Iranian Journal of Numerical Analysis and Optimization
Vol. 14, No. 3, 2024, pp 875–899
https://doi.org/10.22067/ijnao.2024.86362.1375
https://ijnao.um.ac.ir/

Research Article

A pseudo−operational collocation method for optimal control problems of fractal−fractional nonlinear Ginzburg−Landau equation

T. Shojaeizadeh*, E. Golpar-Raboky* and Parisa Rahimkhani

Abstract

The presented work introduces a new class of nonlinear optimal control problems in two dimensions whose constraints are nonlinear Ginzburg−Landau equations with fractal−fractional (FF) derivatives. To acquire their approximate solutions, a computational strategy is expressed using the FF derivative in the Atangana−Riemann−Liouville (A-R-L) concept with the

*Corresponding author

Received 12 January 2024; revised 12 April 2024; accepted 13 May 2024

T. Shojaeizadeh
Department of Mathematics, Qom Branch, Islamic Azad University, Qom, Iran. e-mail: Ta.Shojaeizadeh@iau.ac.ir

E. Golpar-Raboky
Department of Mathematics, University of Qom, Qom, Iran. e-mail: g.raboky@qom.ac.ir

Parisa Rahimkhani
Faculty of Science, Mahallat Institute of Higher Education, Mahallat, Iran. e-mail: rahimkhani.parisa@mahallat.ac.ir

How to cite this article
Shojaeizadeh, T., Golpar-Rabok, E. and Rahimkhani, P., A pseudo−operational collocation method for optimal control problems of fractal−fractional nonlinear Ginzburg−Landau equation. *Iran. J. Numer. Anal. Optim.*, 2024; 14(3): 875-899. https://doi.org/10.22067/ijnao.2024.86362.1375

Mittage-Leffler kernel. The mentioned scheme utilizes the shifted Jacobi polynomials (SJPs) and their operational matrices of fractional and FF derivatives. A method based on the derivative operational matrices of SJR and collocation scheme is suggested and employed to reduce the problem into solving a system of algebraic equations. We approximate state and control functions of the variables derived from SJPs with unknown coefficients into the objective function, the dynamic system, and the initial and Dirichlet boundary conditions. The effectiveness and efficiency of the suggested approach are investigated through the different types of test problems.

AMS subject classifications (2020): Primary 35R30; Secondary 65M32, 35K20.

Keywords: Fractal−fractional (FF) derivative; Shifted Jacobi polynomials (SJPs); Operational matrices; Nonlinear Ginzburg−Landau equation; Optimal control problem.

1 Introduction

The Ginzburg−Landau equation is one of the most studied nonlinear partial differential equations in physics and engineering. This equation describes diverse types of phenomena, including superconductivity, Bose-Einstein, superfluidity, nonlinear waves, second-order phase transitions, condensation, liquid crystals, and strings in field theory [1]. There are many numerical and analytical schemes for solving this equation, for instance, see [10, 12, 18, 20, 25, 31].

Atangana [2] introduced the idea of FF derivation. The FF derivatives have been found very useful in many science and engineering applications. Since the fractals can be realized in nature as a fractal process or fractal media, it is interesting to derive the fractal or FF equations. The fractional partial differential equations appear in chaotic dynamics [32], long-range dissipation [22], and material science [26]. Fractional integrals and derivatives are a robust framework that can be applied to describe processes with various levels of complexity [11].

The fractional generalization of the Ginzburg–Landau equation was introduced in [30]. This equation can be used to describe the dynamical processes in a medium with fractal dispersion and capture some long-range interactions of a system that can not be captured by traditional integer order differential equations. It is well has been evaluated from different aspects of this equation [29, 19, 28, 33]. Recently, Ding et al. studied higher-order numerical algorithm for the two-dimensional nonlinear spatial fractional complex Ginzburg−Landau equation [6].

Orthogonal polynomials have been extensively employed in solving optimal control problems involving fractional partial differential equations [4, 5, 9, 23, 27].

In [15] a numerical method for solving the model of the nonlinear Ginzburg−Landau equation in a FF sense is presented.

Regarding numerical methods for the FF equations, the critical step is the approximation of the fractional or FF derivatives.

Although, some approximate schemes for solving the FF model of nonlinear Ginzburg−Landau equation have been presented, for the first time we propose a scheme for solving the optimal control problem of FF nonlinear Ginzburg−Landau equation. The method uses SJPs for its numerical solution.

Using the FF derivative in the A-R-L concept and fractional derivatives in Caputo and Atangana-Baleanu-Caputo sense, optimal control of FF advection-diffusion-reaction equations is provided. These classes of problems are solved an operational matrix with high accuracy. Here , we consider the following optimal control problem:

$$(P)\min_{\mathbf{u}\in\mathbf{U}_{ad}}\ \mathcal{J}(\mathbf{y},\mathbf{u}) := \|\mathbf{y}(s,t)-\hat{\mathbf{y}}(s,t)\ \|^2_{L^2_{\omega^{(e,f)}}(\Omega)} + \epsilon^2\|\mathbf{u}(s,t)-\hat{\mathbf{u}}(s,t)\ \|^2_{L^2_{\omega^{(e,f)}}(\Omega)}, \tag{1}$$

with a nonlinear FF dynamic equation

$$\begin{aligned} &{}^{FFM}_{\ \ 0}\mathcal{D}^{\alpha,\beta}\mathbf{y}(s,t) - (r_1+i\mu_1)\mathbf{y}_{ss}(s,t) + (r_2+i\mu_2)|\mathbf{y}(s,t)|^2\mathbf{y}(s,t) \\ &- (r(s)+i\mu(s))\mathbf{y}(s,t) = f(s,t) + \mathbf{u}(s,t), \end{aligned} \tag{2}$$

on the domain $(s,t)\in\Omega$ with the initial condition

$$\mathbf{y}(s,0) = v(s), \tag{3}$$

and the boundary conditions

$$\mathbf{y}(0,t)=k(s), \quad \mathbf{y}(1,t)=g(s), \tag{4}$$

where, $\Omega := [0,1]\times[0,1]$. In the above relations, the state variables $\mathbf{y}(s,t)$ and the control variables $\mathbf{u}(s,t)$ are undetermined complex functions $\hat{\mathbf{y}}(s,t)$, $\hat{\mathbf{u}}(s,t)$, v, k and g are complex determined functions, r_1, r_2, μ_1 and, μ_2 are known constants and $r(s)$ and, $\mu(s)$ are real functions; in addition, ϵ in the transition process is the weight of the control action, and $\mathbf{U}_{ad} = \{\mathbf{u} \in L_2(\Omega) : u_1 \le \mathbf{u} \le u_2, u_1, u_2 \in \mathbb{R} \cup \pm\infty\}$ has determined the collection of admissible controls. Here, ${}^{FFM}_{0}\mathcal{D}^{\alpha,\beta}$ denotes the FF derivative operator of order $(\alpha,\beta)\in(0,1)$ in the A-R-L sense with Mittag–Leffler non-singular kernel [2, 3].

In the presented plan, we solve it by converting the main problem into a system of algebraic equations. For this aim, the functions y and u are approximated by SJPs with unknown coefficients. By substituting these approximations into the objective function, a nonlinear algebraic equation with unknown coefficients is derived. By substituting the mentioned approximations in the dynamic system and the initial and boundary conditions and utilizing the FF derivative operational matrix of SJPs, we derive a system of nonlinear algebraic equations. Finally, by using Lagrangian multipliers, we connect the algebraic equations obtained from the nonlinear FF dynamic equation and the initial and boundary conditions with the algebraic equation created by the objective function, and the optimal solution is achieved using the constrained extremum method.

2 Fractal-Fractional calculus

Here, we describe the definitions and basic features of FF calculus in the Atangana-Riemann-Liouville- Caputo sense.

Definition 1. [13]. The two-parameter Mittag-Lefler function is defined as follows:

$$E_{\zeta,\eta}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k\zeta+\eta)}, \tag{5}$$

where $\zeta, \eta \in \mathbb{R}^+$, and $t \in \mathbb{R}$. Please remember that for $\eta = 1$ it is considered as $E_\zeta(t) = E_{\zeta,1}(t)$.

Definition 2. [2, 3]. The FF derivative of the continuous function $z(s,t)$ of order (α, β) in the A-R-L sense with Mittag-Leffler kernel is defined by

$$^{FFM}\quad {}_0\mathcal{D}_t^{\alpha,\beta} z(s,t) = \frac{c(\alpha)}{1-\alpha}\frac{\partial}{\partial t^\beta}\int_0^t z(s,\tau) E_\alpha(\frac{-\alpha(t-\tau)^\alpha}{1-\alpha}) d\tau, \tag{6}$$

where $(\alpha, \beta) \in (0,1)$, $c(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$ and

$$\frac{\partial z(s,t)}{\partial t^\beta} = \lim_{\Delta t \to 0} \frac{z(s,t+\Delta t) - z(s,t)}{(t+\Delta t)^\beta - t^\beta}. \tag{7}$$

Remark 1. The aforementioned definition can be expressed as follows:

$$^{FFM}\quad {}_0\mathcal{D}_t^{\alpha,\beta} z(s,t) = \frac{c(\alpha) t^{1-\beta}}{\beta(1-\alpha)}\frac{\partial}{\partial t}\int_0^t z(s,\tau) E_\alpha(\frac{-\alpha(t-\tau)^\alpha}{1-\alpha}) d\tau. \tag{8}$$

Corollary 1. [14]. Let $\alpha, \beta \in (0,1)$ and $r \in \mathbb{N} \bigcup \{0\}$. Then, we have

$$^{FFM}\quad {}_0\mathcal{D}_t^{\alpha,\beta} t^r = \frac{c(\alpha) r! t^{r-\beta+1}}{\beta(1-\alpha)} E_{\alpha,r+1}(\frac{-\alpha t^\alpha}{1-\alpha}). \tag{9}$$

3 The shifted Jacobi polynomials and their properties

The well-known SJPs on $[0,1]$ can be defined by the following explicit analytic formula: [4, 16]

$$b_i^{(e,g)}(t) = \sum_{k=0}^{i} \pi_k^{(i)} t^k, \tag{10}$$

where

$$\pi_k^{(i)} = (-1)^{i-k} \binom{i+e+f+k}{k} \binom{i+f}{i-k}, \tag{11}$$

$i \in \mathbb{N} \cup \{0\}$, $e, f > -1, e+f \neq -1$. Concerning the weight function $\omega^{e,f}(t) = t^f(1-t)^e$ on $[0,1]$ for SJPs, the orthogonality condition is Demonstrated by

$$\int_0^1 b_i^{(e,f)}(t) b_j^{(e,f)}(t) \omega^{(e,f)}(t) dt = \lambda_i \delta_{ij}, \tag{12}$$

that δ_{ij} is Kronecker's delta function and

$$\lambda_i = \frac{\Gamma(i+e+1)\Gamma(i+f+1)}{(2i+e+f+1)\Gamma(i+e+f+1)\Gamma(i+1)}.$$

Any assumed function $y \in L^2_{\omega^{(e,f)}}[0,1]$ in $(n+1)$ terms of SJPs can be written as follows

$$y(t) \simeq \sum_{i=0}^{n} y_i b_i^{(e,f)}(t) \triangleq Y^T \Phi_n(t), \tag{13}$$

where

$$Y = [y_0,\ y_1,\ \ldots,\ y_n]^T,$$
$$\Phi_n(t) \triangleq \left[b_0^{(e,f)}(t),\ b_1^{(e,f)}(t),\ \ldots,\ b_n^{(e,f)}(t)\right]^T, \tag{14}$$

and

$$y_i = \frac{1}{\lambda_i}\int_0^1 y(t) b_i^{(e,f)}(t)\omega^{(e,f)}(t)dt, \quad i = 0, 1, \cdots, n.$$

In the same way, a bivariate function $y(s,t) \in L^2_{\omega^{(e,f)}}(\Omega)$ can be expanded by the SJPs as

$$y(s,t) \simeq \sum_{i=0}^{m}\sum_{j=0}^{n} y_{ij} b_i^{(e,f)}(s) b_j^{(e,f)}(t) \triangleq \Phi_m^T(s) Y \Phi_n(t), \tag{15}$$

where the entries of the unknown matrix $Y = [y_{ij}]$ (coefficients matrix of $(m+1)\times(n+1)$ dimensional) are obtained from the following equation

$$y_{ij} = \frac{1}{\lambda_i \lambda_j}\int_0^1\int_0^1 y(s,t) b_i^{(e,f)}(s) b_j^{(e,f)}(t)\omega^{(e,f)}(s)\omega^{(e,f)}(t)dxdt, \tag{16}$$

for $i = 0, 1, \cdots, m, \quad j = 0, 1, \cdots, n$. The first-order derivative of the vector $\Phi_n(t)$ can be expressed by [8, 7, 21]

$$\frac{d\Phi_n(t)}{dt} = \mathbf{D}^{(1)}\Phi_n(t) \tag{17}$$

where, $\mathbf{D}^{(1)}$ is the $(n+1)\times(n+1)$ operational matrix of derivative given by

$$\mathbf{D}^{(1)} = (d_{ij}) = \begin{cases} C_1(i,j), & i > j, \\ \\ 0, & \text{otherwise}, \end{cases} , \tag{18}$$

where

$$C_1(i,j) = \frac{(i+e+f+1)(i+e+f+2)_j (j+e+2)_{i-j-1} \Gamma(j+e+f+1)}{(i-j-1)!\Gamma(2j+e+f+1)}$$

$$\times_3 F_2 \left(\begin{matrix} -i+1+j,\, i+j+e+f+2,\, j+e+1 \\ \\ j+e+2, \quad 2j+e+f+2, \end{matrix} ; 1 \right)$$

For example, for even n we have

$$\mathbf{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ C_1(1,0) & 0 & 0 & \dots & 0 & 0 \\ C_1(2,0) & C_1(2,1) & 0 & \dots & 0 & 0 \\ C_1(3,0) & C_1(3,1) & C_1(3,2) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ C_1(n,0) & C_1(n,1) & C_1(n,2) & \dots & C_1(n,n-1) & 0 \end{pmatrix}.$$

Remark 2. [8]. Recall that the shifted factorial $(a)_n$ is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

and the hypergeometric function $\times_3 F_2$ is defined by

$$\times_3 F_2 \left(\begin{matrix} -n,\, a, & b \\ & & ;1 \\ c, \quad 1+a+b-c-n, & \end{matrix} \right) = \frac{(c-a)_n (c-n)_n}{(c)_n (c-a-b)_n}.$$

Remark 3. Generally, the r-derivative operational matrix of SJPs of $\Phi_n(t)$ can be given by:

$$\frac{d^r \Phi_n(t)}{dt^r} = \mathbf{D}^{(r)} \Phi_n(t), \tag{19}$$

where $r \in \mathbb{N}$ and $\mathbf{D}^{(r)}$ denotes the $r-$th power of $\mathbf{D}^{(1)}$.

Theorem 1. [24]. Assume that $\alpha, \beta \in (0,1)$. The FF derivative of order (α, β) in the A-R-L sense of $\Phi_n(t)$ in (14) is achieved as:

$$^{FFM}\quad {}_0\mathcal{D}_t^{\alpha,\beta}\Phi_n(t) \simeq \mathbf{E}^{(\alpha,\beta)}\Phi_n(t), \tag{20}$$

that $\mathbf{E}^{(\alpha,\beta)} = [\epsilon_{ij}^{(\alpha,\beta)}]$ is called FF derivative operational matrix of the SJPs, and its entries for $1 \le i, j \le n+1$ are yielded as follows

$$\begin{aligned} \epsilon_{ij}^{(\alpha,\beta)} = & \frac{c(\alpha)}{\lambda_{j-1}\beta(1-\alpha)} \sum_{m=0}^{i-1}\sum_{r=0}^{j-1}\sum_{l=0}^{\infty}(\frac{\alpha}{1-\alpha})^l \\ & \frac{\pi_m^{(i-1)}\pi_r^{(j-1)} m!\Gamma(e+1)\Gamma(\alpha l - \beta + f + m + r + 2)}{\Gamma(\alpha l + m + 1)\Gamma(\alpha l - \beta + e + f + m + r + 3)}, \end{aligned} \tag{21}$$

in which $\pi_m^{(i-1)}$ and $\pi_r^{(j-1)}$ are presented in (11).

4 Convergence analysis

Here in two dimensions, the convergence analysis of SJPs expansion is explored. Set

$$\mathcal{W}^{(e,f)}(s,t) = \omega^{(e,f)}(s)\omega^{(e,f)}(t), \tag{22}$$

where $\omega^{e,f}(z) = z^f(1-z)^e,\ z \in [0,1]$.

Theorem 2. Suppose $\tau \in C^{m+n+1}(\Omega)$ and $\mid \frac{\partial^{n+m+1}}{\partial x^{n+m+1-i}\partial t^i}\Im(s,t) \mid \leqslant \Delta$, for $0 \le i \le n+m+1$. Let $\mathcal{Y} = span\{b_i^{(e,f)}(s)\ b_j^{(e,f)}(t),\ 0 \le i \le m,\ 0 \le j \le n\}$, be a vector subspace with finite dimension of $L^2(\Omega)$. If $\Im_{mn}(s,t)$ is a unique best approximation of τ out of $\mathcal{Y}$ obtained from the proposed method, then the error upper bound satisfies the following relation:

$$\begin{aligned} \|\Im(s,t) - \Im_{mn}(s,t)\|_{L^2_{\mathcal{W}^{(e,f)}}} &= \int_0^1\int_0^1 (\Im(s,t) - \Im_{mn}(s,t))^2 \mathcal{W}^{(e,f)}(s,t)dsdt \\ &\leqslant \frac{\Delta 2^{2(m+n+1)}}{(n+m+1)!}. \end{aligned} \tag{23}$$

Proof. From Maclaurin's expansion for $\Im(s,t)$, we have

$$\Im(s,t) = \wp(s,t) + \frac{1}{(n+m+1)!}(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^{n+m+1}\Im(\xi_0 s, \xi_0 t), \quad \xi_0 \in (0,1),$$

where

$$\wp(s,t) = \sum_{k=0}^{n+m} \frac{1}{k!}(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^{k} \Im(0,0).$$

Thusly

$$|\Im(s,t) - \wp(s,t)| = |\frac{1}{(n+m+1)!}(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^{n+m+1}\Im(\xi_0 s, \xi_0 t)|, \quad \xi_0 \in (0,1).$$

Since $\Im_{mn}(s,t)$ is the best approximation of $\Im(s,t)$, we acquire

$$\| \Im - \Im_{mn}\|^2{}_{L^2_{\mathcal{W}^{(e,f)}}} \leq \| \tau - p\|^2{}_{L^2_{\mathcal{W}^{(e,f)}}}.$$

By the definition of the L^2-norm and binomial expansion $(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^{n+m+1}$, we will have

$$\begin{aligned}
&\| \Im(s,t) - \wp(s,t) \|^2_{L^2_{\mathcal{W}^{(e,f)}}} \\
&= \int_0^1 \int_0^1 [\frac{1}{(n+m+1)!}(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^{n+m+1}\Im(\xi_0 s, \xi_0 t)]^2 \mathcal{W}^{(e,f)}(s,t)dsdt \\
&= \int_0^1 \int_0^1 [\frac{1}{(n+m+1)!} \sum_{i=0}^{n+m+1} \binom{n+m+1}{i} s^{n+m+1-i} t^i \frac{\partial^{n+m+1}}{\partial s^{n+m+1-i}\partial t^i} \\
&\Im(\xi_0 s, \xi_0 t)]^2 \mathcal{W}^{(e,f)}(s,t)dsdt \\
&\leq \frac{\Delta^2}{(n+m+1)!^2} \int_0^1 \int_0^1 \sum_{i=0}^{n+m+1} \sum_{j=0}^{n+m+1} \binom{n+m+1}{i} \\
&\binom{n+m+1}{j} s^{n+m+1-i} t^i s^{n+m+1-j} t^j \mathcal{W}^{(e,f)}(s,t)dsdt.
\end{aligned}$$

Since, $0 \leq s, t \leq 1$, we have

$$\begin{aligned}
\| \Im - \wp\|^2{}_{L^2_{\mathcal{W}^{(e,f)}}} &\leq \tfrac{\Delta^2}{(n+m+1)!^2} \int_0^1 \int_0^1 \sum_{i=0}^{n+m+1} \sum_{j=0}^{n+m+1} \binom{n+m+1}{i} \\
&\binom{n+m+1}{j} \mathcal{W}^{(e,f)}(s,t)dsdt \\
&= \tfrac{\Delta^2 2^{2(m+n+1)}}{(n+m+1)!^2}
\end{aligned}$$

□

which is the desired result.

Corollary 2. If $\Im(s,t)$ is an infinity differential function on Ω, then

$$\|\Im(s,t)-\Im_{mn}(s,t)\|_{L^2_{\mathcal{W}^{(e,f)}}} \to 0 \; as \; n,m\to\infty$$

5 Expression of the proposed approach

In the present section, we will solve the introduced problem in Eqs. (1)-(4) numerically. For this purpose, we first decompose the complex state and control variables and functions of the problem in their real and imaginary parts as follows

$$\begin{aligned} y(s,t) &= y_1(s,t)+iy_2(s,t), & u(s,t) &= u_1(s,t)+iu_2(s,t),\\ \hat{y}(s,t) &= \hat{y}_1(s,t)+i\hat{y}_2(s,t), & v(s) &= v_1(s)+iv_2(s),\\ g(t) &= g_1(t)+ig_2(t), & k(t) &= k_1(t)+ik_2(t), \end{aligned} \tag{24}$$

where, $y_j(s,t), \hat{y}_j(s,t), u_j(s,t), v_j(s), g_j(t)$ and $k_j(t)$ are real functions for $j=1,2$. Thus, the called problem can be illustrated in a coupled system of nonlinear FF differential equations as

$$\begin{aligned} {}^{FFM}_{\quad 0}\mathcal{D}^{\alpha,\beta} & y_1(s,t) - r_1 {y_1}_{ss}(s,t) + \mu_1 {y_2}_{ss}(s,t)\\ &+r_2({y_1}^2(s,t)+{y_2}^2(s,t))y_1(s,t) - \mu_2({y_1}^2(s,t)+{y_2}^2(s,t))y_2(s,t)\\ &-r(s)y_1(s,t)+\mu(s)y_2(s,t) = u_1(s,t)+f_1(s,t),\\ \\ {}^{FFM}_{\quad 0}\mathcal{D}^{\alpha,\beta} & y_2(s,t) - r_1 {y_2}_{ss}(s,t) - \mu_1 {y_1}_{ss}(s,t)\\ &+r_2({y_1}^2(s,t)+{y_2}^2(s,t))y_2(s,t) + \mu_2({y_1}^2(s,t)+{y_2}^2(s,t))y_1(s,t)\\ &-r(s)y_1(s,t)-\mu(s)y_1(s,t) = u_2(s,t)+f_2(s,t), \end{aligned} \tag{25}$$

for $i=1,2$ with the initial conditions

$$y_i(s,0)=v_i(s), \tag{26}$$

and the boundary conditions

$$\begin{aligned} y_i(0,t) &= g_i(t),\\ y_i(1,t) &= k_i(t). \end{aligned} \tag{27}$$

Now, the state and control variables are approximated in terms of the $SJPs$ as follows for $k = 1, 2$

$$y_k(s,t) \simeq \Phi_m^T(s)\mathbf{Y}_k\Phi_n(t), \tag{28}$$

$$u_k(s,t) \simeq \Phi_m^T(s)\mathbf{U}_k\Phi_n(t), \tag{29}$$

where the vectors $\Phi_m(s)$ and $\Phi_n(t)$ are introduced in Eq. (14), and $\mathbf{Y}_k = ({y^k}_{ij})$, and $\mathbf{U}_k = (u_{ij}^k)$ are the unknown coefficients matrices of $(m+1) \times (n+1)$ dimensional. Set

$$\mathcal{B}(s,t) \triangleq \Big[b_0^{(e,f)}(s)b_0^{(e,f)}(t), \dots, b_m^{(e,f)}(s)b_0^{(e,f)}(t) \mid \dots \mid b_0^{(e,f)}(s)b_n^{(e,f)}(t), \dots, b_m^{(e,f)}(s)b_n^{(e,f)}(t)\Big]^T. \tag{30}$$

Considering Eq. (15), we can write Eqs. (28) and (29) as follows:

$$y_k(s,t) \simeq \Phi_m^T(s)\mathbf{Y_k}\Phi_n(t) = \mathcal{B}^T(s,t)vec(\mathbf{Y_k}), \tag{31}$$

and

$$u_k(s,t) \simeq \Phi_m^T(s)\mathbf{U_k}\Phi_n(t) = \mathcal{B}^T(s,t)vec(\mathbf{U_k}), \tag{32}$$

where $k = 1, 2$, and

$$vec(\mathbf{Y_k}) = \left[y_{00}^k, \dots, y_{m0}^k \mid \dots \mid y_{0n}^k, \dots, y_{mn}^k\right]^T,$$
$$vec(\mathbf{U_k}) = \left[u_{00}^k, \dots, u_{m0}^k \mid \dots \mid u_{0n}^k, \dots, u_{mn}^k\right]^T.$$

The following results are obtained from Eqs. (28) and (20):

$$^{FFM}\quad {}_0\mathcal{D}_t^{\alpha,\beta} y_k(s,t) \simeq \Phi_m^T(s)\mathbf{Y}_k\mathbf{E}^{(\alpha,\beta)}\Phi_n(t). \tag{33}$$

Also, from Remark 3 and two times derivative with respect to s on both sides of Eq. (28) yields

$$y_{k_{ss}}(s,t) \simeq \Phi_m^T(s)(\mathbf{D}^{(2)})^T\mathbf{Y}_k\Phi_n(t). \tag{34}$$

Regarding relations Eqs. (26)–(27) and Eq. (31), for $k = 1, 2$

$$\begin{aligned}\mathcal{B}(s,0)vec(\mathbf{Y_k}) &= v_k(s),\\ \mathcal{B}(0,t)vec(\mathbf{Y_k}) &= k_k(t),\\ \mathcal{B}(1,t)vec(\mathbf{Y_k}) &= g_k(t).\end{aligned} \tag{35}$$

Inserting Eqs. (31)–(34) into Eq. (25) gives

$$\begin{aligned}
\mathcal{Z}_1(s,t,) \triangleq \mathcal{B}(s,t)\Big[&\Big((\mathbf{E}^{(\alpha,\beta)T}\otimes I_{m+1}) - r_1(I_{n+1}\otimes(\mathbf{D}^{(2)})^T) - r(s) + r_2((\mathcal{B}(s,t)vec(\mathbf{Y_1}))^2\\
&+(\mathcal{B}(s,t)vec(\mathbf{Y_2}))^2)\Big)vec(\mathbf{Y_1}) + \Big(\mu_1(I_{n+1}\otimes(\mathbf{D}^{(2)})^T) + \mu(s)\\
&-\mu_2((\mathcal{B}(s,t)vec(\mathbf{Y_1}))^2 + (\mathcal{B}(s,t)vec(\mathbf{Y_2}))^2)\Big)vec(\mathbf{Y_2}) - vec(\mathbf{U_1})\Big] - f_1(s,t) \simeq 0,\\
\mathcal{Z}_2(s,t,) \triangleq \mathcal{B}(s,t)\Big[&\Big((\mathbf{E}^{(\alpha,\beta)T}\otimes I_{m+1}) - r_1(I_{n+1}\otimes(\mathbf{D}^{(2)})^T) - r(s) + r_2((\mathcal{B}(s,t)vec(\mathbf{Y_1}))^2\\
&+(\mathcal{B}(s,t)vec(\mathbf{Y_2}))^2)\Big)vec(\mathbf{Y_2}) - \Big(\mu_1(I_{n+1}\otimes(\mathbf{D}^{(2)})^T) - \mu(s)\\
&-\mu_2((\mathcal{B}(s,t)vec(\mathbf{Y_1}))^2 + (\mathcal{B}(s,t)vec(\mathbf{Y_2}))^2)\Big)vec(\mathbf{Y_1}) - vec(\mathbf{U_2})\Big] - f_2(s,t) \simeq 0,
\end{aligned} \tag{36}$$

where, I_{m+1} and I_{n+1} are identity matrices of $m+1$ and $n+1$ orders, respectively and $\otimes$ represents Kronecker's product [17]. Finally, from Eqs. (35) and (36), a system of $2(m+1)(n+1)$ algebraic equations can be written as for $k=1,2$:

$$\begin{cases}
\mathcal{C}^k_{i,j} \triangleq \mathcal{Z}_k(\xi_i,\eta_j) = 0, & i=2,\dots,m, \quad j=2,\dots,n+1,\\
\tilde{\mathcal{H}}_k \triangleq \mathcal{B}(\xi_i,0)vec(\mathbf{Y_k}) - v_k(\xi_i) = 0 & i=1,\dots,m+1,\\
\tilde{\mathcal{M}}_k \triangleq \mathcal{B}(0,\eta_j)vec(\mathbf{Y_k}) - k_k(\eta_j) = 0, & j=2,\dots,n+1,\\
\tilde{\mathcal{N}}_k \triangleq \mathcal{B}(1,\eta_j)vec(\mathbf{Y_k}) - g_k(\eta_j) = 0, & j=2,\dots,n+1,
\end{cases} \tag{37}$$

where

$$\begin{aligned}
\xi_i &= \frac{1}{2}(1-cos(\frac{2(i-1)\pi}{2m})), & i=1,\dots,m+1,\\
\eta_j &= \frac{1}{2}(1-cos(\frac{2(j-1)\pi}{2n})), & j=1,\dots,n+1.
\end{aligned}$$

Then, the performance index of the examined problem is approximated using SJPs. First, the desired function is approximated as:

$$\hat{y}_k(s,t) \simeq \Phi_m^T(s)\widehat{\mathbf{Y}}_{\mathbf{k}}\Phi_n(t), \quad k=1,2. \tag{38}$$

Inserting Eqs. (28), (29), and (38) into Eq. (1), we get

$$\begin{aligned}
&\mathcal{J}(y,u) \simeq \mathbb{J}_{m,n}(\mathbf{Y_1},\mathbf{Y_2},\mathbf{U_1},\mathbf{U_2})\\
&\quad = \int_0^1\int_0^1 \Big[(\phi_m(s)^T\mathbf{Y_1}\phi_n(t) - \phi_m(s)^T\widehat{\mathbf{Y}}_{\mathbf{1}}\phi_n(t))(\phi_m(s)^T\mathbf{Y_1}\phi_n(t) - \phi_m(s)^T\widehat{\mathbf{Y}}_{\mathbf{1}}\phi_n(t))^T\Big]\\
&\quad \mathcal{W}^{(e,f)}(s,t)dsdt\\
&\quad + \int_0^1\int_0^1 \Big[(\phi_m(s)^T\mathbf{Y_2}\phi_n(t) - \phi_m(s)^T\widehat{\mathbf{Y}}_{\mathbf{2}}\phi_n(t))(\phi_m(s)^T\mathbf{Y_2}\phi_n(t) - \phi_m(s)^T\widehat{\mathbf{Y}}_{\mathbf{2}}\phi_n(t))^T\Big]\\
&\quad \mathcal{W}^{(e,f)}(s,t)dsdt\\
&\quad + \epsilon^2\Big(\int_0^1\int_0^1 \Big[(\phi_m(s)^T\mathbf{U_1}\phi_n(t) - \phi_m(s)^T\widehat{\mathbf{U}}_{\mathbf{1}}\phi_n(t))(\phi_m(s)^T\mathbf{U_1}\phi_n(t) - \phi_m(s)^T\widehat{\mathbf{U}}_{\mathbf{1}}\phi_n(t))^T\Big]\\
&\quad \mathcal{W}^{(e,f)}(s,t)dsdt\\
&\quad + \int_0^1\int_0^1 \Big[(\phi_m(s)^T\mathbf{U_2}\phi_n(t) - \phi_m(s)^T\widehat{\mathbf{U}}_{\mathbf{2}}\phi_n(t))(\phi_m(s)^T\mathbf{U_2}\phi_n(t) - \phi_m(s)^T\widehat{\mathbf{U}}_{\mathbf{2}}\phi_n(t))^T\Big]\\
&\quad \mathcal{W}^{(e,f)}(s,t)dsdt\Big).
\end{aligned}$$

Because expressions $\int_0^1\int_0^1(\phi_m(s)^T\widehat{\mathbf{Y}}_{\mathbf{1}}\phi_n(t))^2\mathcal{W}^{(e,f)}(s,t)dsdt$, $\int_0^1\int_0^1(\phi_m(s)^T\widehat{\mathbf{Y}}_{\mathbf{2}}\phi_n(t))^2\mathcal{W}^{(e,f)}(s,t)dsdt$, $\int_0^1\int_0^1(\phi_m(s)^T\widehat{\mathbf{U}}_{\mathbf{1}}\phi_n(t))^2\mathcal{W}^{(e,f)}(s,t)dsdt$, and $\int_0^1\int_0^1(\phi_m(s)^T\widehat{\mathbf{U}}_{\mathbf{2}}\phi_n(t))^2\mathcal{W}^{(e,f)}(s,t)dsdt$ do not have any effective role in minimization due to being positive, so according to Eq. (12), the above equation is expressed as follows

$$\mathbb{J}_{m,n}((\mathbf{Y_1},\mathbf{Y_2},\mathbf{U_1},\mathbf{U_2})) = vec(\mathbf{Y_1})^T(\Upsilon_n\otimes\Upsilon_m)vec(\mathbf{Y_1}) + vec(\mathbf{Y_2})^T(\Upsilon_n\otimes\Upsilon_m)vec(\mathbf{Y_2}) \tag{39}$$

$$-2vec(\widehat{\mathbf{Y}}_{\mathbf{1}})^T(\Upsilon_n\otimes\Upsilon_m)vec(\mathbf{Y_1}) - 2vec(\widehat{\mathbf{Y}}_{\mathbf{2}})^T(\Upsilon_n\otimes\Upsilon_m)vec(\mathbf{Y_2}) \tag{40}$$

$$+\epsilon^2[vec(\mathbf{U_1})^T(\Upsilon_n\otimes\Upsilon_m)vec(\mathbf{U_1}) + vec(\mathbf{U_2})^T(\Upsilon_n\otimes\Upsilon_m)vec(\mathbf{U_2}) \tag{41}$$

$$-2vec(\widehat{\mathbf{U}}_{\mathbf{1}})^T(\Upsilon_n\otimes\Upsilon_m)vec(\mathbf{U_1}) - 2vec(\widehat{\mathbf{U}}_{\mathbf{2}})^T(\Upsilon_n\otimes\Upsilon_m)vec(\mathbf{U_2})], \tag{42}$$

where

$$\Upsilon_k = diag(\lambda_0,\ldots,\lambda_k).$$

The discussed optimal control problem has now transformed into a finite-dimensional optimization. To solve the obtained optimization problem, we employ the Lagrangian multipliers scheme. First define

$$\mathcal{J}^*(y,u) \simeq \mathbb{J}^*(\mathbf{Y_1},\mathbf{Y_2},\mathbf{U_1},\mathbf{U_2},\Sigma) = \mathbb{J}_{m,n}(\mathbf{Y_1},\mathbf{Y_2},\mathbf{U_1},\mathbf{U_2}) + \Sigma^T\mathcal{C}, \tag{43}$$

where

$$\mathcal{C} = \Big[\mathcal{C}^1_{2,2}, \dots, \mathcal{C}^1_{2,n+1}| \dots |\mathcal{C}^1_{m,2}, \dots, \mathcal{C}^1_{m,n+1}|\mathcal{C}^2_{2,2}, \dots, \\ \mathcal{C}^2_{2,n+1}| \dots |\mathcal{C}^2_{m,2}, \dots, \mathcal{C}^2_{m,n+1}|\tilde{\mathcal{H}}_1|\tilde{\mathcal{H}}_2|\tilde{\mathcal{M}}_1|\tilde{\mathcal{M}}_2|\tilde{\mathcal{N}}_1|\tilde{\mathcal{N}}_2\Big],$$

and

$$\Sigma = \Big[\varsigma_1 \ \varsigma_2 \ \dots \ \varsigma_{2(m+1)(n+1)}\Big]^T,$$

where the Lagrange multipliers vector is denoted with Σ. Now the optimality conditions for $k = 1, 2$ are derived from the following equations:

$$\begin{cases} \dfrac{\partial \mathcal{J}^*(y,u)}{\partial vec(\mathbf{Y_k})} = 0, \\ \\ \dfrac{\partial \mathcal{J}^*(y,u)}{\partial vec(\mathbf{U_k})} = 0, \\ \\ \dfrac{\partial \mathcal{J}^*(y,u)}{\partial \Sigma} = 0. \end{cases}$$

The above system of algebraic equations can be solved by Newton's iterative method or by using Matlab software packages. The numerical solutions of $y(s,t)$ and $u(s,t)$ are specified by determining $\mathbf{Y}_k$ and $\mathbf{U}_k$ and placing them in Eqs. (31) and (32), respectively.

6 Illustrative examples

In this section, using some test problems, the accuracy and efficiency of the described method in Section 5 have been investigated. To achieve this goal, the maximum absolute error (MAE) and root mean square (RMS) are calculated. All calculations and results have been done using the Fmincon package in MATLAB software. The accuracy of the obtained results from the proposed method is calculated using MAE and RMS, which is defined as follows:

$$MAE = \max_{1\le i\le m+1} \max_{1\le j\le n+1} |f(\xi_i,\eta_j) - \widetilde{f}(\xi_i,\eta_j)|,$$

$$RMS = \sqrt{\frac{1}{(m+1)(n+1)}(\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}|f(\xi_i,\eta_j) - \widetilde{f}(\xi_i,\eta_j)|^2)},$$

where $\widetilde{f}$ represents the numerical solution of f in the collocation points of (ξ_i, η_j). Note that in all test problems, the first 29 terms of the infinite series in the Mittag-Leffler functions are used in numerical calculations.

Example 1. Consider the following objective function:

$$\begin{aligned}\mathcal{J}(y,u) = \int_0^1\int_0^1 \Big((y(s,t) - t^4e^{is})^2 - \epsilon^2(u(s,t) - ((1+2i)t^4 + (1+i)t^{12} \\ - s^2(1+is)t^4)e^{is})^2\mathcal{W}^{(e,f)}(s,t)\Big)dsdt, \qquad (44)\end{aligned}$$

where $\epsilon = 1$. Subject to the following nonlinear time FF Ginzburg–Landau equation [14]:

$$\begin{aligned}&{}_0^{FFM}D^{\alpha,\beta}y(s,t) - (1+2i)y_{ss}(s,t) + (1+i)|y(s,t)|^2y(s,t) - s^2(1+is)y(s,t)\\ &\quad = f(s,t) + u(s,t),\end{aligned}$$

where

$$f(s,t) = (\tfrac{c(\alpha)4!t^{5-\beta}}{\beta(1-\alpha)}E_{\alpha,5}(\tfrac{-\alpha t^\alpha}{1-\alpha}))e^{is}.$$

The exact solution of state and control functions is mentioned below,

$$\begin{aligned}&y(s,t) = t^4e^{is},\\ &u(s,t) = ((1+2i)t^4 + (1+i)t^{12} - s^2(1+is)t^4)e^{is}.\end{aligned}$$

From the analytical solution of $y(s,t)$, we acquire the initial and boundary conditions. For the computational solution of this example, we have used the introduced method in Section 5 with values of $m = n$. For some choices (α,β) and $(e,f) = (0,0)$, MAE and RMS values of state and control variables are shown in Figures $1-4$. AE graphs for state and control variables with $m = n = 7,\ \alpha = 0.25, \beta = 0.85$ and $(e,f) = (0,0)$ are depicted in Figure 5.

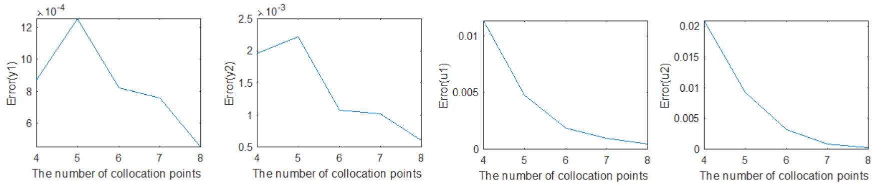

Figure 1: The RMS of the presented method for the state and control functions with $\alpha = 0.35$ and $\beta = 0.35$ in Example 1.

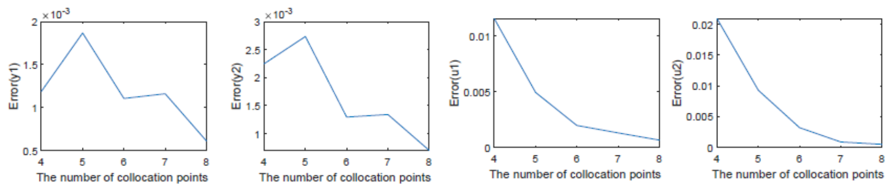

Figure 2: The RMS of the presented method for the state and control functions with $\alpha = 0.55$ and $\beta = 0.35$ in Example 1.

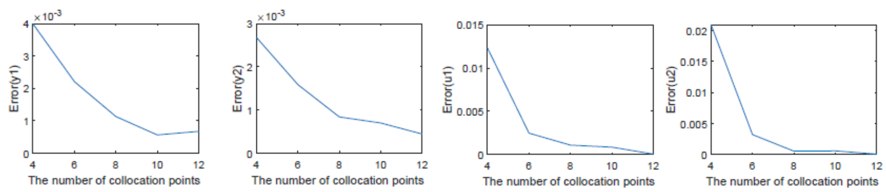

Figure 3: The RMS of the presented method for the state and control functions with $\alpha = 0.75$ and $\beta = 0.35$ in Example 1.

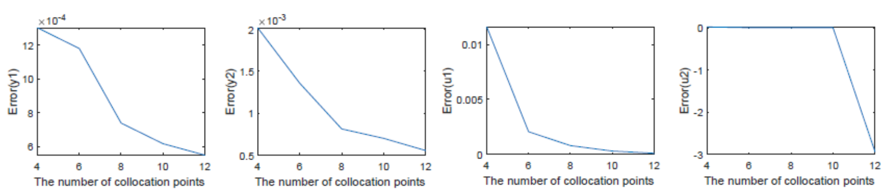

Figure 4: The RMS of the presented method for the state and control functions with $\alpha = 0.75$ and $\beta = 0.65$ in Example 1.

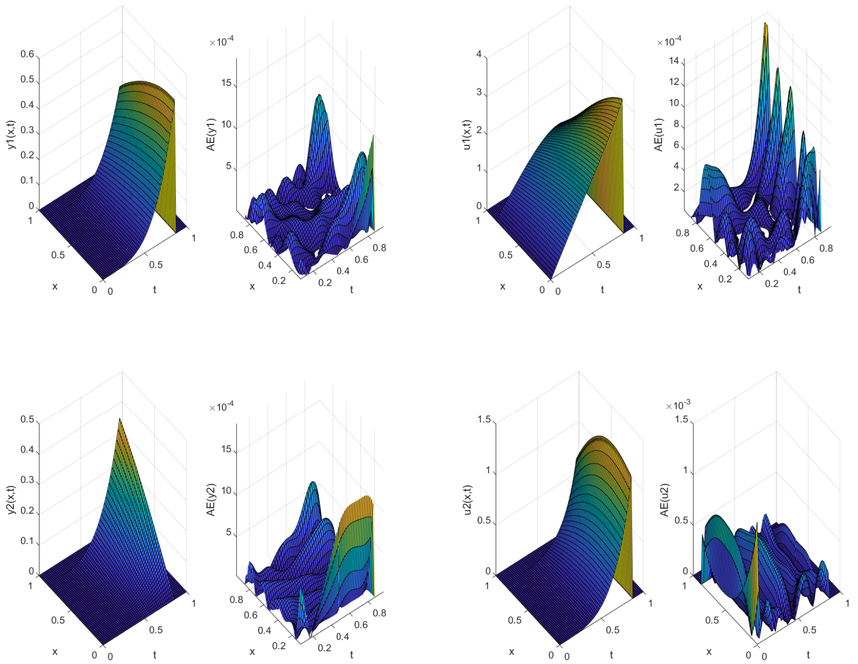

Figure 5: Numerical solution (SJPs) and error function (AE) surfaces of $y(s,t)$ (left) and $u(s,t)$ (right) with (m=n=7) in Example 1.

Example 2. Consider the problem of optimal control with $\epsilon = 10^{-1}$ as follows:

$$\mathcal{J}(y,u) = \int_0^1\int_0^1 \Big((y(s,t) - t^2 sin(t)e^{is})^2 - \epsilon^2(u(s,t) - (5it^2 sin(t) \\ + 2t^6 sin^3(t) - it^2 sin(t)e^{-is})e^{is})^2 \mathcal{W}^{(e,f)}(s,t)\Big)dsdt, \quad (45)$$

with the nonlinear time FF dynamical system: [14]

$${}_0^{FFM}D^{\alpha,\beta}y(s,t) - 5iy_{ss}(s,t) + 2|y(s,t)|^2 y(s,t) - ie^{-is}y(s,t) = f(s,t) + u(s,t),$$

where,

$$f(s,t) = \left(\tfrac{c(\alpha)t^{4-\beta}}{\beta(1-\alpha)}\textstyle\sum_{k=0}^{\infty}(-1)^k(2k+3)(2k+2)t^{2k}E_{\alpha,2k+4}\left(\tfrac{-\alpha t^{\alpha}}{1-\alpha}\right)\right)e^{is}.$$

The exact solution of state and control functions is mentioned following,

$$y(s,t) = t^2 sin(t) e^{is},$$
$$u(s,t) = (5it^2 sin(t) + 2t^6 sin^3(t) - it^2 sin(t) e^{-is}) e^{is}.$$

From the analytical solution of $y(s,t)$, we acquire the initial and boundary conditions. For the computational solution of this example, we have used the introduced method in section 5 with values of $m = n$. For some choices (α, β) and $(e, f) = (0.5, 0.5)$, MAE and RMS values of state and control functions are shown in Figures $6-9$. AE graphs for state and control functions with $m = n = 7$, $\alpha = 0.75, \beta = 0.25$ and $(e, f) = (0.5, 0.5)$ are depicted in Figure 10.

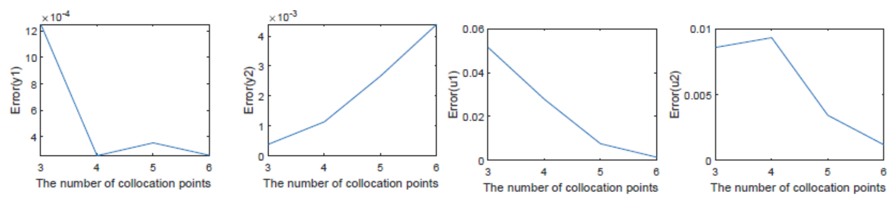

Figure 6: The RMS of the presented method for the state and control functions with $\alpha = 0.35$ and $\beta = 0.35$ in Example 2.

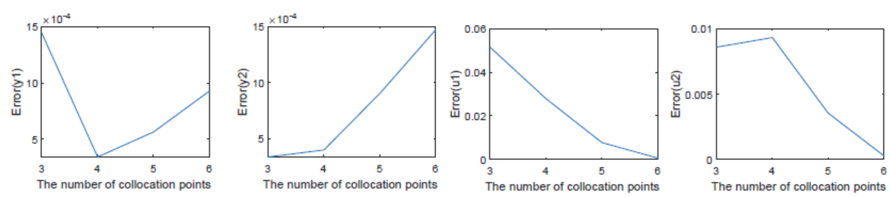

Figure 7: The RMS of the presented method for the state and control functions with $\alpha = 0.65$ and $\beta = 0.35$ in Example 2.

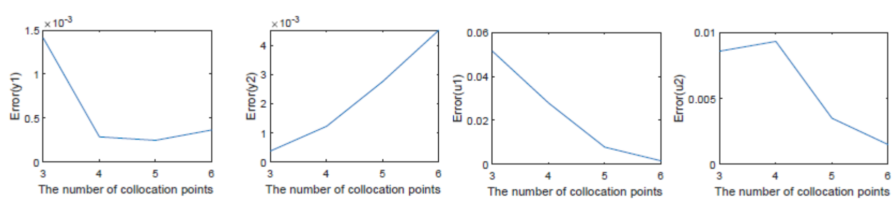

Figure 8: The RMS of the presented method for the state and control functions with $\alpha = 0.75$ and $\beta = 0.15$ in Example 2.

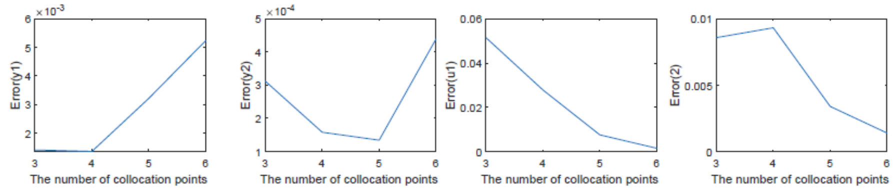

Figure 9: The RMS of the presented method for the state and control functions with $\alpha = 0.75$ and $\beta = 0.45$ in Example 2.

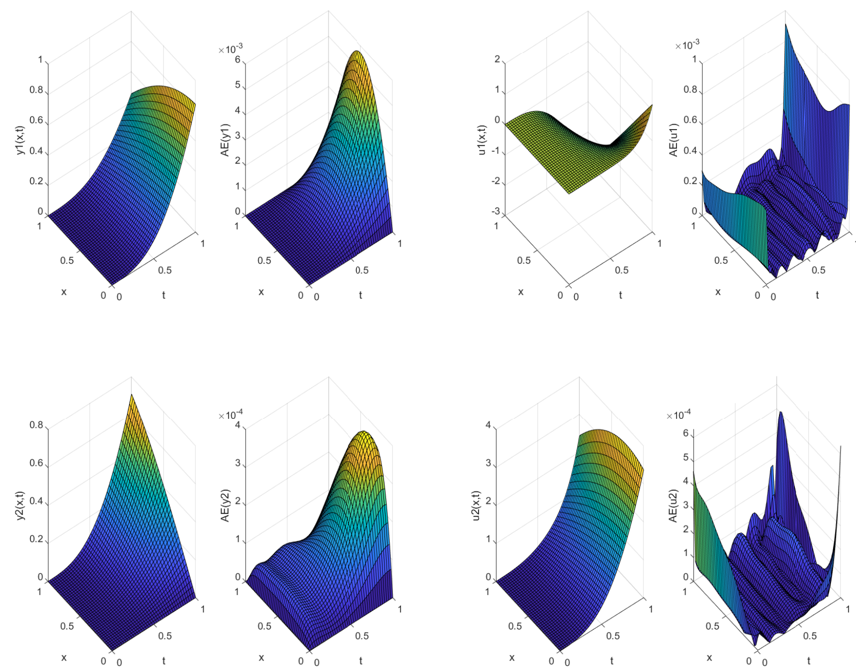

Figure 10: Numerical solution (SJPs) and error function (AE) surfaces of $y(s,t)$ (left) and $u(s,t)$ (right) with (m=n=7) in Example 2.

Example 3. Consider the problem of optimal control with $\epsilon = 1.1$ as follows:

$$\mathcal{J}(y,u) = \int_0^1 \int_0^1 \Big((y(s,t) - it^3 e^{-(t+is)})^2 - \epsilon^2 (u(s,t) - ie^{-is}(2t^3 e^{-t} + 3it^9 e^{-3t} - (2s+1+3is^2)t^3 e^{-t}))^2 \mathcal{W}^{(e,f)}(s,t) \Big) ds dt, \tag{46}$$

with the FF dynamical system: [14]

$$ {}_0^{FFM}D^{\alpha,\beta} y(s,t) - 2y_{ss}(s,t) + 3i|y(s,t)|^2 y(s,t) - (2s+1+3is^2)y(s,t) = f(s,t) + u(s,t), $$

where,

$$ f(s,t) = i\left(\frac{c(\alpha)t^{4-\beta}}{\beta(1-\alpha)} \sum_{k=0}^{\infty} (-1)^k (k+3)(k+2)(k+1)t^k E_{\alpha,k+4}\left(\frac{-\alpha t^\alpha}{1-\alpha}\right)\right)e^{-is}. $$

The exact solution of state and control functions is mentioned following,

$$ y(s,t) = it^3 e^{-(t+is)}, $$

$$ u(s,t) = ie^{-ix}(2t^3e^{-t} + 3it^9e^{-3t} - (2s+1+3is^2)t^3e^{-t}). $$

The homogeneous initial and boundary conditions are obtained from the analytic solution of $y(s,t)$. For the numerical solution of this example, we have used the introduced method in section 5 with values of $m = n$. For some choices (α, β) and (e, f), MAE and RMS values of state and control variables are shown in Figures $11 - 14$. AE graphs for state and control functions with $m = n = 7$, $\alpha = 0.25, \beta = 0.25$ and $(e, f) = (0, 1)$ are depicted in Figure 15.

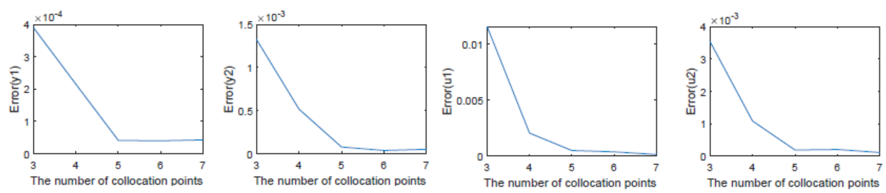

Figure 11: The RMS of the presented method for the state and control functions with $\alpha = 0.25$ and $\beta = 0.25$ in Example 3.

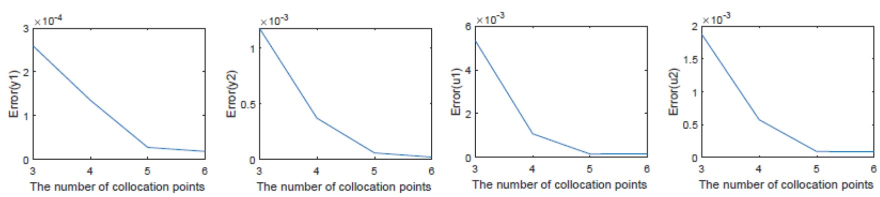

Figure 12: The RMS of the presented method for the state and control functions with $\alpha = 0.65$ and $\beta = 0.25$ in Example 3.

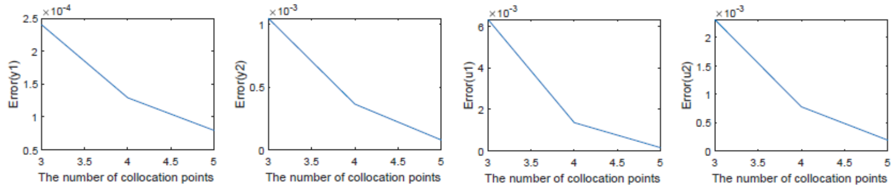

Figure 13: The RMS of the presented method for the state and control functions with $\alpha = 0.80$ and $\beta = 0.25$ in Example 3.

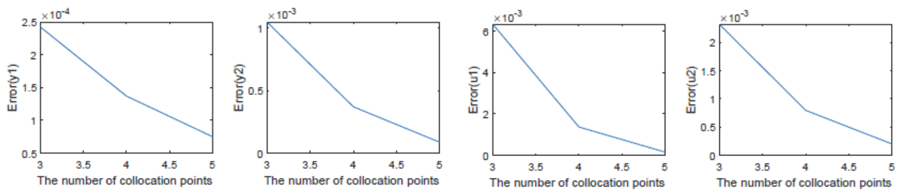

Figure 14: The RMS of the presented method for the state and control functions with $\alpha = 0.80$ and $\beta = 0.65$ in Example 1.

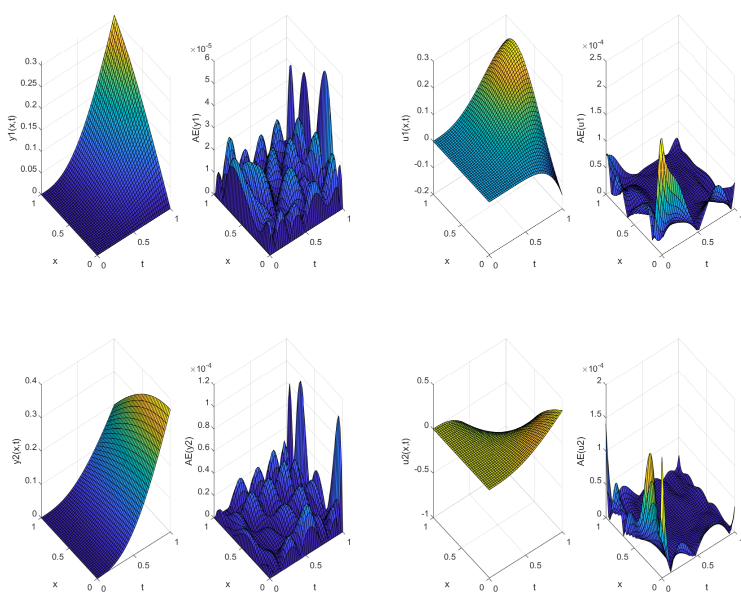

Figure 15: Numerical solution (SJPs) and error function (AE) surfaces of $y(s,t)$ (left) and $u(s,t)$ (right) with (m=n=7) in Example 3.

7 Conclusion

In this paper we introduced a novel class of optimal control with the nonlinear Ginzburg−Landau equation. To express this new class, we have used the FF derivative in the A-R-L sense with Mittag-Leffler non-singular kernel. For the numerical solution of this class of optimal control problems, an efficient method based on the shifted Jacobi polynomials has been proposed. To transform the main problem into a system of nonlinear algebraic equations, we have used the FF derivative operational matrix of SJPs and the collocation method. By presenting three numerical examples, we have investigated and evaluated the accuracy of the mentioned scheme.

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