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# Computing the eigenvalues of fourth order Sturm-Liouville problems with Lie Group method

H. Mirzaei*

**Abstract**

In this paper, we formulate the fourth order Sturm-Liouville problem (FSLP) as a Lie group matrix differential equation. By solving this matrix differential equation by Lie group Magnus expansion, we compute the eigenvalues of the FSLP. The Magnus expansion is an infinite series of multiple integrals of Lie brackets. The approximation is, in fact, the truncation of Magnus expansion and a Gaussian quadrature are used to evaluate the integrals. Finally, some numerical examples are given.

**Keywords:** Lie group method; Fourth order Sturm-Liouville problem; Magnus expansion.

## 1 Introduction

A classical fourth order Sturm-Liouville equation, is a real fourth order linear differential equation of the form

$$(p(x)y''(x))'' - (q_1(x)y'(x))' + q_2(x)y(x) = \lambda w(x)y(x), \tag{1}$$

where $p(x), q_1(x), q_2(x)$ and $w(x)$ are given continuous functions on the finite interval $[a, b]$. Usually the equation (1) isaccompanied with four boundary conditions of the form

$$A_1u(a) + A_2v(a) = 0, \quad B_1u(b) + B_2v(b) = 0, \tag{2}$$

where $u = [y, y'],\ v = [py'', (py'')' + q_1y']$, $A_1, A_2, B_1$ and $B_2$ are real matrices of order 2, such that

*Corresponding author
Received 3 March 2015; revised 9 March 2016; accepted 28 May 2016
H. Mirzaei
Faculty of Basic Sciences, Sahand University of Technology, Tabriz, Iran.
e-mail: h_ mirzaei@sut.ac.ir

$$A_1A_2^T = A_2A_1^T, \quad B_1B_2^T = B_2B_1^T,$$

and the matrices $(A_1 : A_2)$ and $(B_1 : B_2)$ have rank 2, see [8, 9]. The equation (1) with boundary conditions (2) is called fourth-order Sturm-Liouville problem (FSLP). If $\lambda$ is such that the FSLP has a nontrivial solution, then $\lambda$ is called an eigenvalue and nontrivial solution for corresponding to $\lambda$ is called an eigenfunction. The set of all eigenvalues of FSLP called the spectrum. We assume that the interval $(a, b)$ is finite and coefficient functions $r = \frac{1}{p}, w, q_1, q_2$ are real and belong to $L^1(a, b)$. The classical results of self-adjoint Sturm-Liouville problem states that under this assumptions the eigenvalues are bounded below and can be ordered as

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots, \tag{3}$$

where $\lim_{k\to\infty} \lambda_k = \infty$, see [8–10].

FSLP arise in mathematical modeling of motion for the transversely vibrating Beam. In fact, the equation (1) is the standard form of Euler-Bernoulli Beam equation, see [7]. Moan [15] approximates the eigenvalues of second order Sturm-Liouville problems using Lie group methods. Ledoux and et al. [17] introduced a modified Magnus method for numerical solution of one dimensional Schrodinger eigenvalue problem. In [1] the Homotopy analysis method (HAM) is applied to numerically approximate the eigenvalues of the second and fourth order Sturm-Liouville problem. For further schemes to study FSLP, we refer to [2, 5, 10].

Let $u_1 = y,\ u_2 = y',\ u_3 = py'',\ u_4 = (py'')' - q_1y'$. Then we have the matrix representation of equation (1) and boundary conditions (2):

$$\begin{cases} U' = G(x)U \\ AU(a) + BU(b) = 0, \end{cases} \tag{4}$$

where

$$G(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & r(x) & 0 \\ 0 & q_1(x) & 0 & 1 \\ \lambda w(x) - q_2(x) & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

$$U = [u_1, u_2, u_3, u_4],\ A = \begin{pmatrix} A_1 & A_2 \\ \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix},\ B = \begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 \\ B_1 & B_2 \end{pmatrix}.$$

The paper is organized as follows. In Section 2, we introduce Lie group and Magnus expansion. In Section 3, we present the numerical procedure of Magnus expansion for solving matrix differential equations. In Section 4, we apply the Magnus expansion for computing eigenvalues of FSLP. In Section 5, some numerical examples are given.

## 2 Magnus expansion

The concept of Lie group was first proposed by Sophus Lie in 1893 to study transformation groups [13]. A Lie group $(G, .)$ is a differentiable manifold that also has the algebraic structure of a group. For a given Lie group $G$, the tangent space at identity is called the Lie algebra of $G$ and denoted by $\mathbf{g}$. A binary operation $[.,.] : \mathbf{g} \times \mathbf{g} \to \mathbf{g}$ is called the Lie bracket. Let $x, y \in \mathbf{g}$, then the Lie bracket $[x, y]$ is defined as

$$[x, y] = xy - yx. \tag{6}$$

The operator (6) is bilinear, anti symmetric and satisfy the Jacobi identity,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \tag{7}$$

for all $x, y, z \in \mathbf{g}$. The Lie algebra $\mathbf{g}$ is a vector space under Lie bracket (6). An important special case of Lie group is the Special Lie group $(SL(F, n))$. Let $F^{n\times n}$ be the set of all $n \times n$ matrices with entries in $F$. The set

$$SL(F, n) = \{y \in F^{n\times n}, \det y = 1\}. \tag{8}$$

The usual matrix multiplication has the structure of a Lie group and is called the Special Linear group. The corresponding Lie algebra $sl(F, n)$ of this group is the set of all $n \times n$ matrices with zero trace. The Lie bracket in this case is $[A, B] = AB - BA$, for $A, B \in sl(F, n)$. For more details see [4, 11, 16, 22].

The Lie group is the differentiable manifold, therefor we can study ordinary differential equations on a Lie group. A system of linear ordinary differential equations on $SL(F, n)$ have the form [11, 22]

$$y' = G(x)y, \quad tr(G(x)) = 0 \longleftrightarrow \det y = 1. \tag{9}$$

Iserles and Nörset [11] studied the solution of the linear matrix differential equation by the Lie group. In [22] the author introduce the collocation and Runge kutta type methods for the Lie group and present algebraic formulae for coefficients of these methods. Ros et al. [3], studied the Magnus expansion and some of its application in linear system of differential equations. The Magnus expansion represents the solution of (9) in the form

$$y(x) = e^{\sigma(x)} y_0 \tag{10}$$

where $y_0$ is the initial data of the problem (9) and

$$\begin{aligned}\sigma(x) = &\int_a^x G(s)ds + \frac{1}{2}\int_a^x \int_a^s [G(s), G(u)]duds \\ &+ \frac{1}{4}\int_a^x \int_a^s \int_a^u [G(s), [G(u), G(v)]]dvduds\end{aligned}$$

$$+ \frac{1}{12}\int_a^x \int_a^s \int_a^s [[G(s), G(u)], G(v)]dvduds + ... \tag{11}$$

In practice, we have to consider only a finite number of terms in expansion. The following theorem denotes the order of this approximation:

**Theorem 1.** *(Adapted from Theorem 6 in [11]) Let* $\|G(h)\| = O(h^k)$*, then*

$$\|\sigma(h) - \sum_{i=1}^{p} \sigma_i(h)\| = O(h^{(p+1)(k+1)+1}), \tag{12}$$

*where*

$$\begin{aligned}
\sigma_1 &= \int_0^h G(s)ds,\\
\sigma_2 &= \frac{1}{2}\int_0^h \int_0^s [G(s), G(u)]duds,\\
\sigma_3 &= \frac{1}{4}\int_0^h \int_0^s \int_0^u [G(s), [G(u), G(v)]]dvduds\\
&\quad + \frac{1}{12}\int_0^h \int_0^s \int_0^s [[G(s), G(u)], G(v)]dvduds,
\end{aligned}$$

*Moreover, if* $\overline{\sigma}$ *is an approximation of* $\sigma$ *such that* $\|\sigma - \overline{\sigma}\| \leq O(h^q)$ *then*

$$\|e^{\sigma(h)} - e^{\overline{\sigma}(h)}\| \leq O(h^q).$$

In this paper we suppose that $p = 3$, and approximate $\sigma(h)$ as follows

$$\begin{aligned}
\sigma(h) \simeq \overline{\sigma}(h) = &\int_0^h G(s)ds + \frac{1}{2}\int_0^h \int_0^s [G(s), G(u)]duds\\
&+ \frac{1}{4}\int_0^h \int_0^s \int_0^u [G(s), [G(u), G(v)]]dvduds\\
&+ \frac{1}{12}\int_0^h \int_0^s \int_0^s [[G(s), G(u)], G(v)]dvduds.
\end{aligned} \tag{13}$$

We explain the error of Magnus expansion for FSLP by the following Remarks.

**Remark 1.** By Theorem 1, if $\| G(h) \| = O(1)$ then the numerical Magnus series scheme produces an (at least) local order $O(h^5)$ approximation of $\sigma$. A Magnus series integrator with local order $p$ will have global order $p - 1$ [12], thus Theorem 1 guarantees (at least) error of order $O(h^4)$ (if $\| G(h) \| = O(1)$).

**Remark 2.** In FSLP the coefficient matrix $G$ is also a function of eigenvalue

$\lambda$ and for large eigenvalues, the norm of matrix $G$ is : $\|G\| = O(\lambda)$. Thus for large eigenvalues, the error is of order $O(\lambda h^4)$.

## 3 Numerical procedure

The first step for computing the solution $y(x) = e^{\sigma(x)}y_0$ by Magnus series is the truncation of the series (11). As mentioned before we use only four terms (13). Next, we apply a numerical integration method for approximating the integrals (13). Here we use the two point Gaussian integration method. Thus we have

$$\begin{aligned}
&\int_0^h G(x)dx \simeq \tfrac{h}{2}(G(c_1h) + G(c_2h)),\\
&\int_0^h \int_0^k [G(k), G(\xi)]d\xi dk \simeq -\tfrac{\sqrt{3}}{6}h^2[G(c_1h), G(c_2h)],\\
&\tfrac{1}{4}\int_0^h \int_0^s \int_0^u [G(s), [G(u), G(v)]]dvduds \qquad (14)\\
&+\tfrac{1}{12}\int_0^h \int_0^s \int_0^s [[G(s), G(u)], G(v)]dvduds\\
&\simeq \tfrac{h^3}{80}[[G(c_1h), G(c_2h)], G(c_1h) - G(c_2h)],
\end{aligned}$$

Substituting (14) in (13) we obtain

$$\begin{aligned}
\overline{\sigma}(h) = &\frac{h}{2}(G(c_1h) + G(c_2h)) - \frac{\sqrt{3}}{12}h^2[G(c_1h), G(c_2h)]\\
&+ \frac{h^3}{80}[[G(c_1h), G(c_2h)], G(c_1h) - G(c_2h)] \qquad (15)
\end{aligned}$$

where $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$ are the nods of Gaussian quadrature, see [11]. Finally by Matlab software we compute $y(h) \simeq e^{\overline{\sigma}(h)}y_0$. For computing $y(x)$ in interval $[a, b]$ we divide the interval $[a, b]$ into $n$ subintervals with length $h = \frac{b-a}{n}$ and use the following iterative process

$$Y(x_i) = e^{\sigma(x_i)}Y(x_{i-1}), \quad Y(a) = Y_0, \quad x_i = a + ih. \qquad (16)$$

## 4 Eigenvalues of fourth order Sturm-Liouville problem

In this section, we formulate the fourth order Sturm-Liouville problem as the Lie group matrix differential equation and find the eigenvalues. For solving the system (4) and finding the eigenvalues of FSLP, we consider the matrix differential equation

$$Y' = G(x)Y, \quad Y(a) = I_4. \tag{17}$$

**Theorem 2** [14]. *If $\Phi(x)$ is a solution of the matrix differential equation $X' = G(x)X$ on an interval $J$ and if $\tau$ is any point of $J$, then*

$$\forall \tau, x \in J, \det \Phi(x) = \det \Phi(\tau) exp[\int_\tau^x tr(G(s))ds]. \tag{18}$$

From (5) it is clear that $tr(G(x)) = 0$, and by Theorem 2 for any solution $Y(x)$ of system (17) we have

$$\det Y(x) = \det Y(a) exp[\int_a^x tr(G(s))ds] = 1.$$

Thus system (17) is a system of matrix differential equations on the Lie Group $SL(\mathbb{R}^4, 4)$. Therefore we can solve the system (17) by Magnus expansion. Let $Y(x)$ be a solution of (17) for $x \geq a$, then the solution of the system (4) with initial condition $U(a)$ is

$$U(x) = Y(x)U(a). \tag{19}$$

Thus, $U(b) = Y(b)U(a)$. Following $U(b)$ in the boundary condition (4) we obtain

$$AU(a) + BY(b)U(a) = 0 \Longrightarrow [A + BY(b)]U(a) = 0.$$

The initial condition $U(a)$ is nonzero if $U(a) = 0$, then we have the trivial solution $U = 0$, thus determinant of the coefficient matrix $A + BY(b)$ must be zero. This determinant is a function of $\lambda$ and the eigenvalues of the FSLP are roots of equation:

$$F(\lambda) := \det(A + BY(b)) = 0. \tag{20}$$

Equation (20) is the main equation of this paper. In fact, this equation is the characteristic equation of the fourth order Sturm-Liouville problem. For solving system (17) and computing $Y(b)$ by Magnus expansion, we need the brackets in (15). For FSLP these brackets are as follows

$$[G(s), G(u)] =$$

$$\begin{pmatrix} 0 & 0 & r(u)-r(s) & 0 \\ 0 & r(s)q_1(u)-r(u)q_1(s) & 0 & r(s)-r(u) \\ 0 & 0 & r(u)q_1(s)-r(s)q_1(u) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \{\lambda(w(u) - w(s)) + (q_2(s) - q_2(u))\} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$[[G(s), G(u)], G(s) - G(u)] =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2(r(s)-r(u))(r(s)q_1(u)-r(u)q_1(s)) & 0 \\ 0 & 2(q_1(s)-q_1(u))(r(u)q_1(s)-r(s)q_1(u)) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ (r(u) - r(s))(q_1(s) - q_1(u)) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ 2(r(s) - r(u))\{\lambda(w(s) - w(u)) + (q_2(u) - q_2(s))\} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For computing $Y(x_i)$ and finally $Y(b)$ for finding the characteristic equation (20), we should compute $e^{\sigma(x_i)}$ which $\sigma(x_i)$ is a matrix function of unknown parameter $\lambda$. In general, we can not obtain $e^{\sigma(x_i)}$ explicitly. But for any given real number $\lambda$, we can compute it by Matlab or Maple software. For solving this problem, we limit the parameter $\lambda$ into interval $[\lambda_0, \lambda^*]$ and find the eigenvalues of the FSLP in this interval. For this purpose, we divide interval $[\lambda_0, \lambda^*]$ into $m$ subintervals $\lambda_0 < \lambda_1 < ... < \lambda_m = \lambda^*$ with length $h' = \frac{\lambda^* - \lambda_0}{m}$ and consider the following algorithm:

1. $i = 0$,
2. Compute $Y(b)$ for $\lambda = \lambda_i$ with iterative process (8),
3. Compute $D_i = det(A + BY(b))$, if $D_i = 0$, then $\lambda_i$ is an eigenvalue, if $D_i D_{i+1} > 0$ go to step 4. If $D_i D_{i+1} < 0$, we have eigenvalue in interval $[\lambda_i, \lambda_{i+1}]$. By applying Bisection method for function $F(\lambda) = det(A + BY(b))$, we compute the eigenvalues in $[\lambda_i, \lambda_{i+1}]$.
4. $i = i + 1$, if $i \leq m$ go to step 2, if $i > m$ go to step 5,
5. End.

**Remark 3.** We can not obtain the explicit form of characteristic function $F(\lambda)$, but we can compute the values of $F(\lambda)$ for any given real number $\lambda_i$. On the other hand by (3) the roots of $F(\lambda)$ is simple hence the convergence of Bisection method is guaranteed. Thus in step 3 we use the Bisection method. Also, we can apply other numerical methods for computing the roots of $F(\lambda)$.

**Example 1.** Consider the fourth-order Sturm-Liouville problem

$$y^{(4)} - (0.02x^2y')' + (0.0001x^4 - .02)y(x) = \lambda y(x), \quad x \in [0, 5], \tag{21}$$

with the boundary conditions

$$y(0) = 0 = y''(0), \quad y(5) = 0 = y''(5). \tag{22}$$

It is easy to show that the eigenvalues of this problem are the square of the eigenvalues of second order Sturm-Liouville problem

$$\begin{aligned} &-u''(x) + 0.01x^2u(x) = \mu u(x), \quad x \in [0, 5], \\ &u(0) = 0 = u(5). \end{aligned} \tag{23}$$

The eigenvalues of problem (23) can be computed with Matslise [18]. We solve the problem (21) and (22) using Magnus expansion with $n = 500$ ($h = 0.01$) and approximate the eigenvalues in interval $[0, 650]$. In Table 1 we listed the first 8 eigenvalues and compared them with other methods.

In this example, for lower eigenvalues $\|G\| = O(1)$, thus by Theorem 1 and Remark 1, we have error of order (at least) $O(h^4)$. For large eigenvalues $\|G\| = O(\lambda)$ such that we must have error of order (at least) $O(\lambda h^4)$. The absolute errors in Table 1 confirm this order of error. If we compare the results with reference solution [18], it is obvious that our results is better than results of [1, 2, 5]. The Homotopy method [1], for eigenvalue $\lambda_5$ has large error. Also for eigenvalues $\lambda_6$ and $\lambda_7$ the errors on [2] are larger.

Table 1: Eigenvalues and absolute errors $\triangle\lambda_k$ of Example 1

| $\lambda_k$ Matslise [18] | $\triangle\lambda_k$ Our method | $\triangle\lambda_k$ [1] | $\triangle\lambda_k$ [2] | $\triangle\lambda_k$ [5] |
|---|---|---|---|---|
| 0.2150508644 | $2.9E-11$ | $4.4E-9$ | $7.0E-11$ | $2.8E-13$ |
| 2.7548099347 | $1.8E-10$ | $1.1E-6$ | $8.3E-11$ | $3.0E-12$ |
| 13.2153515406 | $2.1E-11$ | $4.4E-5$ | $5.8E-11$ | $4.2E-11$ |
| 40.9508197592 | $4.7E-10$ | $2.6E-3$ | $6.1E-11$ | $4.1E-7$ |
| 99.0534780635 | $5.2E-9$ | $3.5E-1$ | $7.5E-8$ | |
| 204.355732268 | $2.4E-7$ | $\underline{11.3}$ | $1.2E-3$ | |
| 377.430420689 | $7.6E-6$ | | $\underline{12.73}$ | |
| 642.590868170 | $3.4E-5$ | | $\underline{215.5}$ | |

**Example 2.** We consider problem (21) in Example 1 with the different boundary conditions

$$y(0) = 0 = y'(0), \quad y(5) = 0 = y'(5).$$

For the Magnus expansion result with $n = 500$, see Table 2. For this problem, we don't have the exact eigenvalues, but since the equation of this example is the same equation of the Example 1, we expect the same error order of Example 1. Also there is an agreement between the results of our method and the other methods in Table 2.

**Eexample 3.** Consider the following Clamped-Clamped Euler-Bernoulli beam:

Table 2: Eigenvalues of Example 2

| $\lambda_k$ Our method | $\lambda_k$ [5] | $\lambda_k$ [2] | $\lambda_k$ [19] | $\lambda_k$ [20] |
|---|---|---|---|---|
| 0.86690250 | 0.86690250 | 0.86690250 | 0.86690250 | 0.86690250 |
| 6.35768645 | 6.35768645 | 6.35768645 | 6.35768645 | 6.35768645 |
| 23.99274685 | 23.99274698 | 23.99274685 | 23.99274685 | 23.99274685 |
| 64.97866759 | 64.97863592 | 64.97866760 | 64.97866759 | 64.97866761 |
| 144.28062714 | | 144.28062804 | 144.28062684 | 144.28062693 |
| 280.60096225 | | 280.58602048 | 280.60096700 | 280.60096374 |
| 496.38280690 | | | | |

$$\begin{aligned} &((1+x)^{\frac{3}{2}}u''(x))'' = \lambda(1+x)^{\frac{-1}{2}}u(x), \quad 0 \le x \le 1, \\ &u(0) = 0 = u'(0), \quad u(1) = 0 = u'(1). \end{aligned} \tag{24}$$

Problem (24) has the same spectrum as the following uniform Euler-Bernoulli beam problem:

$$\begin{aligned} &v^{(4)}(s) = \lambda v(s), \quad 0 \le s \le L, \quad L = 2(\sqrt{2}-1), \\ &v(0) = 0 = v'(0), \quad v(L) = 0 = v'(L). \end{aligned} \tag{25}$$

We called that the problems (24) and (25) are isospectral. For isospectral beam equation see chapter 12 in [7] and papers [6, 21]. The eigenvalues of uniform beam (25) are the roots of the equation

$$cos(\beta)cosh(\beta) - 1 = 0, \quad \lambda = (\frac{\beta}{L})^4, \quad \beta \neq 0. \tag{26}$$

We compute the eigenvalues of problem (24) by Magnus expansion with $n = 300$ and compare them with exact solution of equation (26) in Table 3. This problem has large eigenvalues such that $\|G\| = O(\lambda)$. Thus the errors are of order $O(\lambda h^4)$ and the results in Table 3 confirm the error order of Theorem 1 and Remarks 1 and 2.

Table 3: Eigenvalues and absolute errors $\triangle\lambda_k$ of Example 3

| $k$ | $\lambda_k$ Our method | $\lambda_k$ Exact | $\triangle\lambda_k$ |
|---|---|---|---|
| 1 | 1062.777339630 | 1062.777339606 | $2.4E-8$ |
| 2 | 8075.518441201 | 8075.518441201 | $4.5E-10$ |
| 3 | 31035.57009892 | 31035.57010021 | $1.3E-6$ |
| 4 | 84807.08316091 | 84807.08315850 | $2.4E-6$ |
| 5 | 189248.7474113 | 189248.7474331 | $2.2E-5$ |

## 5 Conclusion

In this paper, a method based on the Magnus Lie group method developed, which can be used to compute the eigenvalues of the FSLP. Numerical examples were given to confirm the efficiency and accuracy of the method. Although the error grows for large eigenvalues, however the examples show that comparing to the other methods, our method has good accuracy for large eigenvalues. Example 3 shows that this algorithm is applicable for computing the eigenvalues of nonuniform Euler-Bernoulli beam equations. Ledoux [17], presented a modified Magnus method and obtained the large eigenvalues of second order Sturm- Liouville problem with error of order $O(\frac{1}{\lambda})$. It can be an open problem that one may modify the method of this paper in order to improve the results for higher index of eigenvalues.

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## References

1. Abbasbandy, S. and Shirzadi, A. *A new application of the homotopy analysis method: solving the Sturm-Liouville problems,* Commun Nonlinear Sci Numer Simulat, 16 (2011), 112-126.

2. Attili, B.S. and Lesnic, D. *An efficient method for computing eigenelements of Sturm-Liouville fourth-order boundary value problems,* Applied Mathematics and Computation, 182 (2006), 1247-1254.

3. Blanes, S., Casas, F., Oteo, J.A. and Ros, J. *The Magnus expansion and some of its applications,* Physics Reports, 470 (2009), 151-238.

4. Bluman, G.W., Kumei, S. *Symmetries and Differential Equations,* Springer, New York, 1989.

5. Chanane, B. *Accurate solutions of fourth order Sturm-Liouville problems,* Journal of Computational and Applied Mathematics, 234 (2010), 3064-3071.

6. Ghanbari, K. *On the isospectral beams,* Electronic Journal of Differential Equations, 12 (2005), 57-64.

7. Gladwell, G. M. L. *Inverse problems in vibrations,* Kluwer academic, New York, 2004.

8. Greenberg, L. and Marletta, M. *Numerical methods for higher order Sturm-Liouville problems,* Journal of Computational and Applied Mathematics, 125 (2000), 367-383.

9. Greenberg, L. and Marletta, M. *Oscillation theory and numerical solution of fourth order Sturm-Liouville problems,* IMA J. Numer.Anal., 15 (3)(1995), 319-356.

10. Greenberg, L. *A prufer method for calculating eigenvalues of selfadjoint systems of ordinary differential equations,* Parts 1 and 2, University of Maryland, Technical Report TR 91-24, 1991.

11. Iserles, A. and Norsett, S.P. *On the solution of linear differential equations in Lie groups,* Numerical Analysis Report, University of Cambridge, 1997.

12. Iserles, A. *On the global error of discretization methods for highly oscillatory ordinary differential equations,* BIT, 42 (2002), 561-599.

13. Lie, S. and Engel, F. *Theorie de Transformations gruppen,* Teubner, Leipzig, 1893.

14. Miller, K.R. and Michel, N.A. *Ordinary differential equations,* Academic Press, New York, 1982.

15. Moan, P.C. *Efficient approximation of Sturm-Liouville problems using Lie group methods,* Numerical Analysis Report, University of Cambridge, 1998.

16. Olver, P.J. *Applications of Lie Groups to Differential Equations,* Springer, New York, 1986.

17. Ledoux, V., Daele, M.V. and Berghe, G.V. *Efficient numerical solution of the 1D schrodinger problem using Magnus integrators,* IMA journal of Numerical Analysis, 30 (2010), 751-776.

18. Ledoux, V., Daele, M.V. and Berghe, G.V. *Matslise: A Matlab package for the Numerical Solution of Sturm-Liouville and Schrodinger equations,* ACM Transactions on Mathematical Software, 31 (2005), 532-554.

19. Taher, AHS., Malek, A., Masuleh SHM. *Chebychev differentiation matrices for efficient Computation of the eigenvalues of fourth order Sturm-Liouville problems,* Applied Mathematical Modeling, 37 (2013), 4634-4642.

20. Yucel, U. and Boubaker, K. *Differential Quadrature method (DQM) and Boubaker polynomials expansion scheme (BPES) for efficient computation of the eigenvalues of fourth order Sturm-Liouville problems,* Applied Mathematical Modelling, 36 (2012), 158-167.

21. Wafo Soh, C. *Isospectral Euler-Bernoulli beams via factorization and the Lie method,* International Journal of Non-Linear Mechanics, 44 (4)(2009), 396-403.

22. Zanna, A. *On the numerical solution of isospecyral flows,* Ph.D Thesis, University of Cambridge, 1998.

**محاسبه ی مقادیر ویژه ی مسائل اشتورم-لیوویل مرتبه ی چهار به روش گروه لی**

*حنیف میرزایی*

تبریز، دانشگاه صنعتی سهند، دانشکده علوم پایه، گروه ریاضی

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**چکیده :** در این مقاله، مسئله اشتورم-لیوویل مرتبه چهار را به صورت یک معادله دیفرانسیل ماتریسی در گروه لی فرمول بندی می کنیم. با حل این معادله ی ماتریسی به روش بسط مگنوس مقادیر ویژه ی مسئله محاسبه می گردند. بسط مگنوس یک سری نامتناهی از انتگرالهای چندگانه ی براکتهای لی می باشد. روش تقریب، قطع سری به تعداد متناهی جمله و سپس محاسبه ی انتگرالها به روش گاوس می باشد. در پایان چند مثال عددی آورده شده است.

**کلمات کلیدی :** روش گروه لی؛ مسئله اشتورم-لیوویل مرتبه ی چهار؛ بسط مگنوس.