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Research Article

Numerical method for solving fractional Sturm–Liouville eigenvalue problems of order two using Genocchi polynomials

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Abstract

A new numerical scheme based on Genocchi polynomials is constructed to solve fractional Sturm–Liouville problems of order two in which the fractional derivative is considered in the Caputo sense. First, the differential equation with boundary conditions is converted into the corresponding integral equation form. Next, the fractional integration and derivation operational matrices for Genocchi polynomials, are introduced and applied for approximating the eigenvalues of the problem. Then, the proposed polynomials are applied to approximate the corresponding eigenfunctions. Finally, some examples are presented to illustrate the efficiency and accuracy of the numerical method. The results show that the proposed method is better than some other approximations involving orthogonal bases.

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1 Introduction

The purpose of solving the Sturm–Liouville problem is to find the eigenvalues and eigenfunctions of a set of equations, which have many applications in mathematics and physics. The description of the vibrations of a string or a quantum mechanical oscillator is modeled as Sturm–Liouville equations [2, 3, 9]. For more information about the application of Sturm–Liouville problems, we refer to [4, 5, 9, 10, 14] and references therein. With the advent of fractional calculus [8, 20, 24], the use of fractional derivatives in Sturm–Liouville equations led to a new class of equations, which are known as fractional Sturm–Liouville problems. Various types of fractional order derivatives are defined [11, 12, 18, 21, 33], but most researchers use Caputo's concept for Sturm–Liouville equations because Caputo fractional derivative is more compatible with practical application in physics and engineering [22, 23, 25, 26, 30]. The analytical solution to Sturm–Liouville problems is usually not computable. This problem has limited the application of these equations in various fields. In simpler cases, the analytical solutions to this equation can be expressed in terms of specific functions, such as Mittag-Leffler functions, which have their own complexities in terms of calculations. So, in most cases, scientists seek to find numerical methods to solve Sturm–Liouville problems.

In this study, we consider the following fractional Sturm–Liouville problem:

$$ {}_0^C D_x^{\alpha} y(x) + (\lambda r(x) - q(x)) y(x) = 0, \quad 0 \leq x \leq 1, \tag{1} $$

where $q(x)$ and $r(x)$ are real value continuous functions and $r(x) \neq 0$ for $x \in [0,1]$, $1 < \alpha \leq 2$, and ${}_0^C D_x^{\alpha}$ denotes the fractional Caputo derivative and the boundary conditions are as follows:

$$ \begin{cases} ay(0) + by'(0) = 0, \\ cy(1) + dy'(1) = 0, \end{cases} \tag{2} $$

where $a,\ b,\ c$ and d are real constants and $a^2 + b^2 \neq 0$, $c^2 + d^2 \neq 0$.

The authors in [2] established sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problems for fractional differential equations involving the Caputo fractional derivative. Also, the existence and uniqueness of the solution for a fractional Sturm–Liouville boundary value problems based on the Banach fixed point theorem were proved in [15]. We also refer to [6, 13, 34] for more studies on the existence and uniqueness of the solution for boundary value problems of types (1) and (2).

A wide range of numerical methods has been used to solve problems (1) and (2) [1, 4, 5, 7, 18, 19]. The Laplace transform method is applied to convert (1)–(2) to the equivalent integral equation with a weakly singular kernel in [31]. Then, the authors applied a piecewise Lagrange integration method to solve the corresponding integral equation numerically. Inspired by their work, we apply the Genocchi polynomials approximation method to the problem (1)–(2).

Genocchi polynomials, which were introduced in [32], are very important and useful polynomials. These polynomials share some great advantages with Bernoulli and Euler polynomials for approximating an arbitrary smooth function [32]. Genocchi polynomials were applied for solving integer-order delay differential equations [16], fractional optimal control problems [28], and fractional pantograph equation [16]. The numerical solutions obtained by Genocchi polynomials are comparable or even more accurate compared to some well-known existing methods. In this study, motivated by these advantages, we define and successfully apply the operational matrices of Genocchi polynomials to approximate the eigenvalues and the corresponding eigenfunctions of the Sturm–Liouville problem (1) and (2). First, a new approach for calculating the fractional derivative, integration, and product operational matrices is introduced, and then the operational matrices are applied for problems (1) and (2). The structure of the paper is as follows: In Section 2, we recall some definitions and results related to fractional calculus. In Section 3, we construct the new numerical method. Some illustrative examples are provided in Section 4. Finally, the conclusion is given in Section 5.

2 Preliminaries

In this part, the definition of fractional calculus, Genocchi polynomials, and their attributes are explained.

2.1 Fractional calculus

The Riemann–Liouville fractional integral ${}_0I_x^\alpha$ of order $0 \leq \alpha < 1$ is presented with (see [29])

$$ {}_0I_x^\alpha u(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s) ds, & \alpha > 0, \\ u(x), & \alpha = 0. \end{cases} \tag{3} $$

One of the fundamental attributes of the operator ${}_0I_x^\alpha$ is

$$ {}_0I_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} x^{\beta+\alpha}. \tag{4} $$

The Caputo fractional derivative of order $\alpha > 0$ is determined as [29]

$$ {}_{0}^{C}D_{x}^{\alpha}u(x) = {}_{0}I_{x}^{n-\alpha}\frac{d^{n}}{dx^{n}}u(x) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}(x-s)^{n-\alpha-1}u^{(n)}(s)ds, \qquad (5) $$

where n is an integer $(n-1<\alpha\leq n)$ and $u^{(n)}\in L^{1}[0,1]$.

The main relationship between the Riemann–Liouville integral operator and Caputo fractional derivative is as follows:

$$ \begin{aligned} &{}_{0}^{C}D_{x}^{\alpha}\ {}_{0}I_{x}^{\alpha}\ u(x) = u(x),\\ &{}_{0}I_{x}^{\alpha}\ {}_{0}^{C}D_{x}^{\alpha}\ u(x) = u(x) - \sum_{r=0}^{n-1}u^{(r)}(0^{+})\frac{x^{r}}{r!} \qquad (n-1<\alpha\leq n), \end{aligned} \qquad (6) $$

where $u^{(r)}\in L^{1}[0,1], \quad r=0,1,\ldots,n-1.$

2.2 Genocchi polynomials and properties

The Genocchi numbers g_n and the Genocchi polynomials $G_n(x)$ are defined as the coefficients of the exponential generating functions as follows:

$$ \frac{2t}{e^{t}+1} = \sum_{n=0}^{\infty} g_{n}\frac{t^{n}}{n!}, \qquad |t|<\pi, \qquad (7) $$

$$ \frac{2te^{xt}}{e^{t}+1} = \sum_{n=0}^{\infty} G_{n}(x)\frac{t^{n}}{n!}, \qquad |t|<\pi. \qquad (8) $$

The Genocchi polynomial $G_n(x)$ is a polynomial given by

$$ G_{n}(x) = \sum_{k=0}^{n}\gamma_{k}^{n}x^{k}, \quad \gamma_{k}^{n} = \binom{n}{k}g_{n-k}, \quad n=1,2,\ldots. \qquad (9) $$

The Genocchi polynomials have interesting properties, some of which are as follows:

$$ \int_{0}^{1}G_{n}(x)G_{m}(x)dx = \frac{2(-1)^{n}n!m!}{(m+n)!}g_{m+n}, \quad n,m\geq 1, \qquad (10) $$

$$ \frac{d}{dx}G_{n}(x) = nG_{n-1}(x), \quad n\geq 1, \qquad (11) $$

$$ G_{n}(1)+G_{n}(0) = 0, \quad n>1. \qquad (12) $$

The set of $Y=\{G_1(x),G_2(x),\ldots,G_N(x)\}\subset L^2[0,1]$ is a linearly independent set (see[16]). Any $f(x)\in L^2[0,1]$ has a unique best approximation in $Span(Y)$, as $f_N(x)$, which can be represented by Genocchi polynomials as follows:

$$f(x) \simeq f_N(x) = \sum_{n=1}^{N} c_n G_n(x) = C^T G(x), \tag{13}$$

where $C = [c_1, c_2, \ldots, c_N]^T$ is the coefficient vector and $G = [G_1(x), G_2(x), \ldots, G_n(x)]^T$ is the Genocchi polynomials vector. The property of the best approximation requires

$$\langle f(x), G(x) \rangle^T = C^T \langle G(x), G(x) \rangle,$$

and then

$$C^T = \langle f(x), G(x) \rangle^T \mathbf{W}^{-1}, \tag{14}$$

where

$$\langle f(x), G(x) \rangle = [\langle f(x), G_1(x) \rangle, \langle f(x), G_2(x) \rangle, \ldots, \langle f(x), G_N(x) \rangle]^T,$$

and $\mathbf{W} = \langle G(x), G(x) \rangle$ is an $N \times N$ symmetric matrix, and by (10) its entries are calculated as follows:

$$\mathbf{W}_{nm} = \langle G_n(x), G_m(x) \rangle = \int_0^1 G_n(x) G_m(x) dx = \frac{2(-1)^n n! m!}{(m+n)!} g_{m+n}. \tag{15}$$

The following lemma provides the upper bound for the error of function approximation by Genocchi polynomials.

Lemma 1. Suppose that $f(x) \in C^{N+1}[0,1]$ is approximated by truncated Genocchi polynomials $f_N(x)$ in (13). Then

$$\|f(x) - f_N(x)\|_2 \le \frac{R}{(N+1)!\sqrt{2N+3}}, \tag{16}$$

where $R = \max_{x \in [0,1]} \left| f^{(N+1)}(x) \right|$.

Proof. We refer the reader to [17]. □

2.3 Genocchi operational matrices

By (11), the derivative of $G(x)$ can be expressed as follows:

$$\frac{d}{dx} G(x) = \mathbf{D} G(x), \tag{17}$$

where

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 3 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & N-1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & N & 0 \end{bmatrix}, \tag{18}$$

is called the operational matrix of integration for Genocchi polynomials. It is obvious that

$$\frac{d^n}{dx^n} G(x) = \mathbf{D}^n G(x). \tag{19}$$

The operational matrix for fractional order derivative of order α in Caputo sense is defined as follows:

$${}_0^C D_x^\alpha G(x) = \mathbf{D}^\alpha G(x), \tag{20}$$

where $\mathbf{D}^\alpha$ is an $N \times N$ matrix. By using (14), we can write

$$\mathbf{D}^\alpha = \mathbf{Q}^\alpha \mathbf{W}^{-1}, \tag{21}$$

where $\mathbf{Q}^\alpha = \langle D^\alpha G(x), G(x) \rangle$ is an $N \times N$ matrix whose entries are defined by

$$\begin{aligned} \mathbf{Q}_{ij}^\alpha &= \langle D^\alpha G_i(x), G_j(x) \rangle \\ &= \left\langle \sum_{k=n}^{i} \frac{\gamma_k^i k!}{\Gamma(k+1-\alpha)} x^{k-\alpha}, \sum_{r=0}^{j} \gamma_r^j x^r \right\rangle \\ &= \sum_{k=n}^{i} \sum_{r=0}^{j} \frac{\gamma_k^i \gamma_r^j k!}{\Gamma(k+1-\alpha)} \int_0^1 x^{k+r-\alpha} dx \\ &= \sum_{k=n}^{i} \sum_{r=0}^{j} \frac{\gamma_k^i \gamma_r^j k!}{\Gamma(k+1-\alpha)(k+r+1-\alpha)}, \quad i \geq n. \end{aligned} \tag{22}$$

It is obvious that the first $n-1$ rows for $\mathbf{Q}^\alpha$ are equal to zero, where $n-1 < \alpha \leq n$.

The operational matrix for fractional order Riemann–Liouville integration of order α, $(0 < \alpha \leq 1)$ is defined as follows:

$$I^\alpha G(x) = \mathbf{P}^\alpha G(x), \tag{23}$$

where $\mathbf{P}^\alpha$ is an $N \times N$ matrix. By using (14), we can write

$$\mathbf{P}^\alpha = \mathbf{V}^\alpha \mathbf{W}^{-1}, \tag{24}$$

where $\mathbf{V}^\alpha = \langle I^\alpha G(x), G(x) \rangle$ is an $N \times N$ matrix whose entries are defined by

$$
\begin{aligned}
\mathbf{V}^{\alpha}_{ij} &= \langle I^{\alpha} G_i(x), G_j(x) \rangle \\
&= \left\langle \sum_{k=0}^{i} \frac{\gamma^i_k k!}{\Gamma(k+1+\alpha)} x^{k+\alpha}, \sum_{r=0}^{j} \gamma^j_r x^r \right\rangle \\
&= \sum_{k=0}^{i} \sum_{r=0}^{j} \frac{\gamma^i_k \gamma^j_r k!}{\Gamma(k+1+\alpha)} \int_0^1 x^{k+r+\alpha} dx \\
&= \sum_{k=0}^{i} \sum_{r=0}^{j} \frac{\gamma^i_k \gamma^j_r k!}{\Gamma(k+1+\alpha)(k+r+1+\alpha)}.
\end{aligned} \tag{25}
$$

The product of two Genocchi vectors can be expressed by Genocchi vector as follows:

$$
G(x) G^T(x) C = \widetilde{\mathbf{C}} G(x), \tag{26}
$$

where $\widetilde{\mathbf{C}}$ is called the operational matrix for the product. By using (14), the product operational matrix $\widetilde{\mathbf{C}}$ can be defined as follows:

$$
\widetilde{\mathbf{C}} = \mathbf{S}\mathbf{W}^{-1}, \tag{27}
$$

where $\mathbf{S} = \big\langle G(x) G^T(x) C, G(x) \big\rangle$. The entries of $\mathbf{S}$ are determined as

$$
\begin{aligned}
\mathbf{S}_{ij} &= \big\langle (G(x) G^T(x) C)_i, G_j(x) \big\rangle \\
&= \left\langle G_i(x) \sum_{k=0}^{N} c_k G_k(x), G_j(x) \right\rangle = \sum_{k=1}^{N} c_k q_{ijk},
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
q_{ijk} &= \int_0^1 G_i(x) G_k(x) G_j(x) dx \\
&= \sum_{r=0}^{i} \sum_{s=0}^{j} \sum_{t=0}^{k} \frac{\gamma^i_r \gamma^j_s \gamma^k_t}{r+s+t+1}.
\end{aligned} \tag{29}
$$

3 Method of solution

In this section, we apply Genocchi polynomials to solve problem (1). Consider the fractional Sturm–Liouville problem (1) with the boundary conditions (2). The following Lemma converts this problem to the equivalent integral equation.

Lemma 2. The function $y(x)$ is a solution for the fractional Sturm–Liouville problem (1)–(2) if and only if $y(x)$ satisfies the following fractional integral equation:

$$y(x) = g(x)\left[h_1\ {}_0I_x^\alpha\ f(x)|_{x=1} + h_2\ D\ {}_0I_x^\alpha\ f(x)|_{x=1}\right] - {}_0I_x^\alpha f(x), \tag{30}$$

where $f(x) = (\lambda r(x) - q(x))y(x)$ and

$${}_0I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt.$$

The operator D in (30) is the classical derivative operator and $g(x)$, h_1, and h_2 are defined considering the values of $a,\ b,\ c$, and d as follows:

$$\begin{aligned}
g(x) &= \frac{acx-bc}{ad-bc+ac}, \quad h_1 = 1,\ h_2 = \frac{d}{c}, \qquad \text{if } a \neq 0,\ c \neq 0,\\
g(x) &= \frac{adx-bd}{ad-bc+ac}, \quad h_1 = \frac{c}{d},\ h_2 = 1, \qquad \text{if } a \neq 0,\ d \neq 0,\\
g(x) &= \frac{-adx+bd}{-ad+bc-ac}, \quad h_1 = \frac{c}{d},\ h_1 = 1, \qquad \text{if } b \neq 0,\ d \neq 0,\\
g(x) &= \frac{-acx+bc}{-ad+bc-ac}, \quad h_1 = 1,\ h_2 = \frac{d}{c}, \qquad \text{if } b \neq 0,\ c \neq 0.
\end{aligned} \tag{31}$$

Proof. We refer to [31].

□

To solve the Sturm–Liouville problem (1)–(2), we apply the Genocchi polynomials method to the integral equation (30). First, we approximate $y(x)$, $r(x)$, and $q(x)$ in (30) by the truncated Genocchi polynomials (13) as follows:

$$y(x) \simeq C^T G(x), \quad r(x) \simeq R^T G(x), \quad q(x) \simeq Q^T G(x). \tag{32}$$

To apply (32) in (30), we first approximate the elements in (30) as follows:

$$\begin{aligned}
f(x) &= y(x)(\lambda r(x) - q(x))\\
&\simeq C^T G(x)(\lambda G^T(x)R - G^T(x)Q)\\
&= C^T \left(\lambda \widetilde{\mathbf{R}} - \widetilde{\mathbf{Q}}\right) G(x) = C^T \widetilde{\mathbf{F}} G(x),
\end{aligned} \tag{33}$$

where $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{Q}}$ are the product operational matrices corresponding to the vectors R and Q, respectively, and $\widetilde{\mathbf{F}} = \lambda\widetilde{\mathbf{R}} - \widetilde{\mathbf{Q}}$. By (33), the fractional integration part of (30) can be written as follows:

$${}_0I_x^\alpha f(x) \simeq C^T \widetilde{\mathbf{F}}{}_0I_x^\alpha G(x) = C^T \widetilde{\mathbf{F}} \mathbf{P}^\alpha G(x), \tag{34}$$

where $\mathbf{P}^\alpha$ is the operational matrix for fractional integration of order α. The other part of (30) can be represented as

$$D\ {}_0I_x^\alpha f(x) \simeq C^T \widetilde{\mathbf{F}} \mathbf{P}^\alpha D G(x) = C^T \widetilde{\mathbf{F}} \mathbf{P}^\alpha \mathbf{D} G(x), \tag{35}$$

where $\mathbf{D}$ is the operational matrix for regular derivative of order one.

Substituting (32), (33), (34), and (35) in (30) and neglecting the truncation errors yield

$$C^T G(x) = C^T \left\{ \left[h_1 \widetilde{\mathbf{F}}\mathbf{P}^\alpha + h_2 \widetilde{\mathbf{F}}\mathbf{P}^\alpha \mathbf{D} \right] G(1) G^T - \widetilde{\mathbf{F}}\mathbf{P}^\alpha \right\} G(x). \tag{36}$$

The property of linear independency for Genocchi polynomials requires

$$(\mathbf{I} - \mathbf{A}(\lambda))C = 0, \tag{37}$$

where

$$[\mathbf{A}(\lambda)]^T = \left[h_1 \widetilde{\mathbf{F}}\mathbf{P}^\alpha + h_2 \widetilde{\mathbf{F}}\mathbf{P}^\alpha \mathbf{D} \right] G(1) G^T - \widetilde{\mathbf{F}}\mathbf{P}^\alpha, \tag{38}$$

and $\mathbf{I}$ is the $N \times N$ identity matrix.

To have nontrivial solutions for (1) and (2) we have to find nonzero solutions for (37). Therefor, we need to solve the following root-finder problem:

$$\det\left(\mathbf{I} - \mathbf{A}(\lambda)\right) = 0, \tag{39}$$

which can be done by mathematical softwares such as MATLAB and Maple.

After computing eigenvalues, we approximate the eigenfunction $y_i(t)$ corresponding to the eigenvalue λ_i by solving the following linear system:

$$(\mathbf{I} - \mathbf{A}(\lambda_i))C = 0. \tag{40}$$

Since $\det(\mathbf{I} - \mathbf{A}(\lambda_i)) = 0$ and λ_i is the simple root, then we set $C_N = c$ and solve (40) to find C_k, $k = 1, 2, \ldots, N-1$ with respect to c. Then the eigenfunction $y_i(t)$ is obtained by

$$y_i(t) \simeq \sum_{k=1}^{N} C_{k,c} G_k(t), \qquad i = 0, 1, \ldots. \tag{41}$$

By using an appropriate auxiliary condition (for example, $y_i'(0) = 1$ or $y_i''(0) = 1$), $y_i(t)$ can be determined uniquely.

4 Numerical results

In this section, we present some illustrative examples and show the efficiency of the proposed method.

Example 1. Consider the following fractional eigenvalue problem:

$$\begin{aligned} {}_0^C D_x^\alpha (x) + \lambda y(x) = 0, \qquad & 0 \le x \le 1, \\ y(0) = y(1) = 0. & \end{aligned} \tag{42}$$

Table 1: Eigenvalues for $\alpha = 2$, Example 1.

	exact value	proposed method $N = 25$	proposed method error	method of [31] $N = 800$	method of [31] error
λ_1	9.86960440108936	9.869604401089359	2.58e-28	9.86960440	1.09e-9
λ_2	39.4784176043574	39.47841760435743	5.22e-19	39.47841760	4.36e-9
λ_3	88.8264396098042	88.82643960980423	1.62e-15	88.82643963	2.02e-8
λ_4	157.913670417430	157.9136704174555	2.58e-11	157.91367057	1.53e-7
λ_5	246.740110027234	246.7401100283036	1.07e-9	246.74011064	6.13e-7
λ_6	355.305758439217	355.3057578574429	5.82e-7	355.30575850	6.08e-8
λ_7	483.610615653379	483.6106104947717	5.16e-6	483.61061573	7.66e-8
λ_8	631.654681669719	631.6551696075792	4.88e-4	631.65468191	2.40e-7

Table 2: Eigenvalues for different values of α, Example 1 ($N = 20$).

	$\alpha = 1.95$	$\alpha = 1.9$	$\alpha = 1.85$	$\alpha = 1.8$
λ_1	9.66077186263	9.51414296571	9.44013733938	9.45685703307
λ_2	36.4045254812	33.5956740224	30.9825046888	28.4768769642
λ_3	80.3290382609	73.0390014094	66.9448021792	62.2003971381
λ_4	140.076253643	124.418589764	110.360397400	97.0631636666
λ_5	216.597587060	191.145884721	170.311838185	155.450547906
λ_6	308.346500930	267.943639675	232.015405069	196.595011516

This corresponds to the case $a \neq 0$ and $c \neq 0$ in (31). Then, we set $a = c = 1$, $b = d = 0$, $h_1 = 1$, $h_2 = 0$, and $g(x) = x$. For $\alpha = 2$, (42) is converted to the classical Sturm–Liouville problem corresponding to the eigenvalues $\lambda_n = (n\pi)^2$.

In this example, we use the auxiliary condition $y_i'(0) = 1$ to approximate the eigenfunctions. In Table 1, the numerical results are compared with the results of [31]. Table 2 represents the approximated eigenvalues obtained by this method. The results in Tables 1 and 2 clearly show that the proposed algorithm is accurate. Figure 1 shows the first four eigenfunctions for $\alpha = 1.85,\ 1.9,\ 1.95,\ 2$ and $N = 15$. In Figure 2, the graphs of same order eigenvalues are compared for different values of α.

Example 2. For the second example, we solve the following eigenvalue problem:

$$
\begin{aligned}
&{}_0^C D_x^\alpha y(x) + \frac{\lambda}{(1+x)^2} y(x) = 0, \qquad 0 \leq x \leq 1,\\
&y(0) = 0,\ y(1) = 0.
\end{aligned}
\tag{43}
$$

For $\alpha = 2$, the eigenvalues of (43) are $\lambda_n = (n\pi/\ln 2)^2 + 1/4$, and the corresponding eigenfunctions are $y_n(x) = c_n\sqrt{1+x}\sin\left(\frac{n\pi \ln(1+x)}{\ln 2}\right)$, where c_n is constant coefficient [31]. The entries are chosen $h_1 = 1$, $h_2 = 0$, and $g(x) = x$. We use the auxiliary condition $y_i'(0) = 1$ to compute four first eigenfunctions of Example 2, which are plotted in Figure 3. Tables 3 and 4 represent the numerical results obtained for Example 2, which were compared with the results in [31].

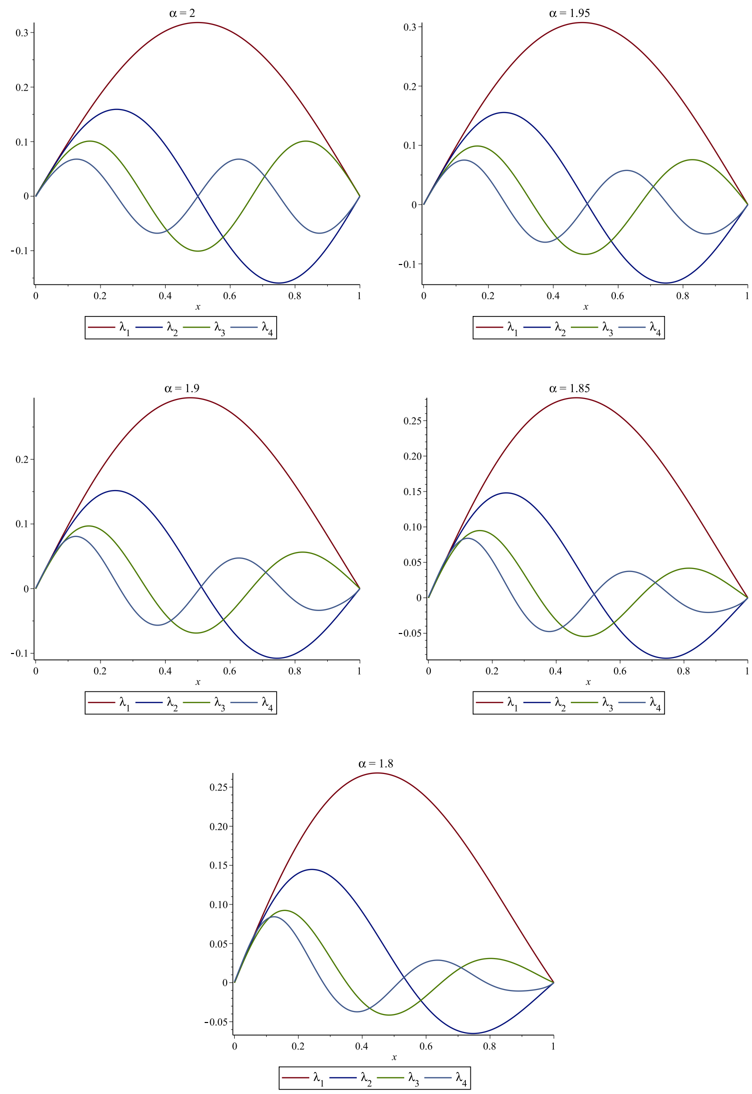

Figure 1: First four eigenfunctions for different values of α, Example 1.

Example 3. For the next example, we solve the following eigenvalue problem:

$$\begin{aligned} {}_0^C D_x^{\alpha} y(x) + (\lambda + 10 \sin \pi x) y(x) = 0, \qquad & 0 \leq x \leq 1, \\ y(0) = 0, \ y(1) = 0. \qquad & \end{aligned} \tag{44}$$

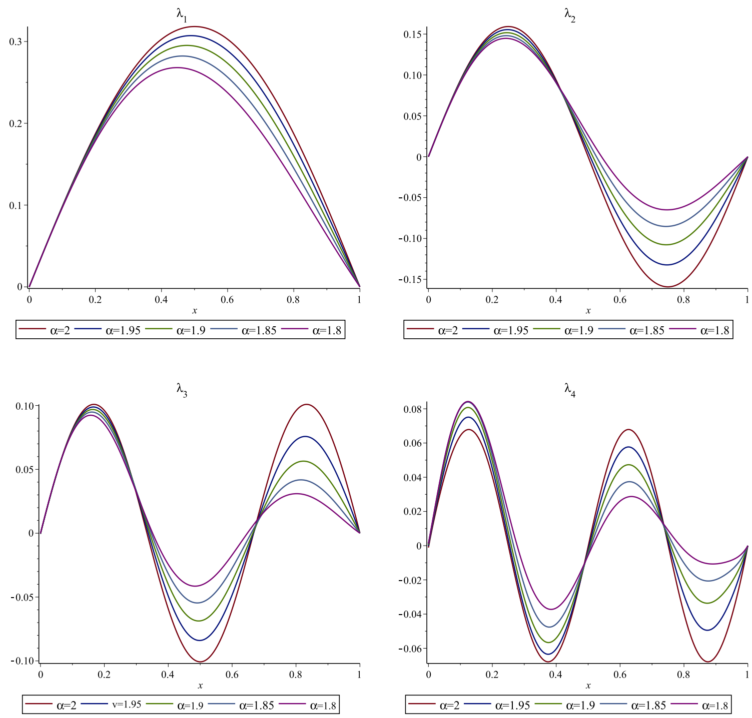

Figure 2: First four eigenfunctions for Example 1.

Table 3: Eigenvalues for $\alpha = 2$, Example 2.

		proposed method		method of [31]	
	exact value	$N = 25$	error	$N = 500$	error
λ_1	20.7922884552238	20.7922884552238	2.70e-17	20.79228809	3.65e-7
λ_2	82.4191538208953	82.4191538208952	6.85e-14	82.41914815	5.67e-6
λ_3	185.130596097014	185.130596097016	1.91e-12	185.13056780	2.83e-5
λ_4	328.926615283581	328.926615287143	3.56e-9	328.92652741	8.79e-5
λ_5	513.807211380596	513.807211123754	2.57e-7	513.80700157	2.10e-4
λ_6	739.772384388058	739.772393148381	8.76e-6	739.77196118	4.23e-4
λ_7	1006.82213430597	1006.82198782297	1.46e-4	1006.82137649	7.58e-4
λ_8	1314.95646113432	1314.95690886405	4.48e-4	1314.95521911	1.24e-3
λ_9	1664.17536487313	1664.19294987497	1.76e-2	1664.17346838	1.90e-3
λ_{10}	2054.47884552238	2054.28429017518	1.95e-1	2054.47611287	2.73e-3

Setting $h_1 = 1$, $h_2 = 0$, and $g(x) = x$, we use the proposed method to solve (44). With $N = 20$ and 40 digits for computation in Maple software, some first eigenvalues are listed in Table 5 for different values of α.

We use the auxiliary condition $y_i'(0) = 1$ to compute four first eigenfunctions of Example 3, which are plotted in Figure 4 for different values of α.

Example 4. Consider the following eigenvalue problem:

Table 4: Eigenvalues for different values of α, $N = 20$, Example 2.

	$\alpha = 1.95$	$\alpha = 1.9$	$\alpha = 1.85$	$\alpha = 1.8$
λ_1	20.6677971198	20.7298859439	21.0420113791	21.7289820388
λ_2	75.7710019802	69.4887851211	63.3252982915	56.9307959084
λ_3	167.652678491	153.001698048	141.496620930	134.502772594
λ_4	291.242134899	257.628271127	226.249088945	193.872634358
λ_5	451.065004856	398.913421123	358.715501294	345.306193742
λ_6	641.015025258	554.923522781	474.802195243	379.243396666
λ_7	867.315229409	752.921297326	669.708580955	2260.68651346
λ_8	1122.28888722	956.431286954	798.315596410	3571.46938188
λ_9	1414.21783814	1212.08349835	1084.45731615	8833.11202980
λ_{10}	1732.91424589	1458.60978678	1178.69721277	80410.2798175

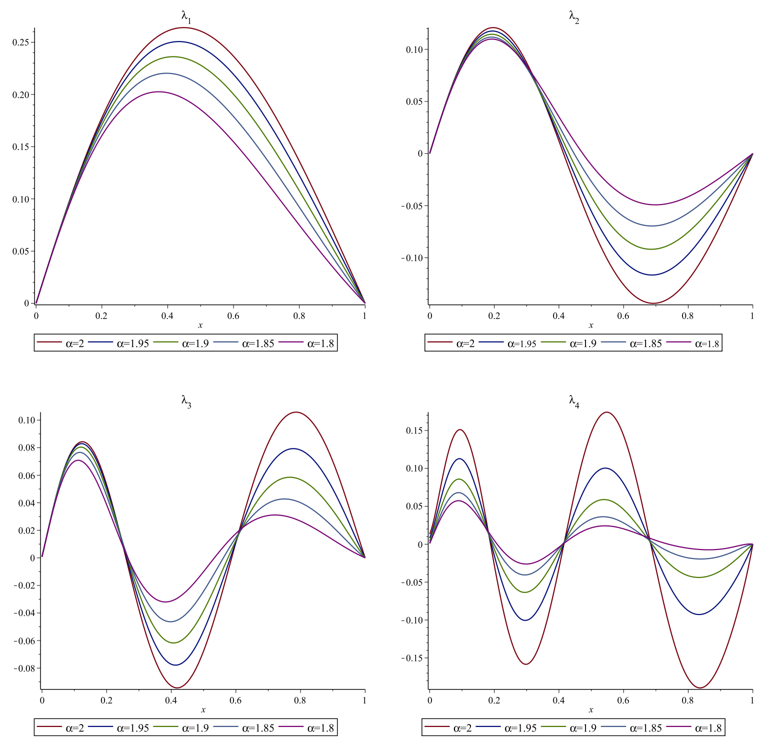

Figure 3: First four eigenfunctions for Example 2.

$$\begin{gathered} {}_0^C D_x^\alpha y(x) + (2\lambda e^x - 5\sin\pi x)y(x) = 0, \qquad 0 \le x \le 1, \\ y(0) = y'(0) \neq 0, \ y(1) = 0. \end{gathered} \tag{45}$$

We set $a = 1$, $b = -1$, $c = 1$, and $d = 0$. Then $h_1 = 1$, $h_2 = 0$ and $g(x) = (x+1)/2$ (see (33)). With $N = 25$ and 40 digits for computation, some first eigenvalues are presented and compared in Table 6 for different values of α.

Table 5: Eigenvalues of Example 3 ($N = 20$).

	$\alpha = 2$	$\alpha = 1.95$	$\alpha = 1.9$	$\alpha = 1.8$
λ_1	18.3200943208	18.1463655269	18.0384472376	18.0792788380
λ_2	46.2367305574	43.1880344976	40.4110340394	35.3810395900
λ_3	95.3857838965	86.8856296038	79.5840346405	68.6611728543
λ_4	164.391065891	146.560791927	130.915815214	103.643270952
λ_5	253.178225348	223.035468745	197.577387685	161.771332663
λ_6	361.722182212	314.767445388	274.375010614	203.160705578
λ_7	490.014088674	423.155049789	367.477687375	305.476138569
λ_8	637.940873962	546.469154163	468.674770446	315.791118150

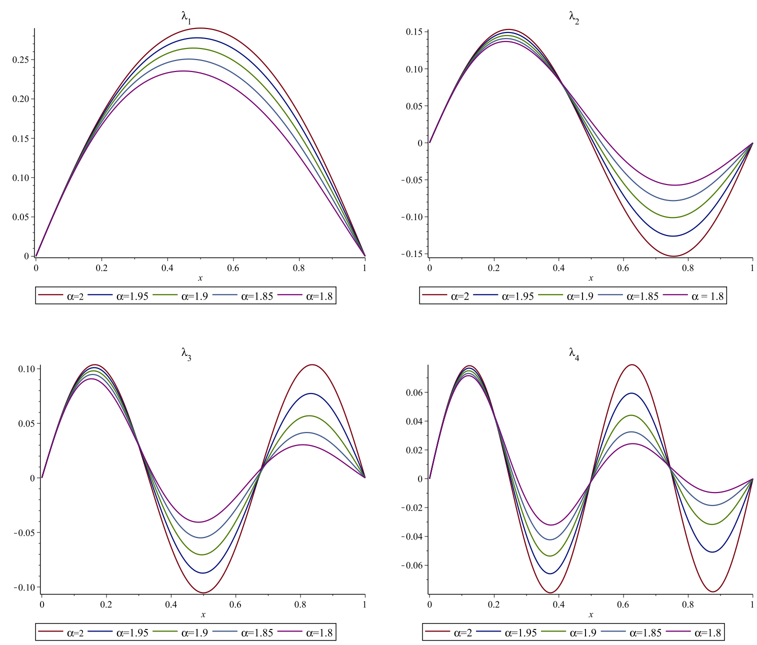

Figure 4: First four eigenfunctions for Example 3.

We use the auxiliary condition $y_1'(1) = 1$ and $y_i'(1) = -1,\ i \neq 1$ to compute four first eigenfunctions of Example 4, which are plotted in Figure 5.

Example 5. Consider the following forth-order eigenvalue problem: [27]

$$\begin{aligned} v^{(4)}(t) - (0.02t^2 v'(t))' + (0.0001t^4 - 0.02)v(t) &= \lambda\, v(t), \qquad 0 \le t \le 5, \\ v(0) = v''(0) = 0, \quad v(1) = v''(1) = 0. & \end{aligned} \tag{46}$$

The eigenvalues of this problem are the square of the eigenvalues of the following second order Sturm–Liouville problem (see [27]):

Table 6: Eigenvalues of Example 4.

	proposed method, $N = 25$				
	$\alpha = 2$	$\alpha = 1.95$	$\alpha = 1.9$	$\alpha = 1.85$	$\alpha = 1.8$
λ_1	2.6213001864	2.5785602459	2.5408885358	2.5083120348	2.4809701676
λ_2	8.3495214073	7.7826681845	7.2770461324	6.8263714594	6.4250121537
λ_3	20.160277342	18.278658822	16.631507344	15.191185649	13.934739577
λ_4	37.779499263	33.634640153	30.049170175	26.945962302	24.258034906
λ_5	61.245497320	53.803913449	47.432736730	41.977052803	37.309595288
λ_6	90.568134413	78.731592840	68.678948337	60.132525229	52.854782758
λ_7	125.75080891	108.37362381	93.720958762	81.357289626	70.931030267
λ_8	166.79492842	142.69052813	122.48797531	105.53364534	91.272428145
	method of [31], $N = 800$				
	$\alpha = 2$	$\alpha = 1.95$		$\alpha = 1.85$	
λ_1	2.62130018	2.57856033		2.50831250	
λ_2	8.34952147	7.78266742		6.82636707	
λ_3	20.16027739	18.27866231		15.19120752	
λ_4	37.77949944	33.63463032		26.94588552	
λ_5	61.24549782	53.80393654		41.97727381	

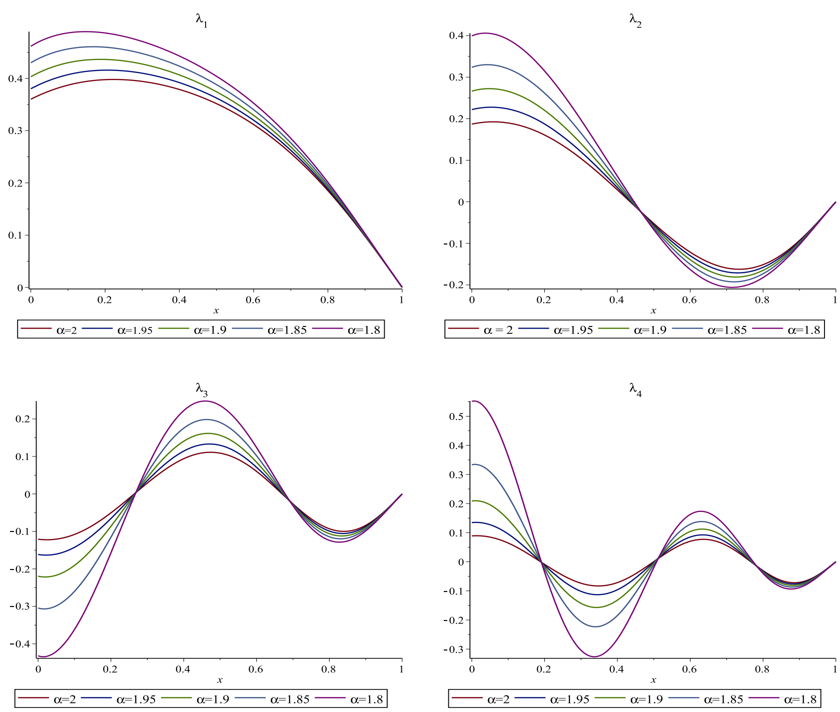

Figure 5: First four eigenfunctions for Example 4.

$$
\begin{aligned}
-u''(t) + 0.01t^2 u(t) = \lambda\, u(t), \qquad 0 \le t \le 5,\\
u(0) = 0,\ u(5) = 0.
\end{aligned}
\tag{47}
$$

By taking the variable change $t = 5x$, the following problem is achieved:

$$y''(x)+(25\lambda - 6.25x^2)y(x) = 0, \qquad 0 \le x \le 1,$$
$$y(0) = 0,\ y(1) = 0. \tag{48}$$

Taking $h_1 = 1$, $h_2 = 0$, $g(x) = x$, $r(x) = 25$, and $q(x) = 6.25x^2$, with $N = 25$ and 40 digits for computation in Maple software, some first eigenvalues are presented and compared with the results of [27] in Table 7. We use the auxiliary condition $y_i'(0) = 1$ to compute four first eigenfunctions of Example 5, which are plotted in Figure 6.

Table 7: Eigenvalues of Example 5.

	proposed method	[27]
λ_1	0.215050864368	0.2150508644
λ_2	2.754809934683	2.7548099347
λ_3	13.21535154056	13.2153515406
λ_4	40.95081975918	40.9508197592
λ_5	99.05347806596	99.0534780635
λ_6	204.3557315637	204.355732268
λ_7	377.4304091791	377.430420689
λ_8	642.5918663327	642.590868170

5 Conclusion

In this paper, the eigenvalues of the second-order fractional Sturm–Liouville problem were approximated by using Genocchi polynomials. The operational matrix for fractional integration, fractional derivative, and product was evaluated. Then, the operational matrices were applied to the fractional Sturm–Liouville problem to convert it into a homogeneous system of linear equations. The eigenvalues of the coefficient matrix corresponded to the eigenvalues of the main Sturm–Liouville problem. The presented illustrative examples verified that the proposed method generated more accurate approximations compared to the results, which were reported in other papers. Like other methods in the literature, the results were accurate for lower indices but not very accurate for higher indices. After computing the eigenvalues, the corresponding eigenfunctions were approximated by applying an auxiliary condition. The results were in good agreement with the exact solutions or the results of other papers.

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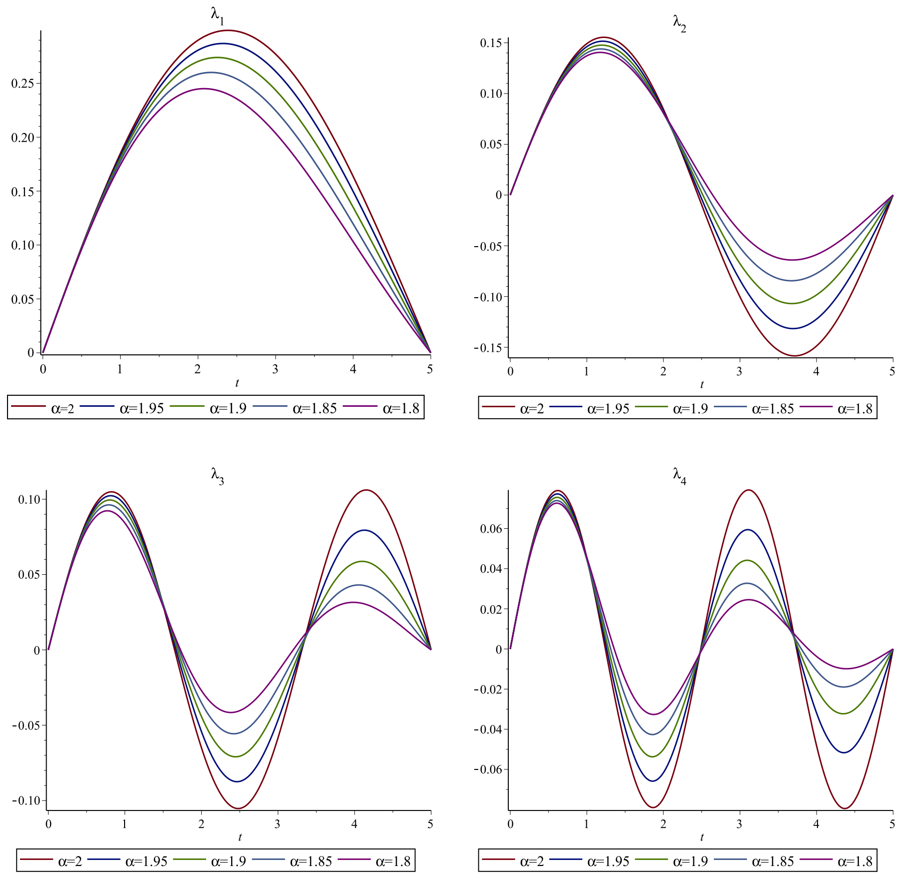

Figure 6: First four eigenfunctions for Example 5.

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