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Research Article

New class of hybrid explicit methods for numerical solution of optimal control problems

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Abstract

Forward-backward sweep method (FBSM) is an indirect numerical method used for solving optimal control problems, in which the differential equation arising from this method is solved by the Pontryagin's maximum principle. In this paper, a set of hybrid methods based on explicit 6th-order Runge–Kutta method is presented for the FBSM solution of optimal control problems. Order of truncation error, stability region, and numerical results of the new hybrid methods were compared with those of the 6th-order Runge–Kutta method. Numerical results show that new hybrid methods are more accurate than the 6th-order Runge–Kutta method and that their stability regions are also wider than that of the 6th-order Runge–Kutta method.

AMS subject classifications (2020): 65K10; 65L20.

Keywords: FBSM; OCP; Stability analysis; Hybrid methods.

1 Introduction

Numerical methods used for solving optimal control problems (OCPs) are generally divided into two categories, direct and indirect methods. Indirect methods solve an OCP numerically based on the Pontryagin's maximum

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principle. The Pontryagin's maximum principle was proposed in 1954 by Pontryagin, a Russian mathematician. In the indirect method, an OCP is converted into a two-point boundary-value problem (TPBVP). The forward-backward sweep method (FBSM) is one of indirect numerical methods, which was first proposed in 2007 in a book entitled as "Optimal Control Applied to Biological Models" written by Lenhart and Workman [13]. In 2009, Silveira et al. [23] suggested six methods for classifying skin lesions in medical images and concluded that the FBSM performs better than the other methods. In 2012, McAsey, Moua, and Han [16] proved the convergence of the FBSM. In 2015, Moualeu et al. [17] used the FBSM to treat and control a type of tuberculosis with unknown cases in Comeroon. In 2015, Rose in his thesis [20] reported that the FBSM is more accurate than the direct shooting method and optimization method of MATLAB . In 2015, Sana et al. [22] used trapezoidal and Euler methods, instead of Runge–Kutta method through the FBSM to solve an OCP, compared them with the 4th-order Runge–Kutta method and concluded that trapezoidal and Runge–Kutta methods have the same performance in solving the OCP. Indeed, the performance of these two methods was improved by increasing step length compared to the Euler method; see [21]. In 2017, Lhous et al. [15] proposed a discrete mathematical model and optimal control to reduce the divorce rate. They used the Pontryagin's maximum principle and FBSM. They informed the people of the community about advantages of marriage and disadvantages of divorce, and in this way, they were able to reduce the divorce rate. In 2018, Kheiri and Jafari [11] proposed a general formulation for a fractional optimal control problem (FOCP), in which state and co-state equations are given in terms of left fractional derivatives. They used an improved FBSM, by the Adams-type method to solve the FOCP . In 2019, Duran, Candelo, and Ortiz [3] used a modified FBSM for reconfiguring unbalanced distribution systems. In 2019, Kongjeen et al. [12] proposed a modified FBSM for analyzing electrical charge of microgrids. In 2020, Ameen, Hidan, and Mostefaoui [1] proposed a mathematical model for studying the relationship between fish consumption and the prevalence of chronic heart disease (CHD). They used an improved FBSM based on a predictor-corrector method and concluded that eating fish reduces the risk of CHD and its mortality. In 2020, Bhih et al. [2] proposed a new model of rumor on social media and determined three optimal controls theoretically minimizing the number of spreader users, fake pages, and related costs. They used the FBSM to solve their optimization system in a duplicate process. In 2021, Ebadi et al. [9, 8] presented hybrid methods to numerically solve the OCP by the FBSM. They concluded that their proposed methods are more accurate than the FBSM in the presence of the explicit Runge–Kutta methods.

In this work, we present new hybrid methods for the FBSM solution of OCP. The paper is organized as follows: In Section 2, new methods and their order of truncation errors are presented. Stability analysis of the methods discussed in Section 3. We review the OCP, FBSM, and its convergence in

Sections $4-6$. The results and final conclusions are presented in Sections 7 and 8, respectively.

2 Hybrid methods and order of truncation errors

Hybrid methods used for solving stiff differential equations are mostly efficient and have better response than other numerical methods. Consider the following initial value problem (IVP):

$$x' = f(t,x)\,,\ x \in \mathbb{R}^n\,, \quad x(t_0) = x_0\,,\ t_0\ \le t \le t_f, \tag{1}$$

where $f : [t_0, t_f] \times \mathbb{R}^n \to \mathbb{R}^n$. There are several explicit and implicit methods for solving such problems, but hybrid methods are more accurate than Runge–Kutta and backward differential formulas methods and have a wide range of stability; see [7, 5, 6, 4, 10]. Let us consider IVP in the form of (1). Linear k-step methods of the following form have $2k+1$ arbitrary parameter:

$$x_{n+1} = \alpha_1 x_n + \alpha_2 x_{n-1} + \cdots + \alpha_k x_{n-k+1} + h\{\beta_0 f_{n+1} + \beta_1 f_n + \cdots + \beta_k f_{n-k+1}\}, \tag{2}$$

where $f_{n+1} = f(t_n + h, x_{n+1})$, $f_{n-m} = f(t_n - mh, x_{n-m})$ and $f_n = f(t_n, x_n)$ for $m = 1, 2, \ldots, k-1$. For increasing order of k-step methods in the form of (2), a linear combination of the slopes is used at several points between t_n and t_{n+1}, where $t_{n+1} = t_n + h$ in which h is the step length on $[t_0, t_f]$. Then, the modified form of (2) with m slopes is given by

$$x_{n+1} = \sum_{j=1}^{k} \alpha_j x_{n-j+1} + h \sum_{j=0}^{k} \beta_j f_{n-j+1} + h \sum_{j=1}^{m} \mu_j f_{n+v_j}, \tag{3}$$

where α_j, β_j, and μ_j are $2k+m+1$ arbitrary parameters; see [10]. Methods of form (3) with m off-step points are called hybrid methods, where

$$0 < v_j < 1, \quad v_j \in \mathbb{R}, \quad j = 1, 2, \ldots, m,$$

and herein, we set $k = 1$ and $m = 4$. Hence, we write (3) as

$$\begin{aligned} x_{n+1} =& \alpha_1 x_n \\ &+ h\{\beta_0 f_{n+1} + \beta_1 f_n\} + h\{\mu_1 f_{n+v_1} + \mu_2 f_{n+v_2} + \mu_3 f_{n+v_3} + \mu_4 f_{n+v_4}\}, \end{aligned} \tag{4}$$

where $\alpha_1, \alpha_2, \beta_0, \beta_1$, and $v_i s$ are arbitrary parameters and v_i is not equal to 0 or 1 for $i = 1, 2, 3, 4$. Expanding each term in (4), in Taylor's series about t_n, we can obtain a family of the 6th-order methods if the equations

Table 1: Values of v_is in cases No. 1, 2, 3, 4, and 5 of new proposed methods

v_i	case 1	case 2	case 3	case 4	case 5
v_1	2.020e+00	2.020e+00	2.020e+00	2.020e+00	2.020e+00
v_2	4.000e-01	2.000e-01	4.000e-01	4.000e-01	1.000e-01
v_3	-2.000e-01	-3.000e-01	-2.000e-01	-2.000e-01	-2.000e-01
v_4	-1.840e+00	-1.540e+00	-1.440e+00	-1.140e+00	-1.840e+00

Table 2: Values of v_is in cases No. 6, 7, 8, 9, and 10 of new proposed methods

v_i	case 6	case 7	case 8	case 9	case 10
v_1	2.020e+00	2.090e+00	2.020e+00	2.020e+00	2.020e+00
v_2	4.000e-01	4.000e-01	3.000e-01	3.000e-01	4.000e-01
v_3	-2.000e-01	-5.000e-01	-3.000e-01	-2.000e-01	-2.0000e-01
v_4	-1.840e+00	-1.840e+00	-1.840e+00	-9.540e-02	-5.240e-01

$$\begin{aligned}
&\alpha_1 = 1,\\
&\beta_0 + \beta_1 + c_1 + c_2 + c_3 + c_4 = 1,\\
&\beta_0 + c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \tfrac{1}{2},\\
&\tfrac{1}{2}(\beta_0 + c_1 v_1^2 + c_2 v_2^2 + c_3 v_3^2 + c_4 v_4^2) = \tfrac{1}{6},\\
&\tfrac{1}{6}(\beta_0 + c_1 v_1^3 + c_2 v_2^3 + c_3 v_3^3 + c_4 v_4^3) = \tfrac{1}{24},\\
&\tfrac{1}{24}(\beta_0 + c_1 v_1^4 + c_2 v_2^4 + c_3 v_3^4 + c_4 v_4^4) = \tfrac{1}{120},\\
&\tfrac{1}{120}(\beta_0 + c_1 v_1^5 + c_2 v_2^5 + c_3 v_3^5 + c_4 v_4^5) = \tfrac{1}{720},\\
&\tfrac{1}{720}(\beta_0 + c_1 v_1^6 + c_2 v_2^6 + c_3 v_3^6 + c_4 v_4^6) = \tfrac{1}{5040},
\end{aligned}$$

are satisfied, where the principal term of the truncation error is

$$\frac{1}{7!}c_7 h^7 x^{(7)}(t_n) + o(h^8), \;\; c_7 = 1 - 7(\beta_0 + c_1 v_1^7 + c_2 v_2^7 + c_3 v_3^7 + c_4 v_4^7).$$

In this paper, the coefficients of (4) are proposed and obtained by searching and using coefficients that are more stable than the explicit Runge–Kutta methods. Values of v_i are determined with a wide stability region by searching for the interval of $[-2.5, 2.5]$. The v_i values are presented in ten cases as shown in Tables 1 and 2. They are called as the new proposed methods in this paper:

$$x_{n+1} = x_n + h\{\beta_1 f_n + \mu_1 f_{n+v_1} + \mu_2 f_{n+v_2} + \mu_3 f_{n+v_3} + \mu_4 f_{n+v_4} + \beta_0 f_{n+1}\}, \quad (5)$$

where $f_{n+1} = f(t_n + h, x_{n+1})$, $f_{n+v_i} = f(t_n + v_i h, x_{n+v_i})$, and $f_n = f(t_n, x_n)$ for $i = 1, 2, 3, 4$. Note that x_{n+1}, x_{n+v_i}, and x_n are numerical approximations according to the exact values of the solution $x(t)$ at $t_{n+1} = t_n + h$, $t_{n+v_i} = t_n + v_i h$. For converting (5) into explicit methods at each step, the values of x_{n+1} and x_{n+v_i} are predicted and used on right-hand side of the proposed methods using the 6th-order explicit Runge–Kutta (RK6) method, respectively, as follows:

$$
\begin{aligned}
&x_{n+1} = x_n + \tfrac{h}{180}(9k_1 + 64k_3 + 49k_5 + 49k_6 + 9k_7),\\
&k_1 = hf(x_n, y_n),\\
&k_2 = hf(x_n + h, y_n + k_1),\\
&k_3 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{8}(3k_1 + k_2),\\
&k_4 = hf(x_n + \tfrac{2}{3}h, y_n + \tfrac{1}{27}(8k_1 + 2k_2 + 8k_3),\\
&k_5 = hf(x_n + \tfrac{h}{14}(7 - \sqrt[2]{21}), y_n + k_{5h}),\\
&k_{5h} = \tfrac{1}{392}(3(3\sqrt[2]{21} - 7)k_1 - 8(7 - \sqrt[2]{21})k_2 + 48(7 - \sqrt[2]{21})k_3)\\
&\qquad + \tfrac{1}{392}(-3(21 - \sqrt[2]{21})k_4),\\
&k_6 = hf(x_n + \tfrac{h}{14}(7 + \sqrt[2]{21}), y_n + \tfrac{1}{1960}(-5(231 + 5\sqrt[2]{21})k_1\\
&\qquad -40(7 + \sqrt[2]{21})k_2 + k_{6h})\\
&k_{6h} = -320(\sqrt[2]{21})k_3 + 3(21 + 121\sqrt[2]{21})k_4 + 392(6 + \sqrt[2]{21})k_5),\\
&k_7 = hf(x_n + h, y_n + \tfrac{1}{180}(15(22 + 7\sqrt[2]{21})k_1 + 120k_2\\
&\qquad +40(7\sqrt[2]{21} - 5)k_3 + k_{7h}),\\
&k_{7h} = -63(3\sqrt[2]{21} - 2)k_4 - 14(49 + 9\sqrt[2]{21})k_5 + 70(7 - \sqrt[2]{21})k_6.
\end{aligned} \tag{6}
$$

$$
\bar{x}_{n+1} = x_n + \frac{h}{180}(9k_1 + 64k_3 + 49k_5 + 49k_6 + 9k_7), \tag{7}
$$

$$
\bar{x}_{n+v_i} = x_n + \frac{v_i h}{180}(9k_1 + 64k_{3i} + 49k_{5i} + 49k_{6i} + 9k_{7i}), \quad i = 1, 2, 3, 4.
$$

$$
x_{n+1} = x_n + h\{\beta_1 f_n + \mu_1 \bar{f}_{n+v_1} + \mu_2 \bar{f}_{n+v_2} + \mu_3 \bar{f}_{n+v_3} + \mu_4 \bar{f}_{n+v_4} + \beta_0 \bar{f}_{n+1}\}, \tag{8}
$$

where $k_{3i}, k_{5i}, k_{6i}, k_{7i}$, $i = 1, 2, 3, 4$, can be obtained by using the method 6 in which h is replaced by $v_i h$ and

$$
f_{n+1} = f(t_n + h, x_{n+1}), \quad f_{n+v_i} = f(t_n + v_i h, x_{n+v_i}), \quad f_n = f(t_n, x_n).
$$

Now suppose that the order of (8) is 6 similar to (6). Thus, the difference between exact and numerical solutions would be as follows:

$$
x(t_{n+m}) - x_{n+m} = C_7 h^7 x^{(p_1)}(t_n) + O(h^8). \tag{9}
$$

The difference operator associated with the 6th-order (8) can be written as

$$
x(t_{n+1}) - x_{n+1} = C h^7 x^{(7)}(t_n) + O(h^8),
$$

where C is the error constant of (8). Therefore, we have the following theorem.

Theorem 1. Given that (8) is of order p, then, p is equal to 6.

Proof. Suppose that $i = 1, 2, 3, 4$ and that x_n is exact. According to (7) and (8), one can write

$$\begin{aligned} x(t_{n+1}) - x_{n+1} =& h\sum_{i=1}^{4}\mu_i[f(t_{n+v_i}, x(t_{n+v_i})) - f(t_{n+v_i}, \overline{x}_{n+v_i})] \\ &+ h\beta_0[f(t_{n+1}, x(t_{n+1})) - f(t_{n+1}, \overline{x}_{n+1})] \\ &+ Ch^p x^{(p)}(t_n) + O(h^{p+1}). \end{aligned}$$

Considering the properties of the IVPs in (1), some values such as η_{v_i} and η_1 belong to intervals of $(\overline{x}_{n+v_i}, x(t_{n+v_i}))$ and $(\overline{x}_{n+1}, x(t_{n+1}))$, respectively. Thus, we can write

$$\begin{aligned} f(t_{n+v_i}, x(t_{n+v_i})) - f(t_{n+v_i}, \overline{x}_{n+v_i}) &= \tfrac{\partial f}{\partial x}(t_{n+v_i}, \eta_{n+v_i})(x(t_{n+v_i}) - \overline{x}_{n+m}), \\ f(t_{n+1}, x(t_{n+1})) - f(t_{n+1}, \overline{x}_{n+1}) &= \tfrac{\partial f}{\partial x}(t_{n+1}, \eta_{n+1})(x(t_{n+1}) - \overline{x}_{n+1}). \end{aligned}$$

Therefore, using (9), we have

$$\begin{aligned} x(t_{n+1}) - x_{n+1} =& h\sum_{i=1}^{4}\mu_i\left[\tfrac{\partial f}{\partial x}(t_{n+v_i}, \eta_{n+v_i})(x(t_{n+v_i}) - \overline{x}_{n+v_i})\right] \\ &+ h\beta_0\left[\tfrac{\partial f}{\partial x}(t_{n+1}, \eta_{n+1})(x(t_{n+1}) - \overline{x}_{n+1})\right] \\ &+ Ch^p x^{(p)}(t_n) + O(h^{p+1}). \end{aligned}$$

Applying (9) to this, we have

$$\begin{aligned} x(t_{n+1}) - x_{n+1} =& h\sum_{i=1}^{4}\mu_i\left[\tfrac{\partial f}{\partial x}(t_{n+v_i}, \eta_{n+v_i})C_{v_i}h^6 x^{(6)}(t_n) + O(h^{6+1})\right] \\ &+ h\beta_0\left[\tfrac{\partial f}{\partial x}(t_{n+1}, \eta_{n+1})C_1 h^6 x^{(6)}(t_n) + O(h^{p_1+1})\right] \\ &+ Ch^p x^{(p)}(t_n) + O(h^{p+1}) \\ =& h^6\left\{\sum_{i=1}^{4}\mu_i\left[\tfrac{\partial f}{\partial y}(t_{n+v_i}, \eta_{n+v_i})C_{v_i}h^{6-p+1}x^{(p_1)}(t_n)\right]\right\} \\ &+ h^6\left\{\beta_0\left[\tfrac{\partial f}{\partial x}(t_{n+1}, \eta_{n+1})C_1 h^{6-p+1}x^{(p_1)}(t_n)\right]\right. \\ &\left.+ Cx^{(7)}(t_n)\right\} + O(h^8). \end{aligned}$$

Thus, it can be concluded that the order of (8) is 6. □

3 Stability analysis of the new methods

In this section, stability analysis is done on new methods. Dahlquist test problem is considered to investigate stability region of the methods presented in this study. Applying the Dahlquist test problem to (5) and inserting $p = 6$, the following equations can be obtained:

$$\bar{x}_{n+m} = \left(1 + m\bar{h} + \frac{(m\bar{h})^2}{2!} + \frac{(m\bar{h})^3}{3!} + \frac{(m\bar{h})^4}{4!} + \frac{(m\bar{h})^5}{5!} + \frac{(m\bar{h})^6}{6!}\right) x_n, \tag{10}$$

$$m = 1,\, v_1,\, v_2,\, v_3,\, v_4,$$

where $\bar{h} = h\lambda$ and $\mu_j \in \mathbb{R}$. Now, let us consider the new hybrid method presented in this work:

$$x_{n+1} = x_n + h\{\beta_1 f_n + \beta_0 \bar{f}_{n+1} + \mu_1 \bar{f}_{n+v_1} + \mu_2 \bar{f}_{n+v_2} + \mu_3 \bar{f}_{n+v_3} + \mu_4 \bar{f}_{n+v_4}\}. \tag{11}$$

Substituting (10) into (11), the following equation is obtained:

$$\begin{aligned} x_{n+1} =& x_n + \bar{h}\left\{\beta_1 x_n + \beta_0 x_n \left(1 + \bar{h} + \frac{(\bar{h})^2}{2!} + \frac{(\bar{h})^3}{3!} + \frac{(\bar{h})^4}{4!} + \frac{(\bar{h})^5}{5!} + \frac{(\bar{h})^6}{6!}\right)\right\} \\ &+ \bar{h}\left\{x_n \sum_{i=1}^{4} \mu_i \left(1 + (v_i\bar{h}) + \frac{(v_i\bar{h})^2}{2!} + \frac{(v_i\bar{h})^3}{3!} + \frac{(v_i\bar{h})^4}{4!}\right.\right. \\ &\left.\left. + \frac{(v_i\bar{h})^5}{5!} + \frac{(v_i\bar{h})^6}{6!}\right)\right\}, \end{aligned}$$

Inserting $x_n = r^n$ into (11) and dividing it by r^n, we can obtain

$$r^{n+1} = r^n \left\{1 + a_1\bar{h} + a_2\bar{h}^2 + a_3\bar{h}^3 + a_4\bar{h}^4 + a_5\bar{h}^5 + a_6\bar{h}^6 + a_7\bar{h}^7\right\}.$$

$$\Rightarrow r = 1 + a_1\bar{h} + a_2\bar{h}^2 + a_3\bar{h}^3 + a_4\bar{h}^4 + a_5\bar{h}^5 + a_6\bar{h}^6 + a_7\bar{h}^7,$$

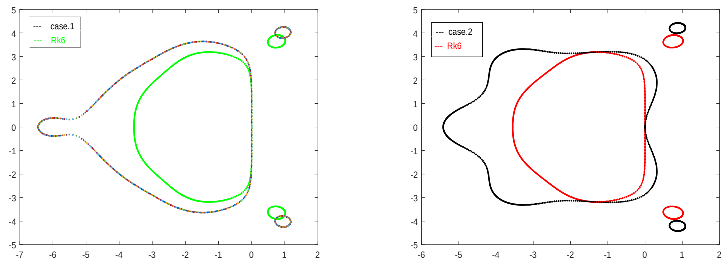
$$a_1 = \beta_1 + \beta_0 + \sum_{i=1}^{4} \mu_i, \qquad a_2 = \beta_0 + \sum_{i=1}^{4} (v_i\mu_i),$$

$$a_3 = \frac{1}{2}(\beta_0 + \sum_{i=1}^{4} (v_i^2\mu_i)), \qquad a_4 = \frac{1}{6}(\beta_0 + \sum_{i=1}^{4} (v_i^3\mu_i)),$$

$$a_5 = \frac{1}{24}(\beta_0 + \sum_{i=1}^{4} (v_i^4\mu_i)), \qquad a_6 = \frac{1}{120}(\beta_0 + \sum_{i=1}^{4} (v_i^5\mu_i)),$$

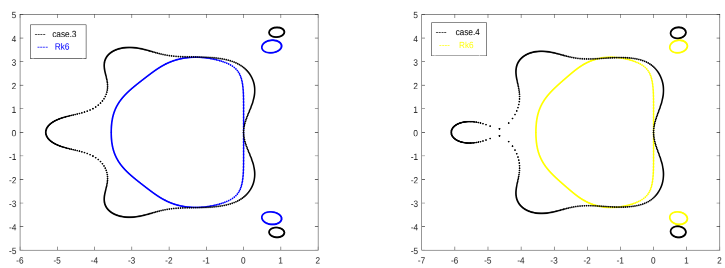
$$a_7 = \frac{1}{720}(\beta_0 + \sum_{i=1}^{4} (v_i^6\mu_i)),$$

which is a stability polynomial of (11). Figure 1 compares the stability region of methods 1 and 2 with that of the 6th-order Runge–Kutta (RK6) method (note that method i means that the new method related to the case i, $i = 1, 2, 3, 4$). As can be clearly seen, the stability region of the proposed



(a) (b)

Figure 1: (a) Stability region of the method 1 and RK6 method. (b) Stability region of the method 2 and RK6 method.

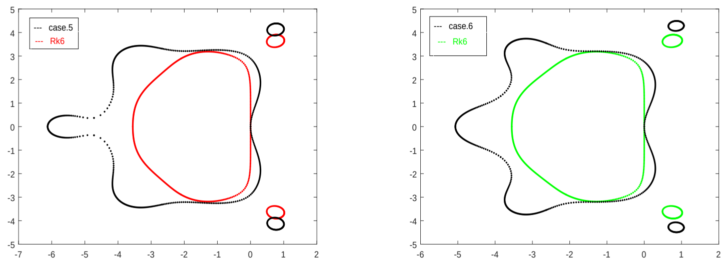


(a) (b)

Figure 2: (a) Stability region of the method 3 and RK6 method. (b) Stability region of the method 4 and RK6 method.

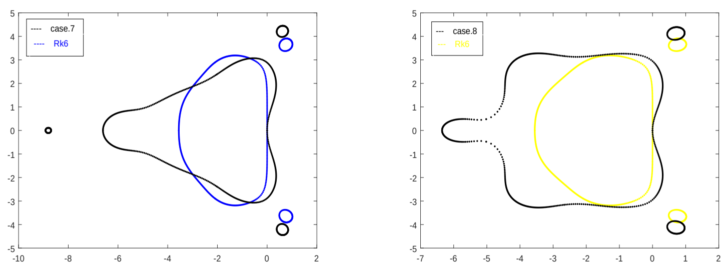
methods 1 and 2 is much wider than that of the RK6 method. Stability region of the proposed methods 3 and 4 is presented in Figure 2 and is compared with that of the RK6 method. According to Figure 2, methods 3 and 4 presented in this paper have a wider range of stability than the RK6 method, showing the efficiency of the proposed methods. Methods 5 and 6 are compared with RK6 method in terms of stability region in Figure 3. As demonstrated in Figure 3, the proposed methods 5 and 6 have a wider stability region than the RK6 method, showing that the efficiency of the proposed methods is higher than the RK6 method.

Figure 4 compares the stability region of methods 7 and 8 with that of the RK6 method. As can be seen, methods 7 and 8 have a wider range of stability than the RK6 method. Figure 5 also shows the superiority of the methods proposed in this paper over the RK6 method. As depicted in Figure 5, the width of stability region of methods 9 and 10 is higher compared to the RK6 method.



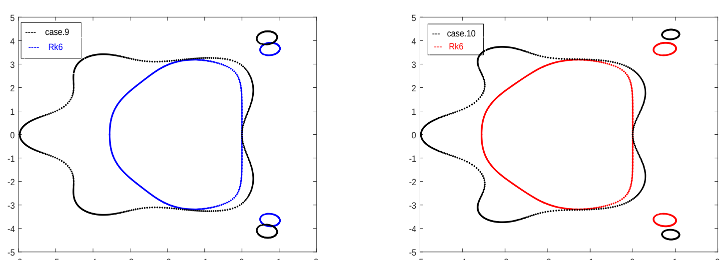
(a) (b)

Figure 3: (a) Stability region of the method 5 and RK6 method. (b) Stability region of the method 6 and RK6 method.



(a) (b)

Figure 4: (a) Stability region of the method 7 and RK6 method. (b) Stability region of the method 8 and RK6 method.



(a) (b)

Figure 5: (a) Stability region of the method 9 and RK6 method. (b) Stability region of the method 10 and RK6 method.

4 Optimal control problems

An OCP includes a cost function $J(x,u)$, a set of state variables, $x \in X$, and a set of control variables, $u \in U$. An OCP is solved to find a piecewise continuous control $u(t)$, $t_0 \leq t \leq t_f$, and the associated continuous state variable $x(t)$, in order to minimize the given objective function. For more explanation, we need a few definitions.

Definition 1 (Lagrange and Bolza problems)**.** The basic problem in Lagrange form is

$$J(x,u) = \int_{t_0}^{t_f} g(t, x(t), u(t))dt. \tag{12}$$

Adding another term to functional (12), the Bolza problem is obtained:

$$J(x,u) = h(t_f, x(t_f)) + \int_{t_0}^{t_f} g(t, x(t), u(t))dt.$$

An OCP is

$$\max_u J = \int_{t_0}^{t_f} g(t, x(t), u(t))dt, \tag{13}$$

$$x'(t) = f(t, x(t), u(t)),$$
$$x(t_0) = x_0.$$

Note that $\min\{J\} = -\max\{-J\}$; see [19].

Definition 2 (Hamiltonian)**.** Consider the OCP (13). The function $H(t,x,u,\lambda)$ is called as the Hamiltonian function and is equal to

$$H(t,x,u,\lambda) = g(t,x,u) + \lambda f(t,x,u)$$

where λ is an adjoint variable.

Theorem 2 (Pontryagin Maximum Principle)**.** Consider the OCP (13). Suppose that $g(t,x,u)$ and $f(t,x,u)$ are both continuously differentiable functions in their three arguments and concave in x and u. If u^* is a control with associated state x^* and λ is a piecewise differentiable function such that u^*, x^*, and λ together are satisfied

$$g_u + \lambda f_u = 0 \Leftrightarrow \frac{\partial H}{\partial u} = 0,$$

$$\lambda' = -(g_x + \lambda f_x) \Leftrightarrow \lambda' = -\frac{\partial H}{\partial x},$$

$$\lambda(t_f) = 0,$$

$$\lambda(t) \geq 0,$$

on $t_0 \leq t \leq t_f$, then
$$J(x^*, u^*) \geq J(x, u),$$
for any admissible pair (x, u).

Proof. We refer readers to [13]. □

Theorem 3. [13]. Let the set of controls for problem (13), be Lebesgue integrable functions and let $t_0 \leq t \leq t_f$ in $\mathbb{R}$. Suppose that $f(t, x, u)$ is concave in u and there exist constants $c_1, c_2, c_3 > 0, c_4$ and $\beta > 1$, such that

$$\begin{aligned} f(t, x, u) &= \alpha(t, x) + \beta(t, x)u, \\ |f(t, x, u)| &\leq c_1(1 + |x| + |u|), \\ |f(t, x_1, u) - f(t, x, u)| &\leq c_2|x_1 - x|(1 + |u|), \\ g(t, x, u) &\leq c_3|u|^\beta - c_4, \end{aligned}$$

for all t with $t_0 \leq t \leq t_f$, $x_1, x_2, u \in \mathbb{R}$.
Then there exists an optimal pair (x^*, u^*) maximizing J, with finite $J(x^*, u^*)$.

5 Forward-backward sweep method

For numerically solving the OCP (13) using the indirect method, an algorithm is introduced according to the literature [13]. For solving such problems numerically first, an algorithm that generates an approximation to an optimal piecewise continuous control u^*, must divide the time interval of $[t_0, t_f]$ into pieces with specific points of interest $t_0 = b_1, b_2, \ldots, b_N, b_{N+1} = t_f$; and these points will usually be equally spaced. Approximation will be a vector $\overrightarrow{u} = (u_1, u_2, \ldots, u_{N+1})$, where $u_i \approx u(b_i)$. Any solution to the above OCP must also be satisfied:

$$\begin{aligned} x'(t) = f(t, x(t), u(t)), \quad x(t_0) &= x_0, \\ \lambda' = -\frac{\partial H}{\partial x}, \quad \lambda(t_f) &= 0, \\ \frac{\partial H}{\partial u} = 0 \quad at \quad u^*&. \end{aligned}$$

The third equation, namely, optimality condition, can usually be manipulated to find a representation of u^* in terms of t, x, and λ. Then, the first two equations form a TPBVP. The generalized problem can be solved using indirect methods, which are numerical techniques used for solving. The FBSM is one of these methods. A rough outline of the algorithm is given below.

Here, $\overrightarrow{x} = (x_1, x_2, \ldots, x_{N+1})$ and $\overrightarrow{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{N+1})$ are the vector approximations for the state and adjoint.

1. Make an initial guess for $\overrightarrow{u}$ over the interval.
2. Using the initial condition $x_1 = x(t_0) = a$ and the value for $\overrightarrow{u}$, solve $\overrightarrow{x}$ forward in time according to its differential equation in the optimality system.
3. Using the transversality condition $\lambda_{N+1} = \lambda(t_f) = 0$ and the values for $\overrightarrow{u}$ and $\overrightarrow{x}$, solve $\overrightarrow{\lambda}$ backward in time according to its differential equation in optimality system.
4. Update $\overrightarrow{u}$ by entering the new $\overrightarrow{x}$ and $\overrightarrow{\lambda}$ values into the characterization of the optimal control.
5. Check convergence. If values of variables in this iteration and the last iteration are negligibly close, then the current values are considered as output solutions. If values are not close, then return to Step 2.

6 Convergence of FBSM

For notational simplicity, we express the problem as finding $(x(t), \lambda(t), u(t))$ such that

$$\begin{aligned}
&x'(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0,\\
&\lambda'(t) = k_1(t, x(t), u(t)) + \lambda(t)k_2(t, x(t), u(t)), \quad \lambda(t_f) = 0,\\
&u(t) = k_3(t, x(t), u(t)).
\end{aligned}$$

Here, $x_0 \in \mathbb{R}^n$ and $t_0 < t_f$ are the given real numbers. For a convergence analysis of the FBSM, we will make the following assumptions:
(T) The functions of $f, k_1,\ k_2,$ and k_3 are Lipschitz continuous with respect to their second and third arguments, with Lipschitz constants of $L_f, L_{k_1}, L_{k_2}, L_{k_3}$. Moreover, $\quad \Lambda = \|\lambda\|_\infty$ and $H = \|k_2\|_\infty < \infty$.

Theorem 4. Under the assumptions (T), if

$$\begin{aligned}
c_0 \equiv & L_{k_3}\left\{[\exp(L_f(t_f - t_0)) - 1]\right\} + L_{k_3}\Big\{(L_{k_1} + \Lambda L_{k_2})\\
&\frac{1}{H}[\exp(H(t_f - t_0)) - 1][\exp(L_f(t_f - t_0)) + 1]\Big\} < 1,
\end{aligned}$$

then the FBSM is convergent, that is, as $n \to \infty$,

$$\max_{t_0 \le t \le t_f} |x(t) - x^{(n)}(t)| + \max_{t_0 \le t \le t_f} |\lambda(t) - \lambda^{(n)}(t)| + \max_{t_0 \le t \le t_f} |u(t) - u^{(n)}(t)| \to 0.$$

Proof. We refer the reader to [16]. □

7 Numerical results

In this section, examples of various types of OCPs are solved using the proposed methods, and their numerical results are compared with those of the FBSM using the RK method of order 6 (FBSM−RK6). One of the most important OCPs is the linear regulator problem, which is generally defined as follows.

Example 1. Let E, $Q(t)$, and $R(t)$ be symmetric and nonnegative definite matrices of appropriate dimensions. The so-called linear regulator problem (with a linear state-space description) involves a cost functional of the form

$$F(u) = \frac{1}{2}x^T(t_f)Ex(t_f) + \frac{1}{2}\int_{t_0}^{t_f}[x^T Q(t)x(t) + u^T(t)R(t)u(t)]dt.$$

For example, we consider an OCP as follows [16]:

$$\begin{aligned}\min_u \frac{1}{2}\int_0^1 [x(t)^2 + u(t)^2]dt\\ s.t.\ x'(t) = -x(t) + u(t),\ \ x(0) = 1.\end{aligned}$$

The Pontryagin's maximum principle can be used to construct an analytic solution

$$\begin{aligned}H(t,x,u,\lambda) &= \frac{1}{2}(x(t)^2 + u(t)^2) + \lambda(-x(t) + u(t)),\\ \frac{\partial H}{\partial u} &= 0 \quad at \quad u^* \quad \Rightarrow u^* + \lambda = 0 \Rightarrow u^* = -\lambda,\\ \lambda' &= -\frac{\partial H}{\partial x} = -x + \lambda, \quad \lambda(1) = 0.\end{aligned}$$

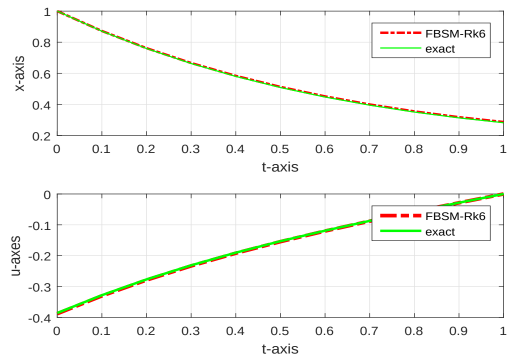
Together with the state equation, the result will be the following linear differential algebraic system:

$$\begin{aligned}x^{'}(t) =& -x(t) + u(t), \quad x(0) = 1,\\ \lambda^{'}(t) =& \lambda(t) - x(t), \quad \lambda(1) = 0, \quad u(t) = -\lambda(t).\end{aligned}$$

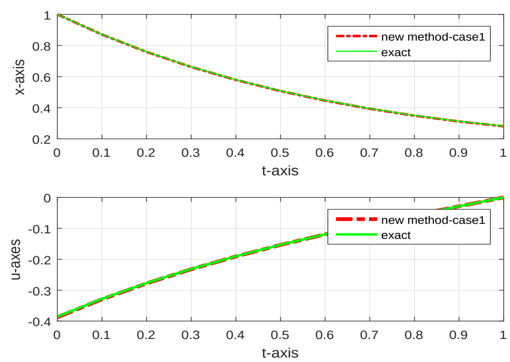
The solution is

$$\begin{aligned}x^*(t) =& \frac{\sqrt{2}\cosh(\sqrt{2}(t-1)) - \sinh(\sqrt{2}(t-1))}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})},\\ u^*(t) =& \frac{\sinh(\sqrt{2}(t-1))}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})}.\end{aligned}$$

Optimal value of the objective functional is $J = 0.1929092981$. The final values of the state and optimal are $x(1) = 0.2819695346$ and 0, respectively,



(a)



(b)

Figure 6: (a) Optimal state and control values of Example 1 FBSM−RK6. (b) Optimal state and control values of Example 1 (new proposed method)

and the initial value of the co-state is $\lambda(0) = 0.3858185962$. Numerical results of the problem are shown in Figure 6 with $h = \frac{1}{10}$ and in Tables 3 and 4.

Numerical results presented in Tables 3 and 4 indicate that each of new ten suggested methods calculates the amount of control variable values much more accurately than the FBSM−RK6 method. Figure 6 also indicates that the new suggested method is exactly based on figure of analytical answer. For avoiding overstatement in this paper, one of the diagrams was selected and drawn. The rest of figures are similar to each other. Approximate performance index is calculated for the proposed method, and the results are presented in Tables 5 and 6. According to the results, the precision of performance index of new proposed methods is more than that of the FBSM−RK6 method by two digits. The Pontryagin's Theorem is also used to solve linear-quadratic problems.

Table 3: Error of control values in Example 1 for FBSM−RK6 and new proposed Methods

t	h	FBSM_RK6	case 1	case 2	case 3	case 4	case 5
0.90	$\frac{1}{10}$	3.0520e-4	7.3292e-5	2.5423e-5	7.2638e-5	7.2172e-5	1.4836e-5
0.90	$\frac{1}{50}$	5.3457e-5	2.1141e-5	1.5067e-5	2.1006e-5	2.0916e-5	4.1655e-6
0.90	$\frac{1}{100}$	2.6278e-5	1.0962e-5	1.4606e-6	1.0895e-5	1.0850e-5	2.5154e-6
0.90	$\frac{1}{200}$	1.3036e-5	5.5876e-7	8.4673e-7	5.5537e-6	5.5314e-6	1.3740e-6

Table 4: Error of control values in Example 1 for FBSM−RK6 and new proposed methods

t	h	FBSM_RK6	case 6	case 7	case 8	case 9	case 10
0.90	$\frac{1}{10}$	3.0520e-4	6.7372e-5	1.1476e-4	3.9201e-5	2.9350e-5	7.1123e-5
0.90	$\frac{1}{50}$	5.3457e-5	1.9984e-5	1.5321e-5	5.6249e-7	1.3363e-6	2.0701e-5
0.90	$\frac{1}{100}$	2.6278e-5	1.0386e-5	7.1974e-6	1.6043e-7	1.1054e-6	1.0742e-5
0.90	$\frac{1}{200}$	1.3036e-5	5.2998e-6	3.4747e-6	1.9879e-7	6.7022e-7	5.4774e-6

Table 5: Errors of the performance index approximation in Example1

h	FBSM_RK6	case 1	case 2	case 3	case 4	case 5
$\frac{1}{50}$	4.0877e-4	2.8316e-5	4.7743e-5	2.7735e-5	2.7293e-5	4.3302e-5
$\frac{1}{100}$	1.9310e-4	6.8092e-6	3.0969e-5	6.5235e-6	6.3012e-6	2.8836e-5
$\frac{1}{200}$	9.2656e-5	4.877e-7	1.8338e-5	3.4612e-7	2.3451e-7	1.7294e-5
$\frac{1}{1000}$	1.8014e-5	2.6262e-7	4.0178e-6	2.9075e-7	3.1314e-7	3.8123e-6

Table 6: Errors of the performance index approximation in Example 1

h	FBSM_RK6	case 6	case 7	case 8	case 9	case 10
$\frac{1}{50}$	4.0877e-4	2.3141e-5	1.0728e-4	6.1400e-5	5.4438e-5	2.6399e-5
$\frac{1}{100}$	1.9310e-4	4.2415e-6	6.0399e-5	3.7814e-5	3.4359e-5	5.8643e-6
$\frac{1}{200}$	9.2656e-5	7.9113e-7	3.2969e-5	2.1765e-5	2.0043e-5	1.8678e-8
$\frac{1}{1000}$	1.8014e-5	5.1760e-7	6.9304e-6	4.7036e-6	4.3604e-6	3.5590e-7

Table 7: Control values errors in Example 2 by using FBSM−RK6 and new proposed methods

t	h	FBSM−RK6	case 1	case 2	case 3	case 4	case 5
0.70	$\frac{1}{10}$	8.2317e-3	1.3560e-4	3.7467e-4	1.3073e-4	1.3148e-4	3.1883e-4
0.70	$\frac{1}{50}$	8.1160e-4	3.7614e-5	6.5252e-5	3.6716e-5	3.6772e-5	5.6347e-5
0.70	$\frac{1}{100}$	4.0487e-4	1.9678e-5	3.1791e-5	1.9236e-5	1.9258e-5	2.7488e-5
0.70	$\frac{1}{200}$	1.0203e-4	1.0221e-5	1.5522e-5	1.0001e-5	1.0010e-5	1.3409-5

Example 2. Consider the following OCP [18]:

$$\begin{array}{l} \min_u \int_0^1 \frac{5}{8}x(t)^2 + \frac{1}{2}x(t)u(t) + \frac{1}{2}u(t)^2 dt \\ st.\ x'(t) = \frac{1}{2}x(t) + u(t),\ \ x(0) = 1. \end{array}$$

For solving the above example, using the FBSM and proposed methods, we should apply the Pontryagin's Theorem as follows:

$$\begin{aligned} H(t,x,u,\lambda) =& \frac{5}{8}x(t)^2 + \frac{1}{2}x(t)u(t) + \frac{1}{2}u(t)^2 + \lambda(\frac{1}{2}x(t) + u(t)), \\ \frac{\partial H}{\partial u} =& 0 \quad at \quad u^* \quad \Rightarrow u^* = -\lambda - \frac{1}{2}x, \\ \lambda' =& -\frac{\partial H}{\partial x} = -\frac{10}{8}x - \frac{1}{2}u - \frac{1}{2}\lambda, \quad \lambda(1) = 0. \end{aligned}$$

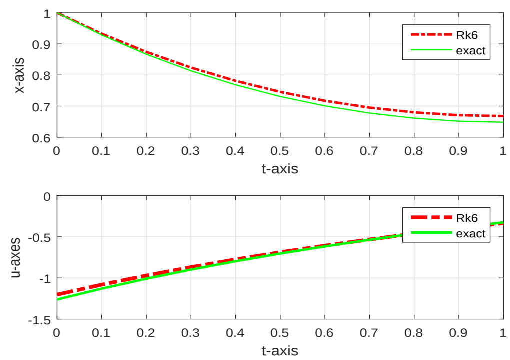
Analytical solutions are as follows [18]:

$$\begin{aligned} u^*(t) =& -\frac{(\tanh(1-t) + .5)\cosh(1-t)}{\cosh(1)}, \\ x^*(t) =& \frac{\cosh(1-t)}{\cosh(1)}. \end{aligned}$$

The state variable at the end point is $x(1) = 6.48054273663 88e-1$. Variable control endpoint is $u(1) = -3.24027136831e - 1$. Optimal value of the objective function is $J^* = 0.3807970779$. The proposed methods of Example 2 were determined as follows in MATLAB environment:

Table 8: Control values errors in Example 2 by using FBSM−RK6 and new proposed methods

t	h	FBSM−RK6	case 6	case 7	case 8	case 9	case 10
0.70	$\frac{1}{10}$	8.2317e-3	9.8758e-5	8.1761e-4	4.4379e-4	3.9543e-4	1.2035e-4
0.70	$\frac{1}{50}$	8.1160e-4	3.0223e-5	5.6542e-5	8.1127e-5	8.5984e-5	3.4750e-5
0.70	$\frac{1}{100}$	4.0487e-4	1.5985e-5	7.3752e-5	3.9860e-5	3.5044e-5	1.8260e-5
0.70	$\frac{1}{200}$	2.0203e-4	8.3745e-6	3.6430e-5	1.9590e-5	1.7183e-5	9.5155e-6



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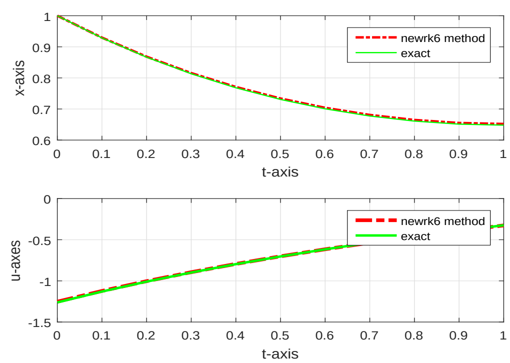


(b)

Figure 7: (a) Optimal state and control values of Example 2 by using FBSM−RK6. (b) Optimal state and control values of Example 2 by using new proposed method

Table 9: Errors of the performance index approximation in Example 2

h	FBSM−RK6	case 1	case 2	case 3	case 4	case 5
$\frac{1}{50}$	5.5920e-4	7.6989e-5	2.1442e-5	7.6139e-5	7.6159e-5	1.2794e-5
$\frac{1}{100}$	2.9187e-4	2.9588e-5	1.9712e-5	2.9168e-5	2.9173e-5	1.5522e-5
$\frac{1}{200}$	1.5038e-4	1.1235e-5	1.3437e-5	1.1026e-5	1.1027e-5	1.1376e-5
$\frac{1}{1000}$	3.0605e-5	1.8661e-6	3.0719e-6	1.8246e-6	1.8246e-6	2.6652e-6

Numerical results presented in Tables 7 and 8 indicate that each of the new ten suggested methods calculates the amount of control variable values much more accurately than the FBSM−RK6 method. Figure 7 also indicates that the new proposed method is exactly based on figure of analytical answer. For simplicity of reporting the results, one of diagrams was selected and drawn. The rest of figures are similar to each other. Numerical results presented in Tables 9 and 10 indicate that the estimated performance index of the new methods is more precise than those of the FBSM−RK6 methods.

Example 3. Consider the following OCP for a fixed T [14]:

$$\begin{aligned} &\min_u \int_0^T \left(\int_0^t x(\eta)d\eta + (u(t))^2\right)dt \\ s.t. \quad &x'(t) = -x(t) + u(t), \quad x(0) = a. \end{aligned}$$

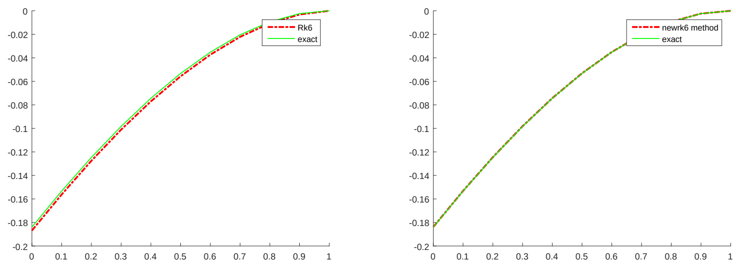
For converting the problem into the standard form, we can add another state and obtain two-dimensional system as follows:

$$\begin{aligned} x_1(t) &= x(t), \\ x_2(t) &= \int_0^t x(\eta)d\eta, \end{aligned}$$

We redefine $x(t) := [x_1(t), x_2(t)]^T$. Thus, we have an OCP of the form:

Table 10: Errors of the performance index approximation in Example 2

h	FBSM−RK6	case 6	case 7	case 8	case 9	case 10
$\frac{1}{50}$	5.5920e-4	6.9979e-5	1.027e5-4	3.6537e-5	2.7287e-5	7.4275e-5
$\frac{1}{100}$	2.9187e-4	2.6081e-5	6.0153e-5	2.7391e-5	2.2764e-5	2.8242e-5
$\frac{1}{200}$	1.5038e-4	9.4814e-6	3.3603e-5	1.73105e-5	1.4995e-5	1.0565e-5
$\frac{1}{1000}$	3.0605e-5	1.5153e-6	7.0964e-6	3.8519e-6	3.3889e-6	1.7326e-6



(a) (b)

Figure 8: (a) Optimal state and control values of Example 3 using FBSM−RK6. (b) Optimal state and control values of Example 3 using new proposed method

$$\begin{aligned} &\min_u \int_0^T (x_2(t) + u(t)^2)dt \\ s.t., \quad & x_1'(t) = -x(t) + u(t), \\ & x_2'(t) = x_1(t), \\ & x(0) = [a, 0]^T. \end{aligned}$$

Analytical solution of the problem is

$$\begin{aligned} H =&(x_2 + u^2) + \lambda_1(-x_1 + u) + \lambda_2 x_1, \\ \frac{\partial H}{\partial u} =&2u + \lambda_1 = 0 \ at \ u^* \Rightarrow u^* = -\frac{1}{2}\lambda_1, \ \lambda_1(T) = \lambda_2(T) = 0, \\ \Rightarrow&\lambda_1' = -\frac{\partial H}{\partial x_1} = \lambda_1 - \lambda_2, \ \lambda_2' = -\frac{\partial H}{\partial x_2} = -1, \\ \Rightarrow&\lambda_1(t) = -(t - T) - 1 + e^{(t-T)}, \\ u^*(t) =& -\frac{1}{2}\lambda_1(t) = \frac{1}{2}(1 + t - T - e^{(t-T)}). \end{aligned}$$

Numerical results for Example 3 are obtained and shown in Figure 8 and Tables 11 and 12.

Table 11: Control values errors in Example 3 using FBSM−RK6 and new proposed method

t	h	FBSM−RK6	case 1	case 2	case 3	case 4	case 5
0.80	$\frac{1}{10}$	1.3027e-3	4.3609e-5	1.9320e-4	4.5033e-5	4.5546e-5	1.7794e-4
0.80	$\frac{1}{50}$	2.4376e-4	1.6351e-6	2.9366e-5	1.8584e-6	1.9768e-6	2.7439e-5
0.80	$\frac{1}{100}$	1.2086e-4	4.1195e-7	1.4147e-5	2.2016e-7	5.8037e-7	1.3244e-5
0.80	$\frac{1}{200}$	6.0174e-5	1.0924e-7	6.9448e-6	1.6251e-7	1.9287e-7	6.5116e-6

Table 12: Control values errors in Example 3 by using FBSM−RK6 and new proposed methods

t	h	FBSM−RK6	case 6	case 7	case 8	case 9	case 10
0.80	$\frac{1}{10}$	1.3027e-3	5.4085e-5	3.2411e-4	2.1473e-4	2.0072e-4	4.8041e-5
0.80	$\frac{1}{50}$	2.4376e-4	3.5454e-6	5.1626e-5	3.4084e-5	3.1524e-5	2.3610e-6
0.80	$\frac{1}{100}$	1.2086e-4	1.3566e-6	2.5050e-5	1.6529e-5	1.5263e-5	7.6572e-7
0.80	$\frac{1}{200}$	6.0174e-5	5.7899e-7	1.2340e-5	8.1415e-6	7.5122e-6	2.8398e-7

Numerical results presented in Tables 11 and 12 indicate that each of the new ten suggested methods calculates the amount of control variable values much more accurately than the FBSM−RK6 method. Figure 8 also indicates that the figure of the new proposed method is quite matched on real answer and is much better than the FBSM−RK6 method. For simplicity of reporting the results, one of diagrams was selected and drawn. The rest of figures are similar to each other.

8 Conclusion

A new class of the 6th-order explicit hybrid methods was presented for which the 6th-order Runge–Kutta method is used as a predictor scheme to gain whole method of the same order, and the order of truncation errors was investigated for the explicit hybrid Runge–Kutta methods. The stability of the methods was discussed, and the results revealed that the stability regions of the proposed methods are wider compared to the 6th-order explicit Runge–Kutta method. Finally, three examples of OCPs were solved using MATLAB, FBSM scheme, and the presented methods and numerical results related to given examples were presented in Tables 3–12. According to the findings, it can be concluded that the new explicit hybrid methods have a good performance in accuracy and performance index approximation compared to the RK6 method.

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