e_\Delta(\xi \wedge \beta_{\Delta,R})| \leq& \varepsilon + \big(\varpi_{\delta,\epsilon}(e_\Delta(\xi \wedge \beta_{\Delta,R}))\big) \\ =& \epsilon + \int_0^{\xi\wedge\beta_{\Delta,R}} \varpi'_{\delta,\epsilon}(e_\Delta(\zeta))\Big[a(x(\zeta)) - \alpha_1 a_\Delta(Y_\Delta(\zeta)) - \alpha_2 a_\Delta(Z_\Delta(\zeta))\Big]d\zeta \\ &+ \frac{1}{2}\int_0^{\xi\wedge\beta_{\Delta,R}} \varpi''_{\delta,\epsilon}(e_\Delta(\zeta))\big[b(x(\zeta)) - b(Z_\Delta(\zeta))\big]^2 d\zeta + S_{\delta,\epsilon,\Delta}(\xi), \end{aligned}$$

where, for $\xi \geq 0$

$$S_{\delta,\epsilon,\Delta}(\xi) = \int_0^{\xi\wedge\beta_{\Delta,R}} \varpi'_{\delta,\epsilon}(e_\Delta(\zeta))\big(b(x(\zeta)) - b(Z_\Delta(\zeta))\big)dW(\zeta). \tag{83}$$

Since $\nu^{(-1)}(h(\Delta)) \geq R$, if $0 \leq \zeta \leq \xi \wedge \beta_{\Delta,R}$, we have $|Y_\Delta(\zeta)| \vee |Z_\Delta(\zeta)| \leq \nu^{(-1)}(h(\Delta))$, which gives $a_\Delta(Y_\Delta(\zeta)) = a(Y_\Delta(\zeta))$ and $a_\Delta(Z_\Delta(\zeta)) = a(Z_\Delta(\zeta))$. If we use the same technique as in the proof of Lemma 7 and apply Assumption 1, then we have

$$|e_\Delta(\xi \wedge \beta_{\Delta,R})| \leq \varepsilon + H_1 \int_0^{\xi\wedge\beta_{\Delta,R}} |e_\Delta(\zeta))|d\zeta \tag{84}$$

$$
\begin{aligned}
&+ \sqrt{K_1}\int_0^{\xi\wedge\beta_{\Delta,R}} \big(1+|y_\Delta(\zeta)|^\rho+|Y_\Delta(\zeta)|^\rho\big)^{\frac{1}{2}}\big|y_\Delta(\zeta))-Y_\Delta(\zeta)\big|d\zeta\\
&+ \alpha_2\sqrt{K_1}\int_0^{\xi\wedge\beta_{\Delta,R}} \big(1+|Y_\Delta(\zeta)|^\rho+|Z_\Delta(\zeta)|^\rho\big)^{\frac{1}{2}}\big|Y_\Delta(\zeta))-Z_\Delta(\zeta)\big|d\zeta\\
&+ \frac{2^{2\alpha}H_2^2\varepsilon^{2\alpha}}{\ln(\delta)}\xi\\
&+ \frac{2^{2\alpha}H_2^2\delta}{\varepsilon\ln(\delta)}\int_0^{\xi\wedge\beta_{\Delta,R}}\big|y_\Delta(\zeta)-Z_\Delta(\zeta)\big|^{1+2\alpha}d\zeta+S_{\delta,\epsilon,\Delta}(\xi).
\end{aligned}
$$

By the Hölder's inequality and Lemmas 3 and 4, we have

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le u\le\xi}|e_\Delta(u\wedge\beta_{\Delta,R})|\Big) \le&\varepsilon+H_1\int_0^{\xi}\mathbb{E}\Big(\sup_{0\le u\le\zeta}|e_\Delta(u\wedge\beta_{\Delta,R})|\Big)d\zeta+C\Delta^{\frac{1}{2}}h(\Delta)\\
&+\frac{2^{2\alpha}H_2^2\varepsilon^{2\alpha}}{\ln(\delta)}\xi+\frac{2^{2\alpha}H_2^2\delta}{\varepsilon\ln(\delta)}\mathbb{E}\int_0^{\xi}\big|y_\Delta(\zeta)-Z_\Delta(\zeta)\big|^{1+2\alpha}d\zeta\\
&+\mathbb{E}\Big(\sup_{0\le u\le\xi}|S_{\delta,\epsilon,\Delta}(u)|\Big). \qquad (85)
\end{aligned}
$$

Next, we try to estimate the value $\mathbb{E}\Big(\sup_{0\le u\le\xi}|S_{\delta,\epsilon,\Delta}(u)|\Big)$ in (85). By the Burkholder–Davis–Gundy inequality and Lemma 6, we can write

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le u\le\xi}|S_{\delta,\epsilon,\Delta}(u)|\Big) \le&4\sqrt{2}\mathbb{E}\Big(\int_0^{\xi\wedge\beta_{\Delta,R}}|\varpi'_{\delta\epsilon}(e_\Delta(\zeta))|^2\big|b(x(\zeta))-b(Z_\Delta(\zeta))\big|^2d\zeta\Big)^{\frac{1}{2}}\\
\le&2^{\alpha+\frac{5}{2}}H_2\mathbb{E}\Big(\int_0^{\xi\wedge\beta_{\Delta,R}}|x(\zeta)-x_\Delta(\zeta)|^{1+2\alpha}d\zeta\Big)^{\frac{1}{2}}\\
&+2^{\alpha+\frac{5}{2}}H_2\mathbb{E}\Big(\int_0^{\xi\wedge\beta_{\Delta,R}}|x_\Delta(\zeta)-Z_\Delta(\zeta)|^{1+2\alpha}d\zeta\Big)^{\frac{1}{2}}.
\end{aligned}
$$

In fact, by Assumption 2 we have

$$
\begin{aligned}
\big|b(x(\zeta))-b(Z_\Delta(\zeta))\big|^2\le& H_2|x(\zeta)-Z_\Delta(\zeta)|^{1+2\alpha}\\
\le& 2^\alpha H_2|x(\zeta)-x_\Delta(\zeta)|^{1+2\alpha}+2^\alpha H_2|x_\Delta(\zeta)-Z_\Delta(\zeta)|^{1+2\alpha}.
\end{aligned}
$$

Therefore, by Lemma 3 we have

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le u\le\xi}|S_{\delta,\epsilon,\Delta}(u)|\Big) \le&C\Big(\Delta^{\frac{1}{4}}\sqrt{h(\Delta)}\Big)^{1+2\alpha}\\
&+2^{\alpha+\frac{5}{2}}H_2\mathbb{E}\Big(\int_0^{\xi\wedge\beta_{\Delta,R}}|x(\zeta)-x_\Delta(\zeta)|^{1+2\alpha}d\zeta\Big)^{\frac{1}{2}} \qquad (86)
\end{aligned}
$$

In the rest, we divide the proof into two cases.

Case 1: Suppose that $\alpha = 0$. In this case, according to Lemma 7 and from the relation (86), it follows that

$$\mathbb{E}\Big(\sup_{0\le u\le\xi}|S_{\delta,\epsilon,\Delta}(u)|\Big) \le C\left(\Delta^{\frac{1}{4}}\sqrt{h(\Delta)} + \Big(\frac{1}{\ln(\Delta^{-1})} + \Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\Big)^{\frac{1}{2}}\right). \quad (87)$$

Substituting (87) into (85) and choosing $\delta = \Delta^{-\frac{1}{8}}$ and $\varepsilon = -1/\ln(\Delta)$, we get

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le u\le\xi}|e_\Delta(u\wedge\beta_{\Delta,R})|\Big) \le & H_1\int_0^\xi \mathbb{E}\Big(\sup_{0\le u\le\zeta}|e_\Delta(u\wedge\beta_{\Delta,R})|\Big)d\zeta \\ & + C\left(\Delta^{\frac{1}{2}}h(\Delta) + \Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}} - \frac{1}{\ln(\Delta)} + \Delta^{\frac{3}{8}}h(\Delta)\right. \\ & \left. + \Big(\frac{1}{\ln(\Delta^{-1})} + \Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\Big)^{\frac{1}{2}}\right),\end{aligned}$$

which implies

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le u\le\xi}|e_\Delta(u\wedge\beta_{\Delta,R})|\Big) \le & H_1\int_0^\xi \mathbb{E}\Big(\sup_{0\le u\le\zeta}|e_\Delta(u\wedge\beta_{\Delta,R})|\Big)d\zeta \\ & + C\Big(\frac{1}{\ln(\Delta^{-1})} + \Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\Big)^{\frac{1}{2}}. \quad (88)\end{aligned}$$

Case 2: Suppose $\alpha \in (0, \frac{1}{2})$. In this case from (86) we have

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le u\le\xi}|S_{\delta,\epsilon,\Delta}(u)|\Big) \le & C\Big(\Delta^{\frac{1}{4}}\sqrt{h(\Delta)}\Big)^{1+2\alpha} \\ & + 2^{\alpha+\frac{5}{2}}H_2\mathbb{E}\left(\Big(\sup_{0\le u\le\xi}|e_\Delta(\xi\wedge\beta_{\Delta,R})|\Big)^{\frac{1}{2}}\Big(\int_0^{\xi\wedge\beta_{\Delta,R}}|e_\Delta(\zeta)|^{2\alpha}d\zeta\Big)^{\frac{1}{2}}\right).\end{aligned}$$

By Young's inequality and Lemma 7 we have

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le u\le\xi}|S_{\delta,\epsilon,\Delta}(u)|\Big) \le & C\left(\Big(\Delta^{\frac{1}{4}}\sqrt{h(\Delta)}\Big)^{1+2\alpha} + \big(\sqrt{\Delta}h(\Delta)\big)^{4\alpha^2}\right) \\ & + \frac{1}{2}\mathbb{E}\Big(\sup_{0\le u\le\xi}|e_\Delta(\xi\wedge\beta_{\Delta,R})|\Big)\Big). \quad (89)\end{aligned}$$

Inserting (89) in (85) and by choosing $\varepsilon = \Delta^{\frac{1}{2}}h(\Delta)$ and $\delta = 2$, we get

$$\mathbb{E}\Big(\sup_{0\le u\le\xi}|e_\Delta(u\wedge\beta_{\Delta,R})|\Big) \le 2H_1\int_0^\xi \mathbb{E}\Big(\sup_{0\le u\le\zeta}|e_\Delta(u\wedge\beta_{\Delta,R})|\Big)d\zeta + C\big(\sqrt{\Delta}h(\Delta)\big)^{4\alpha^2}. \quad (90)$$

Hence, by using the Gronwall inequality on (88) and (90) we have the required assertion. □

Theorem 4. Let $\nu(r) = \eta_h r^{1+\bar{\rho}}$ and $h(\Delta) = \eta_h \Delta^{-\omega\epsilon}$ in which $\bar{\rho} \geq \frac{\rho}{2}$, and

$$\epsilon \in \Big(0, \min\{\frac{1}{4\omega}, \frac{3}{4-2\omega}\}\Big),$$

for any $\omega \in (0, 2)$. Let Assumptions 1 and 2 hold, and also let Δ^* be sufficiently small such that $\Delta^{\frac{1-2\omega\epsilon}{4}} \leq \frac{1}{\ln \Delta^{-1}}$ for all $\Delta \in (0, \Delta^*]$. Then

$$\mathbb{E}\Big(\sup_{0\leq\zeta\leq T}|x(\zeta)-y_\Delta(\zeta)|\Big) \leq \begin{cases} C\Big(\frac{1}{\ln(\Delta^{-1})}\Big)^{\frac{1}{2}}, & \text{if} \quad \alpha = 0, \\ C\Big(\Delta^{2\alpha^2(1-2\omega\epsilon)} + \Delta^{\frac{1-\epsilon}{2}}\Big), & \text{if} \quad \alpha \in (0, \frac{1}{2}). \end{cases} \tag{91}$$

Proof. For any $R > |x_0|$, we can decompose the left side of (91) into two parts as below:

$$\mathbb{E}\Big(\sup_{0\leq\zeta\leq T}|x(\zeta)-y_\Delta(\zeta)|\Big) \leq \mathbb{E}\Big(\sup_{0\leq u\leq\xi}|e_\Delta(u\wedge\beta_{\Delta,R})|\Big)+\mathbb{E}\Big(\sup_{0\leq\zeta\leq T}|e_\Delta(\zeta)|I_{\{\beta_{\Delta,R}\leq T\}}\Big). \tag{92}$$

Let $p > 1 + \frac{1+\bar{\rho}}{2\omega\epsilon}$ be fixed. By the Young inequality, that is

$$z_1 z_2 \leq \frac{\delta}{p} z_1^p + \frac{p-1}{p\delta^{1/(p-1)}} z_2^{1/(p-1)}, \qquad \text{for all } z_1, z_2 \in [0, +\infty), \qquad \text{for all } \delta > 0,$$

and Lemmas 1 and 5, we can deduce

$$\begin{aligned} \mathbb{E}\Big(\sup_{0\leq\zeta\leq T}|e_\Delta(\zeta)|I_{\{\beta_{\Delta,R}\leq T\}}\Big) \leq& \frac{\delta}{p}\mathbb{E}\Big(\sup_{0\leq\zeta\leq T}|e_\Delta(\zeta)|^p\Big) + \frac{p-1}{p\delta^{1/p-1}}\mathbb{P}(\beta_{\Delta,R} \leq T) \\ \leq& \frac{C\delta}{p} + \frac{p-1}{p\delta^{1/p-1}R^p}. \end{aligned} \tag{93}$$

By choosing

$$\delta = \Delta^{(1-\epsilon)/2}, \qquad R = \Delta^{-(1-\epsilon)/2(p-1)},$$

we see that $\nu^{(-1)}(h(\Delta)) \geq R$. Therefore, by substituting (93) into (92) and applying Lemma 9, we have

$$\mathbb{E}\Big(\sup_{0\leq\zeta\leq T}|x(\zeta)-y_\Delta(\zeta)|\Big) \leq \begin{cases} C\Big(\big(\frac{1}{\ln(\Delta^{-1})} + \Delta^{\frac{1-2\omega\epsilon}{4}}\big)^{\frac{1}{2}} + \Delta^{\frac{1-\epsilon}{2}}\Big), & \text{if} \quad \alpha = 0, \\ C\Big(\Delta^{2\alpha^2(1-2\omega\epsilon)} + \Delta^{\frac{1-\epsilon}{2}}\Big), & \text{if} \quad \alpha \in (0, \frac{1}{2}), \end{cases}$$

which completes the proof of the theorem. □

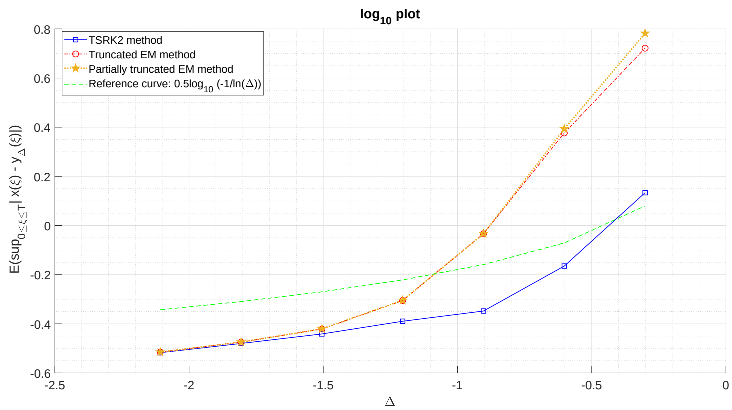

Figure 1: The supremum convergence norm (95) in $[0, 2]$ as a function of step size Δ for Example 1 with $\epsilon = 0.0001$, $\omega = 0.5$, and $\alpha = 0$.

5 Numerical experiments

In Section 3, we derived TSRK2 methods with free parameters α_1, α_2, and θ. In this section, we follow [17] and set them to below:

$$\alpha_1 = \alpha_2 = \frac{1}{2} \qquad \text{and} \qquad \theta = \frac{\omega_1}{\omega_0}, \tag{94}$$

in which $\omega_0 = 1.9$ and $\omega_1 = 0.8184$. We confirm the performance of the new method in terms of accuracy and stability in contrast to the truncated EM method [21, 32] and the modified partially truncated EM method [33]. In this context, we investigate

$$\mathbb{E}\Big(\sup_{0\le\zeta\le T} |x(\zeta) - y_\Delta(\zeta)|\Big) \approx \frac{1}{2000}\sum_{i=1}^{2000} \max_{n=1,\dots,M} \big|x^{(i)}(t_n) - y_\Delta^{(i)}(t_n)\big|, \tag{95}$$

for a given $\Delta = T/N$ to measure the accuracy of the methods.

Example 1. Consider the following SDE with Hölder continuous diffusion coefficient

$$\begin{aligned} dx(\xi) &= \Big(\lambda_1 x(\xi) - \lambda_2 x^3(\xi)\Big)d\xi + \mu|x(\xi)|^{\frac{1}{2}+\alpha} dW(\xi), \quad \xi \in [0, T], \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned} \tag{96}$$

where $\lambda_1 \in \mathbb{R}$ and $\lambda_2, \mu \in \mathbb{R}^+$.

The drift and diffusion coefficients are $a(z) = \lambda_1 z - \lambda_2 z^3$ and $b(z) = \mu|z|^{\frac{1}{2}+\alpha}$, respectively. We can easily show that, for $z_1, z_2 \in \mathbb{R}$,

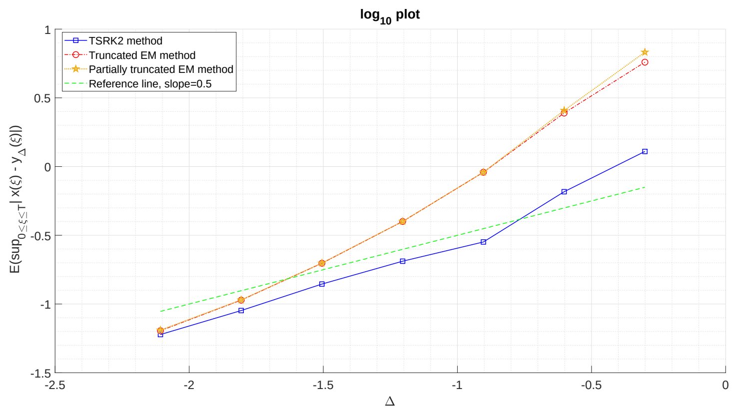

Figure 2: The supremum convergence norm (95) in $[0,2]$ as a function of step size Δ for Example 1 with $\epsilon = 0.0001$, $\omega = 0.5$, and $\alpha = 0.25$.

$$|a(z_1) - a(z_2)|^2 \leq \tilde{K}_1(1 + |z_1|^4 + |z_2|^4)|z_1 - z_2|^2,$$

where $\tilde{K}_1 := 3\max\{\lambda_1^2, \frac{9}{4}\lambda_2^2\}$. In other words, Assumption 1 is satisfied with $\rho = 4$. Moreover, we have

$$(z_1 - z_2)(a(z_1) - a(z_2)) \leq \lambda_1(z_1 - z_2)^2, \quad |b(z_1) - b(z_2)| \leq \mu|z_1 - z_2|^{\frac{1}{2}+\alpha}.$$

That is to say, Assumption 2 is fulfilled. Concerning (10), we set $\eta_h = 4\sqrt{\tilde{K}_1}$. Then, we have

$$\sup_{0<|z_1|\vee|z_2|\leq r} \frac{|a(z_1) - a(z_2)|}{|z_1 - z_2|} \leq \eta_h\Big(\frac{1 + r^2}{2}\Big) \leq \eta_h r^3,$$

and

$$\sup_{|z|\leq r}|a(z)| \leq (|\lambda_1| + \lambda_2)r^3 \leq \eta_h r^3,$$

for all $r \geq 1$. To apply Theorem 4, we set $\nu(r) = \eta_h r^3$ and $h(\Delta) = \eta_h\Delta^{-\omega\epsilon}$. We can therefore conclude that the truncated Runge–Kutta solution (27) satisfies

$$\mathbb{E}\Big(\sup_{0\leq\zeta\leq T}|x(\zeta) - y_\Delta(\zeta)|\Big) \leq \begin{cases} C\Big(\frac{1}{\ln(\Delta^{-1})}\Big)^{\frac{1}{2}}, & \text{if} \quad \alpha = 0, \\ C\Big(\Delta^{2\alpha^2(1-2\omega\epsilon)} + \Delta^{\frac{1-\epsilon}{2}}\Big), & \text{if} \quad \alpha \in (0, \frac{1}{2}). \end{cases}$$

Let us now test the efficiency of the TSRK2 method for problem (96) in comparison with the truncated EM method [21] and the modified partially truncated EM method [33] for different values of the Hölder parameter α. Since we do not know the exact solution of (96), we search for a numerical

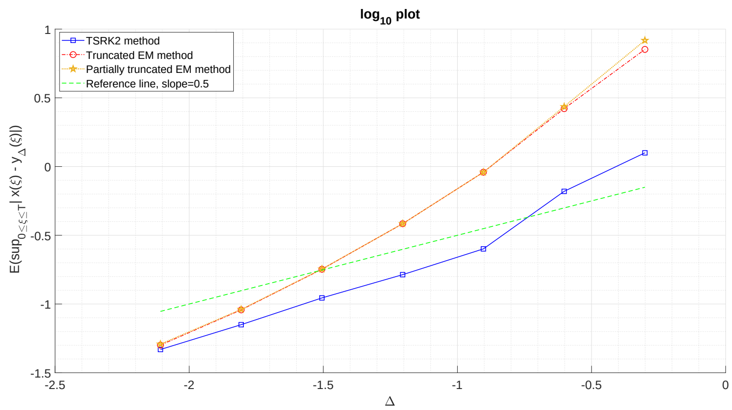

Figure 3: The supremum convergence norm (95) in $[0, 2]$ as a function of step size Δ for Example 1 with $\epsilon = 0.0001$, $\omega = 0.5$, and $\alpha = 0.45$.

solution with $\Delta = 2^{-18}$ using the truncated EM method and use it instead of the exact solution. Figures 1–3 show simulations of (95) for the TSRK2 method with parameters (94), the truncated EM method, and the modified partially truncated EM method with $\omega = 0.5$. Here we take $\lambda_1 = -1.5$, $\lambda_2 = 10$, and $\mu = 1.5$ as parameters of (96) and apply the methods over $0 \le \xi \le 2$ with step sizes, $\Delta = 2^{i-2}$ for $i = 1, 2, \dots, 9$. In these figures, the error of the methods is plotted as a function of seven-step sizes Δ in log_{10} for different values of α $(0, 0.25$ and 0.45, respectively).

Figure 1 shows that, compared to the reference curve $0.5 \log_{10}(-\frac{1}{\ln(\Delta)})$, the TSRK2 method gives better convergence results than the truncated EM method and the modified partially truncated EM method for step size $2^{-5} \le \Delta \le 2^{-1}$. However, for sufficiently small Δ (e.g., $\Delta \in (0, 2^{-5}]$), all methods have the same slope values as in Theorem 4. Increasing the value of the Hölder parameter α, it can be seen in Figures 2 and 3 that for the step size $2^{-5} \le \Delta \le 2^{-1}$, the TSRK2 method has a better convergence rate than the other methods compared to the reference line with slope $1/2$, but for sufficiently small $\Delta \le 2^{-6}$, all methods have the same convergence rate close to 0.5 as in Theorem 4. The simulation results clearly show that the TSRK2 method confirms the convergence results stated in Theorem 4 and that the new method is the most efficient method in terms of accuracy compared to the truncated EM method and the modified partially truncated EM method.

Example 2. Consider the scalar nonlinear Itô SDE with a one-dimensional Wiener process

$$
\begin{aligned}
&dx(t) = \Big(\lambda_1 x(\xi) - \lambda_2 x^5(\xi)\Big)d\xi + \mu|x(\xi)|^{\frac{1}{2}+\alpha} dW(\xi), \quad t \in [0, T], \\
&x(0) = 1, \qquad (97)
\end{aligned}
$$

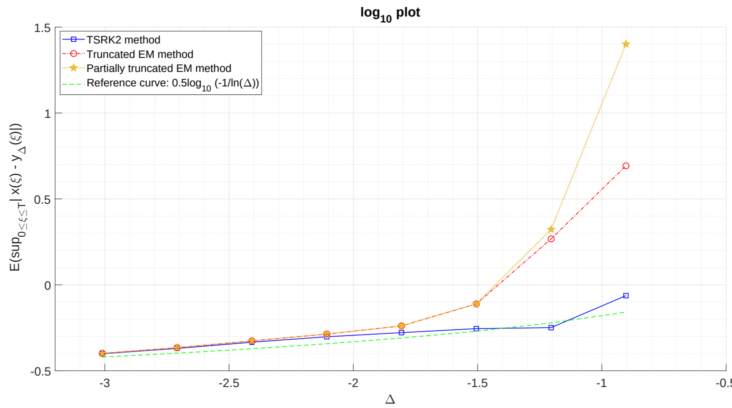

Figure 4: The supremum convergence norm (95) in $[0, 2]$ as a function of step size Δ for Example 2 with $\epsilon = 0.1$, $\omega = 0.5$, and $\alpha = 0$.

where $\lambda_1 \in \mathbb{R}$ and $\lambda_2, \mu \in \mathbb{R}^+$.

The drift and diffusion coefficients are $a(z) = \lambda_1 z - \lambda_2 z^5$ and $b(z) = \mu|z|^{\frac{1}{2}+\alpha}$, respectively. We can easily show that, for $z_1, z_2 \in \mathbb{R}$,

$$|a(z_1) - a(z_2)|^2 \leq \tilde{K}_2(1 + |z_1|^8 + |z_2|^8)|z_1 - z_2|^2,$$

where $\tilde{K}_2 := 3\max\{\lambda_1^2, \frac{25}{4}\lambda_2^2\}$. In other words, Assumption 1 is satisfied with $\rho = 8$. Moreover, we have

$$(z_1 - z_2)(a(z_1) - a(z_2)) \leq \lambda_1(z_1 - z_2)^2, \quad |b(z_1) - b(z_2)| \leq \mu|z_1 - z_2|^{\frac{1}{2}+\alpha}.$$

Concerning (10), we set $\eta_\nu = 4\sqrt{\tilde{K}_2}$. Then, we conclude

$$\sup_{0<|z_1|\vee|z_2|\leq r} \frac{|a(z_1) - a(z_2)|}{|z_1 - z_2|} \leq \eta_h\Big(\frac{1 + r^4}{2}\Big) \leq \eta_h r^5,$$

and

$$\sup_{|z|\leq r} |a(z)| \leq (|\lambda_1| + \lambda_2)r^5 \leq \eta_h r^5,$$

for all $r \geq 1$. To apply Theorem 4, we set $\nu(r) = \eta_\nu r^5$ and $h(\Delta) = \eta_\nu \Delta^{-\omega\epsilon}$. Figures 4–6 show the simulation of (95) of the TSRK2 method for equation (97) as a function of step size Δ for Hölder $\alpha = 0$, $\alpha = 0.25$, and $\alpha = 0.45$, respectively. Since there is no explicit solution, we use the truncated EM solution with $\Delta = 2^{-18}$ as a suitable approximation to the exact solution. Here we take $\lambda_1 = -15$, $\lambda_2 = 20$, and $\mu = 3$ as stiff parameters of (97) and apply the methods over $0 \leq \xi \leq 2$ with different step sizes $\Delta \in \{2^{-3}, 2^{-4}, \cdots, 2^{-10}\}$.

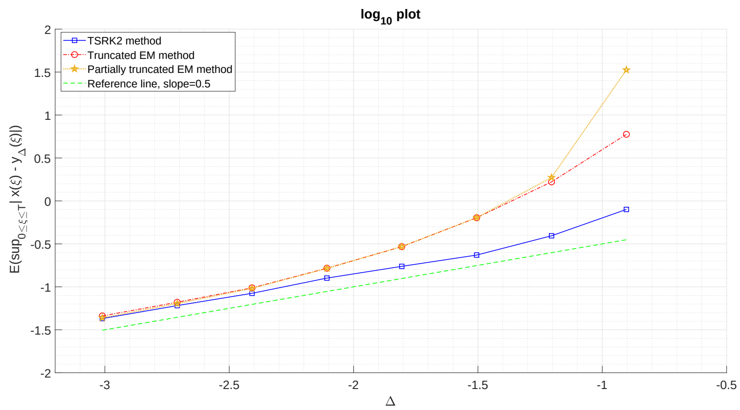

Figure 5: The supremum convergence norm (95) in $[0, 2]$ as a function of step size Δ for Example 2 with $\epsilon = 0.1$, $\omega = 0.5$ and $\alpha = 0.25$.

Figure 4, corresponding to $\alpha = 0$, shows that the TSRK2 method gives better convergence results than the truncated EM method and the modified partially truncated EM method, especially for the step size $2^{-6} \le \Delta \le 2^{-3}$, compared to the reference curve $0.5 \log_{10}(-\frac{1}{\ln(\Delta)})$. For the step size $\Delta = 2^{-3}$, for example, the new method has an error of 0.8655, while the truncated EM method and the modified partially truncated EM method reach an error of 4.9260 and 25.1876, respectively. It should be emphasized that the modified partially truncated method EM is not applicable for $\Delta \ge 2^{-2}$. However, for sufficiently small Δ (e.g., $\Delta \le 2^{-6}$), all methods have the same slope values as in Theorem 4.

In Figures 4 and 5, we present the supremum norm of the methods for larger values of $\alpha = 0.25$ and $\alpha = 0.45$, respectively. These figures also show that the new method has a better convergence rate compared to the other methods mentioned, especially when the step sizes are not very small.

6 Conclusion

In this work, we have developed an explicit TSRK2 scheme for nonlinear one-dimensional Itô-SDEs with one-sided Lipschitz and local Lipschitz drift conditions and continuous Hölder diffusion condition. We proved the moment boundedness and convergence properties of the approximate solutions at a time T and in a finite time interval $[0, T]$, respectively. We have demonstrated the efficiency of the new method using some numerical examples with different Hölder parameters $\alpha \in [0, \frac{1}{2})$ and compared our method with the truncated EM method and the modified partially truncated EM method in terms of their strong convergence performance. The numerical simulations

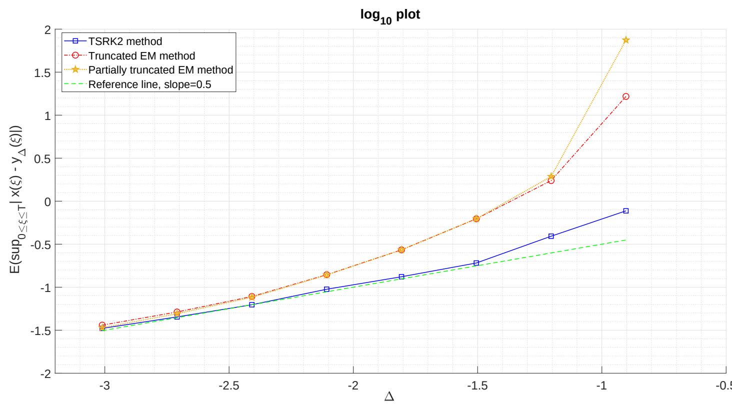

Figure 6: The supremum convergence norm (95) in $[0, 2]$ as a function of step size Δ for Example 2 with $\epsilon = 0.1$, $\omega = 0.5$ and $\alpha = 0.45$.

we performed confirmed the theoretical convergence results and also showed that the TSRK2 method is robust to changes in the Hölder parameters of the SDEs.

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