Iranian Journal of Numerical Analysis and Optimization
Vol. 12, No. 2, pp 483–497
DOI: 10.22067/ijnao.2022.74627.1091
https://ijnao.um.ac.ir/

Research Article

A fourth-order method for solving singularly perturbed boundary value problems using nonpolynomial splines

S. Khan* and A. Khan

Abstract

In this paper, a class of second-order singularly perturbed interior layer problems is examined. A nonpolynomial mixed spline is used to develop the tridiagonal scheme. The developed method is second as well as fourth-order accurate based on the parameters. Error analysis is also carried out. The method is shown to converge point-wise to the true solution with higher accuracy. Linear and nonlinear second-order singularly perturbed boundary value problems have been solved by the presented method. Five numerical illustrations are given to demonstrate the applicability of the proposed method. Absolute errors are given in tables, which show that our method is more efficient than previously existing methods.

Keywords: Singularly perturbed, Second-order problems, Nonpolynomial splines, Convergence

AMS subject classifications (2020): Primary 65L10; Secondary 65D07

1 Introduction

Second-order singularly perturbed boundary-value problems (SPBVPs) have gained more importance for two reasons. Firstly, they occur in many areas of

* Corresponding author

Received 8 January 2022; revised 3 April 2022; accepted 4 April 2022

S. Khan
Department of Mathematics, Sri Venkateswara College, University of Delhi, New Delhi, India. e-mail: khan.shahana5@gmail.com

A. Khan
Department of Mathematics, Jamia Millia Islamia, New Delhi, India. e-mail: akhan1234in@rediffmail.com

science and engineering, like combustion, control theory, nuclear engineering, fluid mechanics, chemical reactor theory, and so on. Secondly, the occurrence of sharp boundary-layers as ϵ, the coefficient of the highest order derivative, approaches zero, makes it difficult for standard numerical methods. The approximate solution to boundary-value problems (BVPs), in which the highest derivative is multiplied with a small positive parameter, is described here. It is a well-known fact that the solution to the SPBVP exhibits a multiscale character. There are a number of methods available in the literature for solving SPBVPs [3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 18]. Some other recent methods for solving nonlinear problems are given in [15, 16, 17].
We consider second-order SPBVPs of the form

$$\epsilon z^{(2)}(x) = G(x, z, z'), \qquad r \leq x \leq s, \tag{1}$$

with

$$z(r) = \eta_0, \qquad z(s) = \eta_n, \tag{2}$$

where $0 < \epsilon \ll 1$ is a parameter multiplied by the highest derivative, $G(x, z, z')$ are continuously differentiable functions on $[r, s]$, and η_0 and η_n are constants.
Consider the following assumptions, for $r < x < s$ and $-\infty < z < z' < \infty$:
(i) G is continuous,
(ii)$\frac{\partial G}{\partial z}$ and $\frac{\partial G}{\partial z'}$ exist and are continuous,
(iii)$\frac{\partial G}{\partial z} > 0$ and $\mid \frac{\partial G}{\partial z} \mid \leq D$, for an arbitrary constant D.

The above assumptions from [11] confirm the existence and uniqueness of the solution to the BVP (1).
Here, we develop a method using a nonpolynomial spline for solving the generalized form of second-order SPBVPs. We use three-point finite difference approximations, which gives in place of give the accuracy of order four. When we implement the method, a tridiagonal system is obtained and solved using MATLAB.
The paper is organized as follows: In Section 2, the derivation of the numerical method is presented. The obtained scheme is applied to the second-order SPBVPs in Section 3. Error analysis of the method is discussed in Section 4. Numerical results are given in Section 5 to prove the efficiency of the proposed scheme. Finally, concluding remarks are presented.

2 Nonpolynomial mixed spline

Let the interval $[r,s]$ is divided into n equal parts as $r = x_0 < x_1 < x_2 < \cdots < x_n = s$, by introducing

$$x_i = r + ih, \quad where\ h = (s - r)/n, \quad i = 0, 1, \ldots, n.$$

Let

$$R_i(x) = a_i e^{\delta(x-x_i)} + b_i[\sin(\delta(x-x_i)) + \cos(\delta(x-x_i))] + c_i, \qquad (3)$$

be a function defined on $[r,s]$, which becomes an ordinary quadratic spline as parameter $\delta \longrightarrow 0$ and $\delta > 0$.

To evaluate $a_i, b_i,$ *and* c_i, we let

$$R_i(x_i) = z_i, \quad R_i(x_{i+1}) = z_{i+1}, \quad R_i^{(2)}(x_i) = \frac{1}{2}(S_i + S_{i+1}), \quad 0 \le i \le n. \qquad (4)$$

Using the above conditions, we get

$$\begin{aligned}
a_i &= \frac{z_{i+1} - z_i}{\zeta} - \frac{1}{2\theta^2}[\frac{e^\theta - 1}{\zeta} - 1]h^2(S_i + S_{i+1}),\\
b_i &= \frac{z_{i+1} - z_i}{\zeta} - \frac{1}{2\theta^2}[\frac{e^\theta - 1}{\zeta}]h^2(S_i + S_{i+1}),\\
c_i &= \frac{z_{i+1} + (2+\zeta)z_i}{\zeta} + \frac{1}{2\theta^2}[\frac{2e^\theta - 1}{\zeta} - 1]h^2(S_i + S_{i+1}),
\end{aligned}$$

where $\zeta = -2 + \sin\theta + \cos\theta + e^\theta$ and $\theta = \lambda h$.
Using the first derivative continuity condition, $R_{i-1}^k(x_i) = R_i^k(x_i),\ k = 0, 1,$ the following scheme is obtained:

$$\phi z_{i-1} + \xi z_i + \psi z_{i+1} = h^2(\phi_1 S_{i-1} + \xi_1 S_i + \psi_1 S_{i+1}), \qquad 1 \le i \le n-1, \quad (5)$$

where

$$\begin{aligned}
\phi &= \frac{\sin\theta + e^\theta + \cos\theta}{2},\\
\xi &= \frac{2 + \sin\theta - e^\theta - \cos\theta}{2},\\
\psi &= 1,\\
\phi_1 &= \frac{(2\sin\theta - 1)e^\theta + \cos\theta - \sin\theta}{4\theta^2},\\
\xi_1 &= \frac{\sin\theta e^\theta - \sin\theta}{2\theta^2},\\
\psi_1 &= \frac{e^\theta - \sin\theta - \cos\theta}{4\theta^2}.
\end{aligned}$$

Remark: Our scheme reduces to [2] when

$$(\phi, \xi, \psi, \phi_1, \xi_1, \psi_1) = (1, -2, 1, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}).$$

Error

After expanding (5), using the Taylor series, we obtain the truncation error (TE) as follows:

$$\begin{aligned} t_i = (\phi + \xi + \psi)z_i + (-\phi + \psi)hz_i' + h^2[\frac{\phi + \psi}{2!} - (\phi_1 + \xi_1 + \psi_1)]z_i^{(2)} \\ +h^3[\frac{-\phi + \psi}{3!} - (-\phi_1 + \psi_1)]z_i^{(3)} + h^4[\frac{\phi + \psi}{4!} - (\frac{\phi_1 + \psi_1}{2!})]z_i^{(4)} \\ +h^5[\frac{-\phi + \psi}{5!} - (\frac{-\phi_1 + \psi_1}{3!})]z_i^{(5)} + h^6[\frac{\phi + \psi}{6!} - (\frac{\phi_1 + \psi_1}{4!})]z_i^{(6)} \\ +h^7[\frac{-\phi + \psi}{7!} - (\frac{-\phi_1 + \psi_1}{5!})]z_i^{(7)} + O(h^8), \qquad 1 \le i \le n-1. \end{aligned} \tag{6}$$

For various values of coefficients obtained in (5), we obtain the second-order method as well as the fourth-order method. For $\phi_1 + \xi_1 + \psi_1 = 1$ and $\phi_1 = \psi_1$, we obtain the second-order method. The TE for $(\phi, \xi, \psi, \phi_1, \xi_1, \psi_1) = (1, -2, 1, 1/6, 4/6, 1/6)$ is given as follows:

$$t_i = -\frac{1}{12}h^4 z_i^{(4)} + O(h^5), \quad 1 \le i \le n-1.$$

For $(\phi, \xi, \psi, \phi_1, \xi_1, \psi_1) = (1, -2, 1, 1/12, 10/12, 1/12)$, we obtain a higher order method, that is, fourth-order. The TE is as follows:

$$t_i = -\frac{1}{120}h^6 z_i^{(6)} + O(h^7), \quad 1 \le i \le n-1.$$

3 Implementation of the scheme

We consider the second-order SPBVP of the generalized form (1) as

$$\epsilon z^{(2)}(x) = G(x, z, z'), \qquad r \le x \le s.$$

In particular, we take the linear second-order SPBVP as follows:

$$\epsilon z^{(2)}(x) = -p(x)z'(x) - q(x)z(x) + g(x), \qquad r \le x \le s. \tag{7}$$

Now, implementing the scheme (5) on problem (7), we get the method, which is of second-order accuracy as follows:

$$\epsilon\phi z_{i-1} + \epsilon\xi z_i + \epsilon\psi z_{i+1} = h^2(\phi_1 G_{i-1} + \xi_1 G_i + \psi_1 G_{i+1}), \quad 1 \le i \le n-1, \tag{8}$$

where $\phi_1 = \psi_1$ and $\phi_1 + \xi_1 + \psi_1 = 1$. The method of fourth-order accuracy is given as follows:

$$\epsilon\phi z_{i-1} + \epsilon\xi z_i + \epsilon\psi z_{i+1} = h^2(\phi_1 G_{i-1} + \xi_1 \tilde{G}_i + \psi_1 G_{i+1}), \quad 1 \le i \le n-1, \tag{9}$$

where $(\phi_1, \xi_1, \psi_1) = (1/12, 10/12, 1/12)$ and

$$\begin{aligned} G_i &= G(x, z_i, z_i'), \\ G_{i-1} &= G(x, z_{i-1}, z_{i-1}'), \\ G_{i+1} &= G(x, z_{i+1}, z_{i+1}'), \\ \tilde{G}_i &= G(x, z_i, \tilde{z}_i'), \end{aligned}$$

in which

$$\begin{aligned} z_i' &= \frac{z_{i+1} - z_{i-1}}{2h}, \\ z_{i-1}' &= \frac{-3z_{i-1} + 4z_i - z_{i+1}}{2h}, \\ z_{i+1}' &= \frac{z_{i-1} - 4z_i + 3z_{i+1}}{2h}, \\ \tilde{z}_i' &= \frac{z_{i+1} - z_{i-1}}{2h} - \frac{h}{20}(G_{i+1} - G_{i-1}). \end{aligned}$$

Using the above approximations, we obtain an expression for the second-order method as follows:

$$\begin{aligned} &(\phi\epsilon - \frac{3}{2}h\phi_1 p_{i-1} - \frac{1}{2}h\xi_1 p_i + \frac{1}{2}h\psi_1 p_{i+1} + h^2\phi_1 q_{i-1})z_{i-1} \\ &+ (\epsilon\xi + 2h\phi_1 p_{i-1} - 2h\psi_1 p_{i+1} + h^2\xi_1 q_i)z_i \\ &+ (\epsilon\psi - \frac{1}{2}h\phi_1 p_{i-1} + \frac{1}{2}h\xi_1 p_i + \frac{3}{2}h\psi_1 p_{i+1} + h^2\psi_1 q_{i+1})z_{i+1} \\ &\quad = h^2(\phi_1 g_{i-1} + \xi_1 g_i + \psi_1 g_{i+1}), \qquad 1 \le i \le n-1. \end{aligned} \tag{10}$$

Similarly for the fourth-order method, it reads as follows:

$$\begin{aligned} &\big[\epsilon\phi - \frac{3}{2}h(\phi_1 - \frac{\xi_1 h}{20}p_i)p_{i-1} - \frac{1}{2}h\xi_1 p_i + \frac{1}{2}h(\psi_1 + \frac{\xi_1 h}{20}p_i)p_{i+1} \\ &\quad + h^2(\phi_1 - \frac{\xi_1 h}{20}p_i)q_{i-1}\big]z_{i-1} \\ &+ \big[\epsilon\xi + 2h\phi_1 p_{i-1} - 2h(\psi_1 + \frac{\xi_1 h}{20}p_i)p_{i+1} + h^2\xi_1 q_i\big]z_i \\ &+ \big[\epsilon\psi - \frac{1}{2}h(\phi_1 - \frac{\xi_1 h}{20}p_i)p_{i-1} + \frac{1}{2}h\xi_1 p_i + \frac{3}{2}h(\psi_1 + \frac{\xi_1 h}{20}p_i)p_{i+1} \\ &\quad + h^2(\psi_1 + \frac{\xi_1 h}{20}p_i)q_{i+1}\big]z_{i+1} \\ &\qquad = h^2\big[(\phi_1 - \frac{\xi_1 h}{20}p_i)g_{i-1} + \xi_1 g_i + (\psi_1 + \frac{\xi_1 h}{20}p_i)g_{i+1}\big], \qquad 1 \le i \le n-1. \end{aligned} \tag{11}$$

4 Convergence

Here, we discuss the error analysis of higher order method, that is, fourth-order. In the matrix form, (11) can be written as

$$MZ = U, \tag{12}$$

where $M = (m_{i,j})$ is the tridiagonal matrix whose entries are given by (11) and $U = [u_1, u_2, \ldots, u_{n-1}]^T$ is given as

$$\begin{aligned}
u_1 =& h^2[(\phi_1 - \frac{h\xi_1}{20}p_1)g_0 + \xi_1 g_1 + (\psi_1 + \frac{h\xi_1}{20}p_1)g_2] \\
&- (\epsilon\phi - \frac{3}{2}h(\phi_1 - \frac{h\xi_1}{20}p_1)p_0 - \frac{1}{2}h\xi_1 p_1 + \frac{1}{2}h(\psi_1 + \frac{h\xi_1}{20}p_1)p_2 \\
&+ h^2(\phi_1 - \frac{h\xi_1}{20}p_1)q_0)y_0, \\
u_i =& h^2[(\phi_1 - \frac{h\xi_1}{20}p_i)g_{i-1} + \xi_1 g_i + (\psi_1 + \frac{h\xi_1}{20}p_i)g_{i+1}], \qquad 2 \le i \le n-2, \\
u_{n-1} =& h^2[(\phi_1 - \frac{h\xi_1}{20}p_{n-1})g_{n-2} + \xi_1 g_{n-1} + (\psi_1 + \frac{h\xi_1}{20}p_{n-1})g_n] \\
&- (\epsilon\psi - \frac{1}{2}h(\phi_1 - \frac{h\xi_1}{20}p_{n-1})p_{n-2} + \frac{1}{2}h\xi_1 p_{n-1} + \frac{3}{2}h(\psi_1 + \frac{\xi_h}{20}p_{n-1})p_n \\
&+ h^2(\psi_1 + \frac{h\xi_1}{20}p_{n-1})q_n)y_n,
\end{aligned}$$

and $Z = [z_1, z_2, \ldots, z_{n-1}]^T$. We also have

$$M\tilde{Z} = U + T, \tag{13}$$

where $\tilde{Z} = [\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{n-1}]^T$ and $T = [t_1, t_2, \ldots, t_{n-1}]^T$.
Also $t_i = [\tilde{z}_i - z_i]^T, i = 1, 2, \ldots, n-1$, is the local TE. From (12) and (13), we have

$$\begin{gathered}
M(\tilde{Z} - Z) = T, \\
ME = T, \\
E = \tilde{Z} - Z.
\end{gathered}$$

Now we calculate the sum of elements of each row of the matrix M as

$$Sum_i = \begin{cases} \epsilon\xi + \epsilon\psi + \frac{3}{2}h(\phi_1 - \frac{h\xi_1}{20}p_1)p_0 + \frac{1}{2}h\xi_1 p_1 \\ -\frac{1}{2}h(\psi_1 + \frac{h\xi_1}{20}p_1)p_2 + h^2\xi_1 q_1 + h^2(\psi_1 + \frac{h\xi_1}{20}p_1)q_2, \qquad i=1, \end{cases}$$

$$Sum_i = \begin{cases} \epsilon\phi + \epsilon\xi + \epsilon\psi \\ +h^2((\phi_1 - \frac{h\xi_1}{20}p_i)q_{i-1} + \xi_1 q_i + (\psi_1 + \frac{h\xi_1}{20}p_i)q_{i+1}), \qquad 2 \le i \le n-2, \end{cases}$$

$$Sum_i = \begin{cases} \phi\epsilon + \epsilon\xi + \frac{1}{2}h(\phi_1 - \frac{h\xi_1}{20}p_{n-1})p_{n-2} - \frac{1}{2}h\xi_1 p_{n-1} \\ -\frac{3}{2}h(\psi_1 + \frac{h\xi_1}{20}p_{n-1})p_n + h^2(\phi_1 - \frac{h\xi_1}{20}p_{n-1})q_{n-2} + h^2\xi_1 q_{n-1}, \quad i{=}n\text{-}1. \end{cases}$$

Furthermore,

$$\begin{aligned} &Sum_i \geq h^2(\xi_1 q_1 + \psi_1 q_2 - \frac{3}{40}\xi_1 p_0 p_1 - \frac{1}{40}\xi_1 p_1 p_2) + O(h^3), \quad i{=}1, \\ &Sum_i \geq h^2(\phi_1 q_{i-1} + \xi_1 q_i + \psi_1 q_{i+1}), \quad 2 \leq i \leq n-2, \\ &Sum_i \geq h^2(\phi_1 q_{n-2} + \xi_1 q_{n-1} - \frac{3}{40}\xi_1 p_n p_{n-1} - \frac{1}{40}\xi_1 p_{n-1} p_{n-2}) + O(h^3), \quad i{=}n\text{-}1. \end{aligned}$$

Let $0 < K \in Z^+$ be the minimum of $\mid p_i \mid$ and $\mid q_i \mid$. For sufficiently small h, we can say that

$$Sum_1 \geq h^2[(\xi_1 + \psi_1 - \frac{1}{10}\xi_1)K],$$

$$Sum_i \geq h^2[(\phi_1 + \xi_1 + \psi_1)K], \qquad 2 \leq i \leq n-2,$$

$$Sum_{n-1} \geq h^2[(\phi_1 + \xi_1 - \frac{1}{10}\xi_1)K].$$

Therefore, we get

$$\begin{aligned} &\frac{1}{Sum_1} \leq \frac{1}{h^2[(\frac{9}{10}\xi_1 + \psi_1)K]}, \\ &\frac{1}{Sum_i} \leq \frac{1}{h^2[(\phi_1 + \xi_1 + \psi_1)K]}, \qquad 2 \leq i \leq n-2, \\ &\frac{1}{Sum_{n-1}} \leq \frac{1}{h^2[(\phi_1 + \frac{9}{10}\xi_1)K]}. \end{aligned}$$

Furthermore,

$$\frac{1}{Sum_i} \leq \begin{cases} \frac{1}{h^2[(\frac{9}{10}\xi_1 + \psi_1)K]}, & i{=}1 \\ \frac{1}{h^2[(\phi_1 + \xi_1 + \psi_1)K]}, & 2 \leq i \leq n-2 \\ \frac{1}{h^2[(\phi_1 + \frac{9}{10}\xi_1)K]}, & i{=}n\text{-}1. \end{cases}$$

The matrix M is monotone, irreducible and diagonally dominant for suiciently small h. Therefore, M^{-1} exist and is positive. Hence,

$$\|E\| = \|M^{-1}\|\|T\|.$$

Let $M^{-1} = (m^*_{i,j})$. Then by [19], we get

$$\sum_{i=1}^{n-1} m^*_{i,j} Sum_i = 1, \qquad 1 \le j \le n-1.$$

Therefore,

$$m^*_{i,j} \le \frac{1}{Sum_i},$$
$$\|M^{-1}\| = \max_{1\le i\le n-1} \sum_{j=1}^{n-1} |m^*_{i,j}| \le \sum_{i=1}^{n-1} \frac{1}{Sum_i}$$
$$= [\frac{1}{h^2(\frac{9}{10}\xi_1 + \psi_1)K} + \frac{1}{h^2(\phi_1 + \xi_1 + \psi_1)K} + \frac{1}{h^2(\phi_1 + \frac{9}{10}\xi_1)K}],$$
$$\|T_i\| = \max_{1\le i\le n-1} \sum_{i=1}^{n-1} |T_i|, \qquad 1 \le i \le n-1.$$

Finally,

$$\|E\| = \|M^{-1}\|\|T\| \le \frac{1}{h^2K}[\frac{1}{\frac{9}{10}\xi_1 + \psi_1} + \frac{1}{\phi_1 + \xi_1 + \psi_1} + \frac{1}{\phi_1 + \frac{9}{10}\xi_1}]\|T\|.$$

As the fourth-order method, $\|T\| = O(h^6)$ using (6). Then

$$\|E\| \le \frac{1}{h^2K}[\frac{1}{\frac{9}{10}\xi_1 + \psi_1} + \frac{1}{\phi_1 + \xi_1 + \psi_1} + \frac{1}{\phi_1 + \frac{9}{10}\xi_1}]O(h^6) = O(h^4).$$

Hence, the error is of order four. For the second-order method, $\|T\| = O(h^4)$ using (6). Then $\|E\| = O(h^2)$. We can prove the convergence of the second-order method by following the above procedure.

5 Numerical illustrations and discussion

To check the applicability of the developed method to existing problems, we solve three linear and one nonlinear problem of the type (1). The maximum absolute errors (MAEs) are tabulated in Tables 1–8. We have also solved the heat flow problem in Example 5.

Example 1. Consider the nonlinear SPBVP from [18] as

$$\epsilon z^{(2)}(x) + 2z'(x) + e^{z(x)} = 0, \qquad 0 \le x \le 1, \tag{14}$$

with

$$z(0) = z(1) = 0.$$

The analytical solution is

$$z(x) = \ln\Big(2/(1+x)\Big) - e^{-2x/\epsilon}\ln 2.$$

The results of our method along with comparison are given in Table 1. Also we have tabulated the MAE in Table 2.

Table 1: Comparison of approximate solution of Example 1 with the method of [18] for $\epsilon = 10^{-3}$ and $n = 1000$

x	Presented method	[18]	Analytical solution
0.001	0.5933418	0.6913641	0.5983404
0.010	0.6835355	0.6825219	0.6831968
0.020	0.6736751	0.6726859	0.6733446
0.030	0.6639109	0.6629456	0.6635884
0.040	0.6542413	0.6532992	0.6539264
0.050	0.6446647	0.6437448	0.6443570
0.100	0.5981091	0.5972949	0.5978370
0.300	0.4309488	0.4304523	0.4307829
0.500	0.2877780	0.2874905	0.2876821
0.700	0.1625668	0.1624234	0.1625189
0.900	0.0513068	0.0512663	0.0512933
1.000	0.0000000	0.0000000	0.0000000

Table 2: MAE of Example 1 for $(\phi_1, \xi_1, \psi_1) = (1/12, 10/12, 1/12)$

$n \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}
16	3.010×10^{-2}	5.450×10^{-2}	3.950×10^{-2}	3.208×10^{-1}
32	2.900×10^{-2}	2.010×10^{-2}	8.300×10^{-2}	1.275×10^{-1}
64	2.870×10^{-2}	1.600×10^{-2}	4.430×10^{-2}	3.280×10^{-2}
128	2.860×10^{-2}	1.550×10^{-2}	1.310×10^{-2}	7.760×10^{-2}
256	2.860×10^{-2}	1.540×10^{-2}	5.300×10^{-2}	4.150×10^{-2}

Example 2. Consider the linear SPBVP from [14, 18] as

$$\epsilon z^{(2)}(x) + z'(x) = 1 + 2x, \qquad 0 \le x \le 1, \tag{15}$$

with

$$z(0) = 0, \qquad z(1) = 1.$$

The analytical solution is

$$z(x) = x(x + 1 - 2\epsilon) + \frac{(2\epsilon - 1)(1 - e^{-x/\epsilon})}{(1 - e^{-1/\epsilon})}.$$

The results of our method have been compared with the methods of [14, 18] in Table 3. We have also tabulated the MAE in Table 4.

Table 3: Comparison of the approximate solution of Example 2 with the methods of [14, 18] for $\epsilon = 10^{-3}$ and $n = 1000$

x	[18]	[14]	Presented method	Analytical solution
0.001	-1.0009970	-0.6311195	-0.6293169	-0.6298573
0.010	-0.9918800	-0.9898546	-0.9878740	-0.9878747
0.020	-0.9815600	-0.9796000	-0.9776400	-0.9776400
0.030	-0.9710400	-0.9691000	-0.9671600	-0.9671600
0.040	-0.9603199	-0.9584000	-0.9564800	-0.9564800
0.050	-0.9493999	-0.9475000	-0.9456000	-0.9456000
0.100	-0.8918000	-0.8900000	-0.8882000	-0.8882000
0.300	-0.6114000	-0.6100000	-0.6086000	-0.6086000
0.500	-0.2510000	-0.2500000	-0.2490000	-0.2490000
0.700	0.1894000	0.1900000	0.1906000	0.1906000
0.900	0.7098000	0.7099999	0.7102000	0.7102000
1.000	1.0000000	1.0000000	1.0000000	1.0000000

Example 3. Consider the two-parameter SPBVP from [20] as

$$-\epsilon_d z^{(2)}(x) + \epsilon_c z'(x) + z(x) = \cos(\pi x), \qquad 0 \leq x \leq 1, \tag{16}$$

with

$$z(0) = 0, \qquad z(1) = 0.$$

The analytical solution is

$$z(x) = A\cos(\pi x) + B\sin(\pi x) + Ce^{\lambda_1 x} + De^{-\lambda_2(1-x)},$$

where

$$A = \frac{\epsilon_d \pi^2 + 1}{\epsilon_c^2 \pi^2 + (\epsilon_d \pi^2 + 1)^2}, \qquad B = \frac{\epsilon_c \pi}{\epsilon_c^2 \pi^2 + (\epsilon_d \pi^2 + 1)^2},$$

Table 4: MAE of Example 2 for $(\phi_1, \xi_1, \psi_1) = (1/12, 10/12, 1/12)$

$n \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}
16	8.8950×10^{-7}	4.7391×10^{-4}	5.6800×10^{-2}	4.6880×10^{-1}
32	5.5531×10^{-8}	2.8359×10^{-5}	7.3000×10^{-3}	2.2370×10^{-1}
64	3.4736×10^{-9}	1.7529×10^{-6}	5.2469×10^{-4}	5.8100×10^{-2}
128	2.1708×10^{-10}	1.0925×10^{-7}	3.1398×10^{-5}	7.5000×10^{-3}
256	1.3640×10^{-11}	6.8234×10^{-6}	1.9407×10^{-6}	5.3738×10^{-4}

Table 5: Comparison of the approximate solution of Example 3 with the method of [20] for $n = 128$

ϵ_c	[20]	Presented method	[20]	Presented method
	$\epsilon_d = 10^{-2}$	$\epsilon_d = 10^{-2}$	$\epsilon_d = 10^{-4}$	$\epsilon_d = 10^{-4}$
10^{-3}	9.2993×10^{-7}	2.6109×10^{-8}	1.3294×10^{-3}	2.7811×10^{-4}
10^{-4}	1.1557×10^{-7}	2.5996×10^{-8}	3.6708×10^{-4}	2.7239×10^{-4}
10^{-5}	3.4933×10^{-8}	2.5984×10^{-8}	2.8085×10^{-4}	2.7148×10^{-4}
10^{-6}	2.6878×10^{-8}	2.5983×10^{-8}	2.7232×10^{-4}	2.7138×10^{-4}
10^{-7}	2.6072×10^{-8}	2.5982×10^{-8}	2.7147×10^{-4}	2.7139×10^{-4}

$$C = -A\frac{1+e^{-\lambda_2}}{1-e^{\lambda_1-\lambda_2}}, \qquad D = A\frac{1+e^{\lambda_1}}{1-e^{\lambda_1-\lambda_2}},$$
$$\lambda_1 = \frac{\epsilon_c - \sqrt{\epsilon_c^2+4\epsilon_d}}{2\epsilon_d}, \qquad \lambda_2 = \frac{\epsilon_c + \sqrt{\epsilon_c^2+4\epsilon_d}}{2\epsilon_d}.$$

The results of our method along with comparison are given in Table 5.

Example 4. Consider the SPBVP from [14, 18] as

$$\epsilon z^{(2)}(x) + z'(x) - z(x) = 0, \qquad 0 \le x \le 1, \tag{17}$$

with

$$z(0) = 1, \qquad z(1) = 1.$$

The analytical solution is

$$z(x) = \frac{(e^{m_4}-1)e^{m_3 x} + (1-e^{m_3})e^{m_4 x}}{e^{m_4} - e^{m_3}},$$

where

$$m_3 = \frac{-1+\sqrt{1+4\epsilon}}{2\epsilon}, \quad m_4 = \frac{-1-\sqrt{1+4\epsilon}}{2\epsilon}.$$

The results of our method have been compared with methods [14, 18] in Table 6. We have also tabulated the MAE in Table 7.

Example 5. Consider the heat flow problem of viscous incompressible fluid over a stretching plate from [1] as

$$z^{(2)}(x) + P_r[1-e^{-x}]z'(x) + \epsilon[z(x)z^{(2)}(x) + [z'(x)]^2] = 0, \quad 0 \le x \le 1, \tag{18}$$

with

$$z(0) = 1, \qquad z(1) = 0,$$

Table 6: Comparison of approximate solution of Example 4 with the methods of [14, 18] for $\epsilon = 10^{-3}$ and $n = 1000$

x	[18]	[14]	Presented method	Analytical solution
0.001	0.6003270	0.6007916	0.6011354	0.6007918
0.010	0.3712379	0.3716054	0.3719728	0.3719724
0.020	0.3749439	0.3753111	0.3756784	0.3756784
0.030	0.3787160	0.3790830	0.3794502	0.3794502
0.040	0.3825260	0.3828929	0.3832599	0.3832599
0.050	0.3863742	0.3867410	0.3871079	0.3871079
0.100	0.4062043	0.4065697	0.4069350	0.4069350
0.300	0.4962382	0.4965853	0.4969323	0.4969323
0.500	0.6062278	0.6065307	0.6068334	0.6068334
0.700	0.7405963	0.7408182	0.7410400	0.7410400
0.900	0.9047471	0.9048374	0.9049277	0.9049277
1.000	1.0000000	1.0000000	1.0000000	1.0000000

Table 7: MAE of Example 4 for $(\phi_1, \xi_1, \psi_1) = (1/12, 10/12, 1/12)$

$n \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}
16	1.9745×10^{-6}	4.1185×10^{-4}	3.8100×10^{-2}	2.9880×10^{-1}
32	1.2290×10^{-7}	2.4523×10^{-5}	5.0000×10^{-3}	1.4300×10^{-1}
64	7.6864×10^{-9}	1.5138×10^{-6}	3.6087×10^{-4}	3.7300×10^{-2}
128	4.8037×10^{-10}	9.4320×10^{-8}	2.1567×10^{-5}	4.8000×10^{-3}
256	3.0033×10^{-11}	5.9003×10^{-9}	1.3326×10^{-6}	3.4706×10^{-4}

where P_r is the Prandtl number, which can take different values. Problem (18) in [1] is perturbed by the small parameter ϵ with the first part consisting of the terms independent of ϵ and the second part involving ϵ. Here, we take the first part of (18) as

$$z^{(2)}(x) + P_r[1 - e^{-x}]z'(x) = 0, \qquad 0 \le x \le 1, \tag{19}$$

with

$$z(0) = 1, \qquad z(1) = 0.$$

The MAE of (19) is calculated using the double mesh principle due to the unavailability of an exact solution. The MAEs are given in Table 8 for P_r=1.5.

Table 8: MAE of Example 5

n	16	32	64	128	256	512	1024
Second-order method for $(\phi_1, \xi_1, \psi_1) = (1/6, 4/6, 1/6)$	2.1451×10^{-5}	2.7598×10^{-6}	3.4967×10^{-7}	4.3996×10^{-8}	5.5172×10^{-9}	6.9075×10^{-10}	8.6421×10^{-11}
Fourth-order method for $(\phi_1, \xi_1, \psi_1) = (1/12, 10/12, 1/12)$	7.1156×10^{-6}	9.5771×10^{-7}	1.2409×10^{-7}	1.5788×10^{-8}	1.9909×10^{-9}	2.4996×10^{-10}	3.1305×10^{-11}

6 Conclusion

In this paper, we solved second-order SPBVPs using a tridiagonal scheme obtained by a nonpolynomial spline. The presented method is based on a lower degree spline that could reduce computational time and is more feasible working. The error obtained in the developed scheme (5) is of order two as well as four depending upon the choice of parameters. Point-wise convergence and MAE are given in Tables 1–8. Here, linear, nonlinear, and two-parameter perturbed problems are being solved. Comparison with the result of [14, 18, 20] was given in Tables 1,3,5, and 6, which proves the efficiency of the method and also shows that our method is better than these existing methods.

References

1. Ahmad, N., Siddiqui, Z.U. and Mishra, M.K., *Boundary layer flow and heat transfer past a stretching plate with variable thermal conductivity,* Int. J. Non-Linear Mech. 45 (2010) 306–309.

2. Al-Said, E.A., *Smooth spline solution for a system of second order boundary value problems,* J. Nat. Geometry 16 (1999) 19–28.

3. Ascher, U.M., Matheij, R.M.M. and Russell, R.D., *Numerical solution of boundary value problems for ordinary differential Equations,* (Englewood Cliffs, NJ: Prentice-Hall) 1988.

4. Ayalew, M., Kiltu, G.G. and Duressa, G.F., *Fitted numerical scheme for second-order singularly perturbed differential-difference equations with mixed shifts,* Abstr. Appl. Anal. Volume 2021, Article ID 4573847, 11 pages.

5. Aziz, T. and Khan, A., *A spline method for second-order singularly perturbed boundary-value problems,* J. Comput. Appl. Math. 147 (2002) 445–452.

6. Chekole, A.T., Duressa, G.F. and Kiltu, G.G., *Non-polynomial septic spline method for singularly perturbed two point boundary value problems of order three,* J. Taibah Univ. Sci. 13 (2019) 651–660.

7. El-Zahar, E.R., *Approximate analytical solution of singularly perturbed boundary value problems in MAPLE,* AIMS Mathematics, 5 (2020) 2272–2284.

8. Gupta, R., Bhagyamma, D. and SharathBabu, K., *Numerical solution of singularly perturbed two-point boundary value problem using transformation technique using quadrature method,*Turk. J. Comput. Math. Educ. 12 (2021) 2084–2091.

9. Kadalbajoo, M.K. and Bawa, R.K., *Variable mesh difference scheme for singularly-perturbed boundary value problems using splines,* J. Optim. Theory Appl. 90 (1996) 405–416.

10. Kadalbajoo, M.K. and Reddy, Y.N., *Asymptotic and numerical analysis of singular perturbation problems: A survey,* Appl. Math. Comput. 30 (1989) 223–259.

11. Keller, H.B., *Numerical methods for two point boundary value problems*, Blaisdell Publications Co., New York, 1968.

12. Khan, A., Khan I., Aziz, T. and Stojanovic, M., *A Variable-Mesh Approximation Method for Singularly Perturbed Boundary-Value Problems using Cubic Spline in Tension,* Int. J. Comput. Math. 81 (2004) 1513–1518.

13. Kiltu, G.G., Duressa, G.F. and Bullo, T.A., *Computational method for singularly perturbed delay differential equations of the reaction-diffusion type with negative shift,* 'J. Ocean Eng. Sci. 6 (2021) 285–291.

14. Kumar, M., Mishra, H.K. and Singh, P., *A boundary value approach for a class of linear singularly perturbed boundary value problems,* Adv. Eng. Softw. 10(2009) 298–304.

15. Nikan, O., Avazzadeh, Z. and Rasoulizadeh, M.N., *Soliton solutions of the nonlinear sine-Gordon model with Neumann boundary conditions arising in crystal dislocation theory,* Nonlinear Dyn. 106 (2021) 783–813.

16. Rasoulizadeh, M.N., Ebadi, M.J., Avazzadeh, Z. and Nikan, O., *An efficient local meshless method for the equal width equation in fluid mechanics,* Eng. Anal. Bound. Elem. 131 (2021) 258–268.

17. Rasoulizadeh, M.N., Nikan, O. and Avazzadeh, Z., *The impact of LRBF-FD on the solutions of the nonlinear regularized long wave equation,* Math. Sci. 15 (2021) 365–376.

18. Reddy, Y.N. and Chakravarthy, P.P., *An initial-value approach for solving singularly perturbed two-point boundary value problems,* Appl. Math. Comput. 155 (2004) 95–110.

19. Varga, R.S., *Matrix iterative analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1962.

20. Zahra, W.K. and Mhlawy, A.M.El. *Numerical solution of two-parameter singularly perturbed boundary value problems via exponential spline,* J. King Saud Univ. Sci. 25 (2013) 201–208.

How to cite this article

S. Khan and A. Khan A fourth-order method for solving singularly perturbed boundary value problems using nonpolynomial splines. *Iranian Journal of Numerical Analysis and Optimization*, 2022; 12(2): 483-497. doi: 10.22067/ijnao.2022.74627.1091.