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Research Article

Fitted numerical method for singularly perturbed semilinear three-point boundary value problem

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Abstract

We consider a class of singularly perturbed semilinear three-point boundary value problems. An accelerated uniformly convergent numerical method is constructed via the exponential fitted operator method using Richardson extrapolation techniques to solve the problem. To treat the semilinear term, we use quasi-linearization techniques. The numerical results are tabulated in terms of maximum absolute errors and rate of convergence, and it is observed that the present method is more accurate and ε-uniformly convergent for $h \geq \varepsilon$, where the classical numerical methods fail to give a good result. It also improves the results of the methods existing in the literature. The method is shown to be second-order convergent independent of perturbation parameter ε.

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1 Introduction

Singularly perturbed differential equations are typically characterized by the presence of a small positive parameter ε multiplying some or all of the highest order terms in differential equations. Such types of problems arise frequently in mathematical models of different areas of physics, chemistry, biology, engineering science, economics, and even sociology.

Boundary value problems including nonlocal conditions, which connect the values of the unknown solution at the boundary with values in the interior, are known as nonlocal boundary value problems. The study of this kind of problem was initiated by Il'in and Miseev [18, 19], motivated by the work of Bitsadze and Samarskii [4] on nonlocal linear elliptic boundary value problems. These problems have been used to represent mathematical models of a large number of phenomena, such as problems of semiconductors in electronics, the vibrations of a guy wire of a uniform cross-section, heat transfer problems, problems of hydromechanics, catalytic processes in chemistry and biology, the diffusion-drift model of semiconducting devices, and some other physical phenomena; see [1, 17, 25]. The existence and uniqueness of the solutions to nonlocal boundary value problems have been studied by many authors [3, 20]. Some approaches for the numerical solution of singularly perturbed nonlocal boundary value problems have been proposed in [2, 6, 7, 11, 12, 13, 16, 21, 26]. Uniformly convergent numerical methods of order second and high for solving different singularly perturbed problems have been studied in [5, 9, 10, 14, 23, 27]. What makes the problem under consideration challenging is that, it contains the perturbation parameter in both diffusion and convection terms, that it contains semilinear source term, and that it has three-point nonlocal boundary condition. To the best of our knowledge, the problem under consideration has not been done using the exponentially fitted operator method. Motivated by the paper [8], we develop a uniformly convergent numerical method for solving the singularly perturbed problem under consideration.

2 Definition of the problem

Consider the following singularly perturbed problem of the form

$$Ly := \varepsilon^2 y''(x) + \varepsilon a(x) y'(x) - f(x, y(x)) = 0, \quad 0 < x < l, \tag{1}$$

with the given conditions

$$y(0) = A, \tag{2}$$

$$L_0 y := y(l) - \phi(y(l_1)) = 0, \quad 0 < l_1 < l, \tag{3}$$

where $\varepsilon, 0 < \varepsilon \ll 1$ is the perturbation parameter, A is a given constant, and the functions $a(x) \geq a > 0$ and $f(x, y)$ are sufficiently smooth on $[0, l]$ and $[0, l] \times R$, respectively. Moreover $0 < b \leq \frac{\partial f}{\partial y} \leq \beta < \infty$ and $|\frac{d\phi}{dy}| \leq k < 1$.

The solution of $y(x)$ of (1)–(3) has boundary layers at $x = 0$ and $x = l$ for ε near 0.

Equations of this type arise in mathematical problems in many areas of mechanics and physics. Among these are the Navier–Stokes equations of fluid flow at high Reynolds number, mathematical models of liquid crystal materials and chemical reactions, shear in second-order fluids, control theory, electrical networks, and other physical models [23, 24].

To obtain the numerical solution to (1)–(3), the well-known Newton's quasi-linearization techniques were used. This technique allows us to linearize the semilinear problem into a linear problem, whose solution $y^{(p)}(x)$ with a proper initial guess $y^{(0)}$ will converge to the original solution $y(x)$.

So, after applying quasi-linearization technique for $f(x, y(x))$, we rewrite (1)–(3) in the form

$$Ly := \varepsilon^2 y''(x) + \varepsilon a(x) y'(x) - b(x) y(x) = R(x), \qquad x \in \Omega = (0, l), \tag{4}$$

with the given conditions

$$y(0) = A, \tag{5}$$

$$L_0 y := y(l) - \phi(y(l_1)) = 0, \quad 0 < l_1 < l, \tag{6}$$

where $b(x) = \frac{\partial f}{\partial y}(x, \eta y(x))$, $0 < \eta < 1$, and $R(x) = f(x, \eta y(x)) - \eta y(x) b(x)$.

In this article, we analyze an exponentially fitted operator scheme with Richardson extrapolation techniques on uniform mesh for the numerical solution of (4)–(6). Uniform convergence is proved in the discrete maximum norm. Finally, we formulate the algorithm for solving the discrete problem and give the illustrative numerical results.

3 Properties of continuous solution

The following lemmas are necessary for the existence and uniqueness of the solution and for the problem to be well-posed.

Lemma 1. Let $y(x)$ be the solution of (1)–(3). If $a \in C^1[0, l]$, then the following inequality

$$\|y\|_\infty \leq C_0 \tag{7}$$

holds, where $C_0 = (1-\gamma)^{-1}(|A|+|B|+b^{-1}\|R\|_\infty),\ B = \phi(0,)\ R(x) = f(x, 0)$, and $\|y\|_\infty = \max_{[0,l]} |y(x)|$.

Proof. For proof refer [8]. □

Lemma 2. Let y_ε be the solution of (P_ε). Then, for $k = 0, 1, 2, 3, 4$,

$$| y_\varepsilon^{(k)}(x) |\leq C\left(1+\varepsilon^{-k}\left(\exp\left(\frac{-c_0x}{\varepsilon}\right)+\exp\left(\frac{-c_1(l-x)}{\varepsilon}\right)\right)\right), \quad x\in[0,l], \tag{8}$$

holds fore the solution $y(x)$ provided that $\frac{\partial f}{\partial y}-\varepsilon a'(x)\geq b$ and $\frac{\partial f}{\partial x}\leq C$ for $x\in[0,l]$ and $|y|\leq C$, where $c_0=\frac{1}{2}[\sqrt{a^2(0)+4b}+a(0)]$ and $c_1=\frac{1}{2}[\sqrt{a^2(l)+4b}-a(l)]$.

Proof. For proof refer [8]. □

4 Formulation of numerical scheme

We subdivide the domain $\Omega=[0,1]$ into N equal number of subintervals, each of length h. In this article, we develop the fitted operator finite difference method to find a numerical solution to the problem (4). We use the theory in the asymptotic method for developing the exponential factor.

In order to evaluate the fitting factor, we divide both sides of (4) by ε, and we obtain

$$\begin{cases} Ly(x)\equiv\varepsilon y''(x)+a(x)y'+p(x)y(x)=g(x), & 0<x<\ell,\\ y(0)=A, & \\ L_0y:=y(\ell)=\theta, & 0<\ell_1<\ell, \end{cases} \tag{9}$$

where $p(x)=-\frac{b(x)}{\varepsilon}$, $g(x)=\frac{R(x)}{\varepsilon}$, and $\theta=\varphi(y(\ell_1))$.

4.1 Left end boundary layer problem

The boundary layer occurs on the left side of the domain, that is, near $x=0$. To find the numerical solution of (9), we use the theory applied in the asymptotic method for solving singularly perturbed boundary value problems. From the theory of singular perturbations given by O'Malley [24] and using Taylor's series expansion for $a(x)$ about $x=0$ and restriction to their first terms, we get the asymptotic solution as

$$y(x)=y_0(x)+\frac{a(0)}{a(x)}(A-y_0(0))e^{-\int_0^x\left(\frac{a(x)}{\varepsilon}-\frac{p(x)}{a(x)}\right)dx}+O(\varepsilon), \tag{10}$$

where $y_0(x)$ is the solution of the reduced problem (obtained by setting $\varepsilon=0$) of (9) given by $a(x)y_0'(x)+p(x)y_0(x)=g(x)$, with $y(0)=A$ and $y_0(\ell)=\theta$, where $\theta=\varphi(y(l_1))$.

By taking Taylor series expansion for $a(x)$ and $p(x)$ about the point "0" and simplifying them, we obtain

$$y(x) = y_0(x) + (A - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{p(0)}{a(0)}\right)x} + O(\varepsilon). \tag{11}$$

Now we divide the interval $[0,1]$ into N equal subinterval of mesh size $h = \frac{1}{N}$ so that $x_i = ih$, $i = 0, 1, 2 \ldots, N$. From (11), we have

$$y(ih) = y_0(ih) + (A - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{p(0)}{a(0)}\right)ih} + O(\varepsilon).$$

Taking limit as $h \to 0$ on both sides, we have

$$\lim_{h\to 0} y(ih) = y_0(0) + (A - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{p(0)}{a(0)}\right)i\rho} + O(\varepsilon). \tag{12}$$

Also, we discretized $x_{i+1} = (i+1)h = ih + h$ from (11), we obtain

$$y(ih+h) = y_0(ih+h) + (A - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{p(0)}{a(0)}\right)ih+h} + O(\varepsilon).$$

Taking limit as $h \to 0$ on both sides, we have

$$\lim_{h\to 0} y(ih+h) = y_0(0) + (A - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{p(0)}{a(0)}\right)i\rho+\rho} + O(\varepsilon). \tag{13}$$

Similarly for $x_{i-1} = (i-1)h = ih - h$ from (11), we have

$$y(ih-h) = y_0(ih-h) + (A - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{p(0)}{a(0)}\right)ih-h} + O(\varepsilon).$$

Taking limit as $h \to 0$ on both sides, we have

$$\lim_{h\to 0} y(ih-h) = y_0(0) + (A - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{p(0)}{a(0)}\right)i\rho-\rho} + O(\varepsilon). \tag{14}$$

To handle the effect of the perturbation parameter, artificial viscosity (exponentially fitting factor $\sigma(\rho)$) is multiplied on the term containing the perturbation parameter as

$$\varepsilon\sigma(\rho)y_i'' + a(x_i)y_i' + p(x_i)y_i = g(x_i), \tag{15}$$

with boundary conditions $y(0) = A$ and $y(\ell)=\theta$. Next, we consider the difference approximation of (9) on a uniform grid $\overline{\Omega}^N = \{x_i\}_{i=0}^N$ and denote $h = x_{i+1} - x_i$.

Using the Taylor series expansion for $y(x)$ about the point x_i, this can be written as

$$y(x_i - h) \approx \; y_{i-1} = y_i - hy_i' + \frac{h^2}{2!}y_i'' + O(h^3), \tag{16}$$

$$y(x_i + h) \approx \; y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + O(h^3). \tag{17}$$

For any mesh function y_i, from (16) and (17), we obtain the following finite difference operator:

$$\begin{cases} D^- y_i = \frac{y_i - y_{i-1}}{h}, \\ D^+ y_i = \frac{y_{i+1} - y_i}{h}, \\ D^0 y_i = \frac{y_{i+1} - y_{i-1}}{2h}, \\ D^+ D^- y_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}. \end{cases} \tag{18}$$

Then, applying the central and forward finite difference formula for the second and the first derivative, respectively, on (9), we get

$$\varepsilon\sigma(\rho)(D^+D^-y(x_i)) + a(x_i)(D^+y(x_i)) + p(x_i)y(x_i) = g(x_i). \tag{19}$$

Using the difference operator in (19), we have

$$L_\varepsilon^N y_i = g_i, \tag{20}$$

with boundary conditions $y(0) = A$ and $y(\ell) = \theta$.

From (19), we have

$$\varepsilon\sigma(\rho)\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + a(x_i)\left(\frac{y_{i+1} - y_i}{h}\right) + p(x_i)y_i = g(x_i), \quad 1 \le i \le \frac{N}{2}, \tag{21}$$

where $\rho = \frac{h}{\varepsilon}$.

Multiplying (21) by h, considering h small, and truncating the term $(g(x_i) - p(x_i)y_i)h$, we get

$$\frac{\sigma}{\rho}\big(y_{i-1} - 2y_i + y_{i+1}\big) + a(x_i)\big(y_{i+1} - y_i\big) = h(g(x_i) - p_i y_i). \tag{22}$$

By evaluating the limit of (22) as h approaches to zero and $\lim_{h\to 0} h(g_i - p_i y_i) = 0$, we get

$$\sigma = -a(0)\rho \frac{\lim_{h\to 0}(y_{i+1} - y_i)}{\lim_{h\to 0}(y_{i+1} - 2y_i + y_{i-1})}. \tag{23}$$

Substituting (12), (13), and (14) into (23) for solving the fitting parameter σ and simplifying it, we obtain

$$\sigma_1 = \frac{\rho a(0)}{4}\left[\frac{(1 - e^{-(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\rho)})}{\sinh^2[(\frac{a^2(0) - \varepsilon b(0)}{2a(0)}\rho)]}\right]. \tag{24}$$

4.2 Right end boundary layer problem

The boundary layer occurs on the right side of the domain, that is, near $x = 1$. Assume that $\overline{\Omega}^N$ denotes the partition of $[0, \ell]$ into N subintervals such that $0 = x_0,\ x_1, \ldots, x_N = 1$ with $x_i = ih,\ h = \frac{1}{N}\ \ i = 0, 1, 2, \ldots, N$. To find the numerical solution of (9), we use the theory applied in the asymptotic method for solving singularly perturbed boundary value problems. Assume that the problem has the right layer. From the theory of singular perturbations given by O'Malley [24] and using Taylor's series expansion for $a(x)$ about $x = 1$ and restriction to their first terms, we get the asymptotic solution as

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\theta - y_0(1))e^{-\int_x^1 \left(\frac{a(x)}{\varepsilon} - \frac{p(x)}{a(x)}\right)dx} + O(\varepsilon), \tag{25}$$

where $y_0(x)$ is the solution of the reduced problem (obtained by setting $\varepsilon = 0$) of (9) given by $a(x)y_0'(x)+p(x)y_0(x) = g(x)$, with $y_0(1) = \theta$. By taking Taylor series expansion for $a(x)$ and $b(x)$ about the point "1" and simplifying them, we obtain

$$y(x) = y_0(x) + (\theta - y_0(1))e^{-\left(\frac{a(1)(1-x)}{\varepsilon}\right)} + O(\varepsilon). \tag{26}$$

Now we divide the interval $[0, 1]$ into N equal subintervals of mesh size $h = \frac{1}{N}$ so that $x_i = ih$, $i = 0, 1, 2 \ldots, N$. From (26), we have

$$\lim_{h\to 0} y(ih) = y_0(0) + (\theta - y_0(1))e^{\left(-a(1)(\frac{1}{\varepsilon} - i\rho)\right)} + O(\varepsilon). \tag{27}$$

Also, we discretization $x_{i+1} = (i + 1)h = ih + h$ From (26), we have

$$\lim_{h\to 0} y(ih + h) = y_0(0) + (\theta - y_0(1))e^{\left(-a(1)(\frac{1}{\varepsilon} - i\rho)\right)} + O(\varepsilon). \tag{28}$$

Similarly for $x_{i-1} = (i - 1)h = ih - h$, we have

$$\lim_{h\to 0} y(ih - h) = y_0(0) + (\theta - y_0(1))e^{\left(-a(1)(\frac{1}{\varepsilon} - i\rho)\right)} + O(\varepsilon). \tag{29}$$

Now we consider the second-order upwind finite difference scheme from (15). We have

$$\varepsilon\sigma(\rho)\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + a_i\left(\frac{y_i - y_{i-1}}{h}\right) + p_i y_i = g_i. \tag{30}$$

Multiplying (30) by h, considering h small, and truncating the term $(g(x_i) - p(x_i)y_i)h$, we get

$$\frac{\sigma}{\rho}\big(y_{i-1} - 2y_i + y_{i+1}\big) + a_i\big(y_i - y_{i-1}\big) = h(g_i - p_i y_i). \tag{31}$$

By evaluating the limit of (31) as h approaches to zero and using $\lim_{h\to 0} h(g_i - p_i y_i) = 0$, we get

$$\sigma = -a(1)\rho \frac{\lim_{h\to 0}(y_i - y_{i-1})}{\lim_{h\to 0}(y_{i-1} - 2y_i + y_{i+1})}. \tag{32}$$

Substituting (27), (28), and (29) into (32) for solving the fitting parameter σ and rearranging it, we obtain

$$\sigma_2 = \frac{\rho a(1)}{4}\left[\frac{(e^{-(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\rho)} - 1)}{\sinh^2[(\frac{a^2(1)-\varepsilon b(1)}{2a(1)}\rho)]}\right]. \tag{33}$$

Case (1): For the left layer, from the discrete form of (4) on the domain $[1, \frac{N}{2}]$, we have

$$\varepsilon^2 y_i'' + \varepsilon a_i y_i' - b_i y_i = R_i. \tag{34}$$

Using the central and forward finite difference formula on (34), we have

$$\varepsilon^2\sigma_1(\rho)\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \varepsilon a_i\left(\frac{y_{i+1} - y_i}{h}\right) - b_i y_i = R_i.$$

Hence, the required finite difference scheme becomes

$$\left(\frac{\varepsilon^2\sigma_1(\rho)}{h^2}\right)y_{i-1} + \left(\frac{-2\varepsilon^2\sigma_1(\rho)}{h^2} - \varepsilon\frac{a_i}{h} - b_i\right)y_i + \left(\frac{\varepsilon^2\sigma_1(\rho)}{h^2} + \varepsilon\frac{a_i}{h}\right)y_{i+1} = R_i. \tag{35}$$

The numerical scheme in (35) can be written as a three-term recurrence relation by

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, \qquad 1 \le i \le \frac{N}{2}, \tag{36}$$

where

$$E_i = \left(\frac{\varepsilon^2\sigma_1(\rho)}{h^2}\right), \qquad F_i = \left(\frac{-2\varepsilon^2\sigma_1(\rho)}{h^2} - \varepsilon\frac{a_i}{h} - b_i\right),$$
$$G_i = \left(\frac{\varepsilon^2\sigma_1(\rho)}{h^2} + \varepsilon\frac{a_i}{h}\right), \qquad H_i = R_i.$$

Case (2): For the right layer, from the discrete form of (4) on the domain $(\frac{N}{2}, N-1]$, we have

$$\varepsilon^2 y_i'' + \varepsilon a_i y_i' - b_i y_i = R_i. \tag{37}$$

Using the central and backward finite difference formula on (37), we have

$$\varepsilon^2\sigma_2(\rho)\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \varepsilon a_i\left(\frac{y_i - y_{i-1}}{h}\right) - b_i y_i = R_i.$$

Hence, the required finite difference scheme becomes

$$\left(\frac{\varepsilon^2\sigma_2(\rho)}{h^2}-\varepsilon\frac{a(x_i)}{h}\right)y_{i-1}+\left(\frac{-2\varepsilon^2\sigma_2(\rho)}{h^2}+\varepsilon\frac{a_i}{h}-b_i\right)y_i+\left(\frac{\varepsilon^2\sigma_2(\rho)}{h^2}\right)y_{i+1}=R_i. \tag{38}$$

The numerical scheme in (38) can be written as a three-term recurrence relation by

$$E_iy_{i-1}+F_iy_i+G_iy_{i+1}=H_i,\qquad \frac{N}{2}<i\le N-1, \tag{39}$$

where

$$E_i=\left(\frac{\varepsilon^2\sigma_2(\rho)}{h^2}-\varepsilon\frac{a_i}{h}\right),\qquad F_i=\left(\frac{-2\varepsilon^2\sigma_2(\rho)}{h^2}+\varepsilon\frac{a_i}{h}-b_i\right),$$
$$G_i=\left(\frac{\varepsilon^2\sigma_2(\rho)}{h^2}\right),\qquad H_i=R_i.$$

Next, in order to treat the boundary conditions, the following equations are obtained: For $i=1$, (36) becomes

$$F_1y_1+G_1y_2=H_1-E_1A. \tag{40}$$

For $i=N-1$, (36) becomes

$$E_{N-1}y_{N-2}+F_{N-1}y_{N-1}+G_{N-1}y_N=H_{N-1}. \tag{41}$$

From (6) and (41), we obtain

$$E_{N-1}y_{N-2}+F_{N-1}y_{N-1}+G_{N-1}\phi(y_{\frac{N}{2}})=H_{N-1}. \tag{42}$$

Therefore, the problem in (1) with given boundary conditions in (2)–(3), can be solved using the scheme (36), (39), (40), and (42), which forms an $N\times N$ system of algebraic equations.

5 Uniform convergence analysis

In this section, we need to show the discrete scheme in (36) and (39) satisfies the discrete minimum principle and uniform convergence.

Lemma 3 (Discrete Minimum Principle)**.** Let Y_i be any mesh function that satisfies $Y_0\ge 0, Y_N\ge 0$ and $L_\varepsilon Y_i\le 0$, $i=1,2,3,\dots,N-1$. Then $Y_i\ge 0$, for $i=0,1,2,\dots,N$.

Proof. The proof is by contradiction. Let j be such that $Y_j=\min_i Y_i$, and suppose that $Y_j\le 0$. Clearly, $j\notin\{0,N\}$. $Y_{j+1}-Y_j\ge 0$ and $Y_j-Y_{j-1}\le 0$.

Therefore,

$$\begin{aligned} L_\varepsilon^N Y_j =& \varepsilon\left(\frac{Y_{j+1}-2Y_j+Y_{j-1}}{h^2}\right)+a_j\left(\frac{Y_{j+1}-Y_j}{h}\right)+p_jY_j,\\ =&\frac{\varepsilon}{h^2}(Y_{j+1}-2Y_j+Y_{j-1})+\frac{a_j}{h}(Y_{j+1}-Y_j)+p_jY_j,\\ =&\frac{\varepsilon}{h^2}((Y_{j+1}-Y_j)-(Y_j-Y_{j-1}))+\frac{a_j}{h}(Y_{j+1}-Y_j)+p_jY_j,\\ \geq& 0, \end{aligned} \tag{43}$$

where the strict inequality holds if $Y_{j+1}-Y_j>0$. This is a contradiction, and therefore $Y_j \geq 0$. Since j is arbitrary, we have $Y_i \geq 0, \quad \text{for} \quad i = 0,1,2,\ldots,N$. □

We proved above that the discrete operator L_ε^N satisfies the minimum principle. Next, we analyze the uniform convergence analysis.

Theorem 1. Let $y(x_i)$ and Y_i be, respectively, the exact solution of (9) and numerical solutions of (20). Then for sufficiently large N, the following parameter uniform error estimate holds:

$$|L^N(y(x_i)-Y_i)| \leq Ch\left(1+\varepsilon^{-4}\max_{1\leq i\leq N-1}\left(\exp\left(\frac{-a(l-x_i)}{\varepsilon}\right)\right)\right). \tag{44}$$

Proof. Let us consider the local truncation error defined as

$$L^N(y(x_i)-Y_i)=\varepsilon\sigma_2(\rho)(y''(x_i)-D^+D^-y(x_i))+a(x_i)(y'(x_i)-D^-y(x_i)), \tag{45}$$

where $\varepsilon\sigma_2(\rho)=\frac{\rho a(1)}{4}\left[\frac{(e^{-(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\rho)}-1)}{\sinh^2[(\frac{a^2(1)-\varepsilon b(1)}{2a(1)}\rho)]}\right]$, since $\rho=\frac{h}{\varepsilon}$.

For fixed $N=\frac{1}{h}$, taking the limit for $\varepsilon\to 0$, we obtain

$$\lim_{\varepsilon\to 0}\varepsilon\sigma(\rho)=\lim_{\varepsilon\to 0}\frac{\rho a(1)}{4}\left[\frac{(e^{-(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\rho)}-1)}{\sinh^2[(\frac{a^2(1)-\varepsilon b(1)}{2a(1)}\rho)]}\right]=Ch,$$

where C is constant independent of h and ε.

Using Taylor series expansion, the bound for $y(x_{i-1})$ and $y(x_{i+1})$ at x_i is obtained as

$$\begin{cases} y(x_{i-1})=y(x_i)-hy'(x_i)+\frac{h^2}{2!}y''(x_i)-\frac{h^3}{3!}y^{(3)}(x_i)+\frac{h^4}{4!}y^{(4)}(x_i)+O(h^5),\\ y(x_{i+1})=y(x_i)+hy'(x_i)+\frac{h^2}{2!}y''(x_i)+\frac{h^3}{3!}y^{(3)}(x_i)+\frac{h^4}{4!}y^{(4)}(x_i)+O(h^5). \end{cases}$$

We obtain the bound for

$$\begin{cases} |D^+D^-y(x_i)|\leq C|y''(x_i)|,\\ |y''(x_i)-D^+D^-y(x_i)|\leq Ch^2|y^{(4)}(x_i)|. \end{cases} \tag{46}$$

Similarly, for the first derivative term, we have

$$|y'(x_i) - D^+ y(x_i)| \le Ch|y''(x_i)|, \tag{47}$$

where $|y^{(k)}(x_i)| = \sup_{x_i\in(x_0,x_N)} |y^{(k)}(x_i)|, \quad k = 2, 3, 4.$

Using the bounds in (46) and (47), we obtain

$$|L^N(y(x_i) - Y_i) \le Ch^3|y^{(4)}(x_i)| + C|y''(x_i)|.$$

Now, using the bounds for the derivatives of the solution in Lemma (2), we have

$$\begin{aligned}|L^N(y(x_i) - Y_i)| &\le Ch^3 \left(1 + \varepsilon^{-4} \exp\left(\frac{-a(l-x_i)}{\varepsilon}\right)\right)\\ &\quad + Ch \left(1 + \varepsilon^{-2} \exp\left(\frac{-a(l-x_i)}{\varepsilon}\right)\right)\\ &\le Ch \left(1 + \varepsilon^{-4} \max_{1\le i\le N-1} \exp\left(\frac{-a(l-x_i)}{\varepsilon}\right)\right)\end{aligned}$$

since $\varepsilon^{-4} \ge \varepsilon^{-2}$. □

Lemma 4. For a fixed mesh and for $\varepsilon \to 0$, it holds

$$\lim_{\varepsilon\to 0} \max_{1\le i\le N-1} \frac{\exp\left(\frac{-ax_i}{\varepsilon}\right)}{\varepsilon^m} = 0, \quad m = 1, 2, 3, \dots,$$

$$\lim_{\varepsilon\to 0} \max_{1\le i\le N-1} \frac{\exp\left(\frac{-a(1-x_i)}{\varepsilon}\right)}{\varepsilon^m} = 0, \quad m = 1, 2, 3, \dots.$$

Proof. Consider the partition $[0, 1] := \{0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1\}$ for the interior grid points, we have

$$\max_{1\le i\le N-1} \frac{\exp\left(\frac{-ax_i}{\varepsilon}\right)}{\varepsilon^m} \le \frac{\exp\left(\frac{-ax_1}{\varepsilon}\right)}{\varepsilon^m} = \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m},$$

$$\max_{1\le i\le N-1} \frac{\exp\left(\frac{-a(1-x_i)}{\varepsilon}\right)}{\varepsilon^m} \le \frac{\exp\left(\frac{-a(1-x_{N-1})}{\varepsilon}\right)}{\varepsilon^m} = \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m},$$

as $x_1 = 1 - x_{N-1} = h$. The repeated application of L'Hospital's rule gives

$$\lim_{\varepsilon\longrightarrow 0} \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m} = \lim_{r=\frac{1}{\varepsilon}\to\infty} \frac{r^m}{\exp(ahr)} = \lim_{r=\frac{1}{\varepsilon}\to\infty} \frac{m!}{(ah)^m \exp(ahr)} = 0.$$

□

Theorem 2. Let $y(x_i)$ and Y_i be the exact solution of (9) and numerical solutions of (20), respectively. Then, the following error bound holds

$$\sup_{0<\varepsilon<<1} |(y(x_i) - Y_i)| \leq Ch. \tag{48}$$

Proof. By substituting the results in Lemma (4) into Theorem (1) and applying the discrete minimum principle, we obtain the required bound. □

Richardson extrapolation

Here, we apply the Richardson extrapolation technique to accelerate the rate of convergence of the proposed scheme. Richardson Extrapolation is a convergence acceleration technique, which involves a combination of two computed approximations of solution. From (48), we have

$$y(x_i) - Y_i \leq Ch, \tag{49}$$

where $y(x_i)$ and Y_i are exact and approximate solutions, respectively, and C is constant independent of ε and h. Let Ω^{2h} be the mesh obtained by bisecting each mesh interval in Ω^h and denote the approximation of the solution on Ω^{2h} by Y_i^{2h}. From (49), we have

$$y(x_i) - Y_i \cong Ch + R^h, \quad x_i \in \Omega^N, \tag{50}$$

So, this works for any $\frac{h}{2} \neq 0$ gives

$$y(x_i) - {Y_i}^{2h} \cong C\frac{h}{2} + R^{2h}, \quad x_i \in \Omega^{2h}, \tag{51}$$

where the remainders, R^h and R^{2h} are $O(h^2)$. Equations (50) and (51) lead to

$$y(x_i) - (2Y_i^{2h} - Y_i) \cong Ch^2,$$

which gives that

$$Y_i^{ext} = 2Y_i^{2h} - Y_i \tag{52}$$

is also an approximate solution. The total truncation error for the approximate solution in (52) becomes

$$\sup_{0<\varepsilon<<1} |(y(x_i) - Y_i)| \leq Ch^2. \tag{53}$$

6 Numerical example and results

To validate the established theoretical results, we perform numerical experiments using the model problems of the form in (1)–(3).

Example 1. Consider the model singularly perturbed boundary value problem

$$\varepsilon^2 y''(x) + \varepsilon(1 - \frac{x}{2})y'(x) - (1 + x^2 + 2y - \exp(-y)) = 0, \quad 0 < x < 1,$$

subject to the boundary conditions

$$y(0) = 0, \quad \phi(y) = \cos(\frac{\pi y}{4}) + 2, \quad l_1 = \frac{1}{2}.$$

Having $y_j \equiv y_j^h$ (the approximated solution obtained via fitted operator finite difference method) for different values of h and ε, the maximum errors. Since the exact solution is not available, the maximum errors (denoted by E_ε^h) are evaluated using the double mesh principle for fitted operator finite difference methods using formula

$$E_\varepsilon^N := \max_{0 \le j \le N} |y_j^h - y_{2i}^{2h}|.$$

Furthermore, we will tabulate the ε-uniform error

$$E^N = \max_{0 < \varepsilon \le 1} E_\varepsilon^N.$$

The numerical rate of convergence and the ε-uniform rate of convergence are computed using the formulas

$$R^N = \frac{\log(E^N) - \log(E^{2N})}{\log(2)}.$$

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-4}	3.3238e-03	8.4445e-04	2.1268e-04	5.3355e-05	1.3361e-05
10^{-8}	3.3238e-03	8.4445e-04	2.1268e-04	5.3355e-05	1.3361e-05
10^{-12}	3.3238e-03	8.4445e-04	2.1268e-04	5.3355e-05	1.3361e-05
10^{-16}	3.3238e-03	8.4445e-04	2.1268e-04	5.3355e-05	1.3361e-05
10^{-20}	3.3238e-03	8.4445e-04	2.1268e-04	5.3355e-05	1.3361e-05
E^N	3.3238e-03	8.4445e-04	2.1268e-04	5.3355e-05	1.3361e-05
R^N	1.9767	1.9893	1.9950	1.9976	

Table 1: Maximum absolute errors and rates of convergence for Example 1.

	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
Present					
E^N	3.3238e-03	8.4445e-04	2.1268e-04	5.3355e-05	1.3361e-05
R^N	1.9767	1.9893	1.9950	1.9976	
Method in [8]					
E^N	0.01388073	0.00955129	0.00550127	0.00314426	0.00176922
R^N	0.69	0.75	0.81	0.91	

Table 2: Comparison of maximum absolute errors and rate of convergence for Example 1 at different number of mesh points.

	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
After					
E^N	3.3238e-03	8.4445e-04	2.1268e-04	5.3355e-05	1.3361e-05
R^N	1.9767	1.9893	1.9950	1.9976	
Before					
E^N	6.1832e-02	3.2578e-02	1.6711e-02	8.4619e-03	4.2576e-03
R^N	0.9245	0.9631	0.9817	0.9909	

Table 3: Maximum absolute errors and rate of convergence before and after Richardson extrapolation for Example 1 at different number of mesh points.

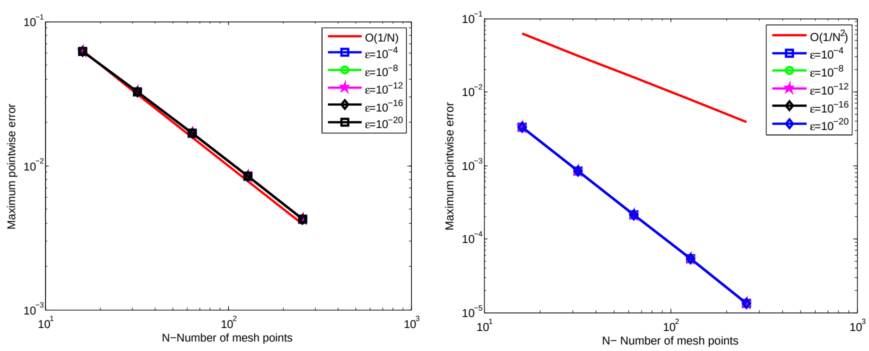

Figure 1: ε-uniform convergence of exponentially fitted operator method in log-log scale before and after Richardson Extrapolation, respectively.

7 Discussion and conclusion

This study introduced an accelerated exponentially fitted finite difference numerical method for solving singularly perturbed semilinear differential equations with nonlocal boundary conditions. The behavior of the continuous solution of the problem was studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. The numerical scheme was developed on the uniform mesh using exponential fitted operator method in the given differential equation. The stability of the

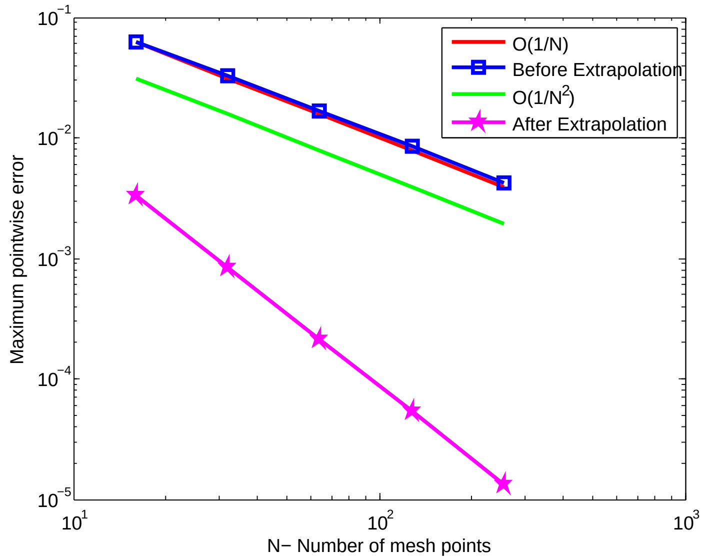

Figure 2: The ε-uniform convergence of the method in log-log scale before and after extrapolation at the same plot for Example 1.

developed numerical method was established, and its uniform convergence was proved. To validate the applicability of the method, a model problem was considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results were tabulated in terms of maximum absolute errors, numerical rate of convergence, and uniform errors (see Table 1) and to show the performance of our scheme after Richardson extrapolation, we compared both the maximum absolute error and the rate of convergence of the scheme before and after Richardson extrapolation (see Table 3). To visualize our result more, we plotted the log-log plot of maximum absolute error before and after Richardson in different plane and in the same plane in Figures 1 and 2, respectively. The method was shown to be ε-uniformly convergent with the order of convergence $O(h^2)$. The performance of the proposed scheme was investigated by comparing it with prior study (see Table 2). The proposed method gave more accurate, stable, and ε-uniform numerical results.

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