

High order immersed interface method for acoustic wave equation with discontinuous coefficients

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Abstract

This paper concerns the numerical solution of the acoustic wave equation that contains interfaces in the solution domain. To solve the interface problems with high accuracy, more attention should be paid to the interfaces. In fact, any direct application of a high order finite difference method to these problems leads to inaccurate approximate solutions with high oscillations at the interfaces. There is however, the possibility of deriving some high order methods to resolve this phenomenon at the interfaces. In this paper, a sixth order immersed interface method for acoustic wave equation is presented. The order of accuracy is also maintained at the discontinuity using the jump conditions. Some numerical experiments are included which confirm the order of accuracy and numerical stability of the presented method.

Keywords: Interface methods; High order methods; Lax-Wendroff method; Discontinuous coefficients; Jump conditions.

1 Introduction

In this paper we develop a class of high order numerical methods for wave equations with discontinuous coefficients. The class of interface problems involves many problems of real world applications in Science and Engineering, such as Seismology, Ocean acoustics, and Electromagnetic. A naive implementation of high order methods fails to achieve high order accuracy and

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produces some spurious oscillations (Gibbs phenomenon) near the discontinuity. There are several approaches to deal with such a problem of accuracy loss. An efficient method for the simulation of these equations should be able to reduce dispersion and dissipation errors in the propagation of the solution [16]. Many researchers are interested in high order methods for hyperbolic problems. Long time behavior of the solution of these problems is an important challenge to the numerical simulation of such problems.

A good deal of literature exists on the numerical solution of interface problems. Two traditional methods are adding viscosity to the problem and using flux limiters [5]. A recent approach is using essentially nonoscillatory (ENO) or weighted ENO methods [6]. We distinguish the interface methods in two sets with or without use of jump conditions. The first set consists of those methods, such as the recent works of Gustafsson and his coworkers [3, 4] and Leveque [8] that do not impose jump conditions in the formulation of the numerical solution of these equations. These methods are based on shock capturing methods and Riemman solvers for conservation laws. The second set, whose history goes back to the pioneering work of Peskin [11] for the simulation of blood flow in the heart, consists of those methods that use some sort of jump condition in their formulation. In fact, to achieve high order results, it is recommended to use a special high order method in the vicinity of the discontinuity. Derivative matching methods is a related issue and have been developed by Driscoll and Fornberg [1] for one and two dimensional Maxwell equations and followed by many others researcher such as Zhao and Wei [19], which derived a derivative matching approach based on the FDTD schemes for Maxwell equations. This scheme is based on fictitious point method and in a vicinity of the interface they introduce original points in one side and ghost or fictitious points (unknown values) in the other side of the interface. Using the derivative matching conditions the values at the ghost points are evaluated. The emphasis in this paper is on the second set of methods that use physical jump conditions at the interface which are usually easily accessible from the physical properties of the problem. Therefore, we perform the numerical solutions to satisfy these jump conditions. The immersed interface method (IIM) considers a standard method for the regular points and imposes a new method for the irregular points to update the solution at the next time level with the same accuracy as the standard method. The implementation of this new method requires the solution of several small linear systems to obtain coefficients of the difference method on the irregular points, without imposing any significant computational cost on the calculations. The explicit relations between two media (in heterogeneous media) through jump conditions eliminate the role of the fictitious points and therefore, this method attains the high order results without ghost points. There is a related class of methods, known as simplified immersed interface methods, that modify explicitly the numerical values at the irregular points [12, 17].

The immersed interface method has been discussed for various kinds of partial differential equations and its implementation to many real world ap-

plications has been successful. A full review of this method for interface problems of parabolic and elliptic type is available in the recent book of Li and Ito [10].

In this paper, a sixth order method for the acoustic wave equation with discontinuous coefficients is presented. This higher order method in some sense is an improvement on the works of Zhang and LeVeque [18] and Farzi and Hosseini [2]. There are several methods for the simulation of the time evolution for the wave equation. The Lax-Wendroff method is a simple time discretization method for implementation. However, to reduce possible dispersion and dissipation errors one can invoke TVD and WENO methods to avoid oscillations near the discontinuity [6, 9]. In this paper, we consider a coupled application of Lax-Wendroff and the immersed interface method on the interface. The contribution of this paper is given in the next sections. The extension of this method to any order is direct and is given in Section 4. The stability analysis and implementation of the method for two dimensional problems is presented in Section 5. Physical jump conditions are demonstrated for the one dimensional acoustic wave equation.

The explicit and closed form discretization formula is obtained in Section 2. The theory and numerical results are also developed for piecewise smooth coefficients (Sections 3, 4 and 5). The numerical results reported in Section 5 confirm the efficiency of the method to approximate the solution with a well presented behavior of wave propagation and also a high order accuracy at the interfaces. The long time behavior of the method is illustrated by Test problem 3 in numerical results. The order of the new immersed interface method is justified in Test problem 4 with numerical order of accuracy. Numerical stability of the method is addressed in Test problem 5.

2 Acoustic Wave Equation

Let $u(x, t)$ and $p(x, t)$ be the acoustic velocity and acoustic pressure, respectively. Then the one-dimensional wave equation can be written as the following model problem

$$U_t + AU_x = 0, \quad (1)$$

where,

$$U(x, t) = \begin{pmatrix} u \\ p \end{pmatrix}, \quad A(x) = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \kappa & 0 \end{pmatrix}, \quad (2)$$

and $A(x)$ is a function of position consisting of physical quantities such as density $\rho(x)$, sound speed $c(x)$ and $\kappa = \rho c^2$. We first focus mainly on problems in which the density and sound speed are piecewise constant functions and have a jump discontinuity at the point $x = \alpha$, which we call the interface,

$$(\rho, c) = \begin{cases} (\rho^-, c^-), & x < \alpha, \\ (\rho^+, c^+), & x > \alpha, \end{cases} \quad (3)$$

Then application of our formulation for problems with more general piecewise smooth coefficients will be addressed. Typical applications in which this assumption is appropriate include long range underwater acoustics, various seismological problems as well as electromagnetic problems. Throughout the paper we use the symbol A^+ for $A(x)$ with $x > \alpha$ and A^- for $A(x)$ with $x < \alpha$. The same meaning will apply to other matrices.

In this part, we derive a high order method for this equation and we postpone the treatment of the nonsmooth solution at the interface to the next sections, where we discuss and derive jump conditions and give an approximation that extends the high accuracy of the solution, obtained for the smooth regions, to the interface. At the left or right of the point of discontinuity we can use any standard method. The Lax-Wendroff method is a simple time evolution explicit method for implementation which uses the values at the current time level and does not need any knowledge of previously calculated values. This method is based on Taylor series expansion in time and substitution of the time derivatives with space derivatives using (1). We consider

$$\begin{aligned} U(x_j, t_{n+1}) \approx & U(x_j, t_n) + \frac{k}{1!} \frac{\partial U}{\partial t}(x_j, t_n) + \frac{k^2}{2!} \frac{\partial^2 U}{\partial t^2}(x_j, t_n) \\ & + \frac{k^3}{3!} \frac{\partial^3 U}{\partial t^3}(x_j, t_n) + \frac{k^4}{4!} \frac{\partial^4 U}{\partial t^4}(x_j, t_n) + \frac{k^5}{5!} \frac{\partial^5 U}{\partial t^5}(x_j, t_n) \\ & + \frac{k^6}{6!} \frac{\partial^6 U}{\partial t^6}(x_j, t_n) \end{aligned} \quad (4)$$

where k is the length of the time steps. Replacing the time derivatives by the space derivatives and discretizing the space derivatives we get

$$\begin{aligned} U(x_j, t_{n+1}) \approx & U(x_j, t_n) - \frac{1}{1!} k A_j Q_6^{(1)} U(x_j, t_n) + \frac{1}{2!} k^2 c_j^2 Q_6^{(2)} U(x_j, t_n) \\ & - \frac{1}{3!} k^3 c_j^2 A_j Q_4^{(3)} U(x_j, t_n) + \frac{1}{4!} k^4 c_j^4 Q_4^{(4)} U(x_j, t_n) \\ & - \frac{1}{5!} k^5 c_j^4 A_j Q_2^{(5)} U(x_j, t_n) + \frac{1}{6!} k^6 c_j^6 Q_2^{(6)} U(x_j, t_n), \end{aligned} \quad (5)$$

where $A_j = A(x_j)$ and $c_j = c(x_j)$, $Q_p^{(q)}$ is central difference formula of order p for $\frac{\partial^q}{\partial x^q}$ [5] and we have used the relation $A^2 = c^2 I$, in which I is the 2×2 identity matrix.

Substitution of $Q_p^{(q)}$ as explained in [2], which deals only with the advection equation, gives a fully discretized representation of the acoustic equation by the following matrix-vector equation

$$U_j^{n+1} = U_j^n + \sum_{l=1}^7 \Gamma_{j,l} U_{j+l-4}^n \quad (6)$$

where the $\Gamma_{j,l}$'s are two by two matrices (coefficient matrices) and can be expressed as

$$\Gamma_{j,l} = w_l A(\lambda A + (4-l)I), \quad j, l = 1, 2, \dots, 7, \quad (7)$$

with

$$\begin{aligned} w_1 &= \frac{1}{6!} \lambda ((2 - \lambda^2 c^2)^2 - \lambda^2 c^2), \\ w_2 &= -\frac{1}{5!} \lambda ((3 - \lambda^2 c^2)^2 - 4\lambda^2 c^2), \\ w_3 &= -\left(\frac{1}{3!} \lambda (7 - \lambda^2 c^2)\right)^2, \\ w_4 &= \frac{1}{2 \times 4!} \lambda ((6 - \lambda^2 c^2)^2 - \lambda^2 c^2), \end{aligned}$$

and $w_7 = w_1, w_6 = w_2, w_5 = w_4$. h is the spacial grid step length and $\lambda = \frac{k}{h}$.

In fact, the high order derivatives are not valid at the interface and consequently the obtained results are of first order or even less. A detailed discussion of this phenomenon of losing the accuracy has already been presented by Sei and Symes [14]. So, obviously, on each side of the discontinuity of $A(x)$ the method is of order six for the acoustic equation, while at the grid points near to the discontinuity the method fails to maintain this order of accuracy.

More precisely, suppose that the interface lies between two adjacent grid points with indices J and $J+1$, i.e. $x_J < \alpha < x_{J+1}$, then this method works well at those grid points at which all the required points to update them are located completely on the left or right of the interface, but it fails to be accurate at the grid points (irregular points) $J-2, J-1, \dots, J+3$. So, we need a new scheme to maintain the same order of accuracy at these irregular points.

For a general piecewise smooth coefficient problem we can derive the same method, but in this case the derivatives of the coefficient $A(x)$ will come into the difference equation. It should be noted, however, that the coefficient matrices are not as simple as those appearing in (7), but we can follow the same procedure to provide a sixth order method for this case as well. Here with an efficient derivation of these formulas we can save more in computations. In the following, we present the time derivative approximations in (4) and approximations of the corresponding space derivatives

$$\begin{aligned}
(U_t)_j^n &\approx -AQ_6^{(1)}U_j^n, \\
(U_{tt})_j^n &\approx B^{(1)}Q_6^{(1)}U_j^n + B^{(2)}Q_6^{(2)}U_j^n, \\
(U_{ttt})_j^n &\approx C^{(1)}Q_4^{(1)}U_j^n + C^{(2)}Q_4^{(2)}U_j^n + C^{(3)}Q_4^{(3)}U_j^n, \\
(U_{tttt})_j^n &\approx D^{(1)}Q_4^{(1)}U_j^n + D^{(2)}Q_4^{(2)}U_j^n + D^{(3)}Q_4^{(3)}U_j^n + D^{(4)}Q_4^{(4)}U_j^n, \\
(U_{ttttt})_j^n &\approx E^{(1)}Q_2^{(1)}U_j^n + E^{(2)}Q_2^{(2)}U_j^n + E^{(3)}Q_2^{(3)}U_j^n + E^{(4)}Q_2^{(4)}U_j^n \\
&\quad + E^{(5)}Q_2^{(5)}U_j^n, \\
(U_{tttttt})_j^n &\approx F^{(1)}Q_2^{(1)}U_j^n + F^{(2)}Q_2^{(2)}U_j^n + F^{(3)}Q_2^{(3)}U_j^n + F^{(4)}Q_2^{(4)}U_j^n \\
&\quad + F^{(5)}Q_2^{(5)}U_j^n + F^{(6)}Q_2^{(6)}U_j^n,
\end{aligned} \tag{8}$$

where,

$$\begin{aligned}
B^{(1)} &= AA', \\
B^{(2)} &= A^2, \\
C^{(1)} &= -B^{(1)}A' - B^{(2)}A'', \\
C^{(2)} &= -B^{(1)}A - 2B^{(2)}A', \\
C^{(3)} &= -B^{(2)}A, \\
D^{(1)} &= -C^{(1)}A' - C^{(2)}A'' - C^{(3)}A''', \\
D^{(2)} &= -C^{(1)}A - 2C^{(2)}A' - 3C^{(3)}A'', \\
D^{(3)} &= -C^{(2)}A - 3C^{(3)}A', \\
D^{(4)} &= -C^{(3)}A, \\
E^{(1)} &= -D^{(1)}A' - D^{(2)}A'' - D^{(3)}A''' - D^{(4)}A''', \\
E^{(2)} &= -D^{(1)}A - 2D^{(2)}A' - 3D^{(3)}A'' - 4D^{(4)}A''', \\
E^{(3)} &= -D^{(2)}A - 3D^{(3)}A' - 6D^{(4)}A'', \\
E^{(4)} &= -D^{(3)}A - 4D^{(4)}A', \\
E^{(5)} &= -D^{(4)}A, \\
F^{(1)} &= -E^{(1)}A' - E^{(2)}A'' - E^{(3)}A''' - E^{(4)}A'''' - E^{(5)}A''''', \\
F^{(2)} &= -E^{(1)}A - 2E^{(2)}A' - 3E^{(3)}A'' - 4E^{(4)}A''' - 5E^{(5)}A'''', \\
F^{(3)} &= -E^{(2)}A - 3E^{(3)}A' - 6E^{(4)}A'' - 10E^{(5)}A''', \\
F^{(4)} &= -E^{(3)}A - 4E^{(4)}A' - 10E^{(5)}A'', \\
F^{(5)} &= -E^{(4)}A - 5E^{(5)}A', \\
F^{(6)} &= -E^{(5)}A.
\end{aligned}$$

A similar standard method has been addressed by Qiu and Shu [13], in which the finite difference WENO schemes with Lax-Wendroff time discretization for solving nonlinear hyperbolic conservation law systems was developed. It uses a WENO scheme instead $Q_6^{(1)}$ in (8) and proceeds to the final formulations.

3 Jump Conditions

We study the jump conditions for a piecewise coefficient problem and then consider a general variable coefficient wave equation. A test problem for this case is also given in the section on numerical results.

3.1 Piecewise constant coefficient

In this section we introduce some jump conditions to serve as a tool for developing a new method for the irregular points. By irregular points, we mean those grid points that in the process of updating to the next time level, use grid points on both sides of the interface. If the interface $x = \alpha$ lies in the interval (x_J, x_{J+1}) then the irregular points for the given method (6) are the grid points $J - 2, J - 1, \dots, J + 3$. So, there exist six irregular points and the method (6) fails to be accurate at these points.

For the acoustic wave equation we impose the jump conditions $[u] = 0$ and $[p] = 0$ that can be denoted by a single statement

$$[U] = 0. \quad (9)$$

Using these conditions and the wave equation (1), we obtain the following relations at the interface [12],

$$\frac{\partial^k U(\alpha^+, t)}{\partial x^k} = D_k \frac{\partial^k U(\alpha^-, t)}{\partial x^k}, \quad k = 0, 1, 2, \dots \quad (10)$$

where,

$$D_{2k} = \left(\frac{c^-}{c^+}\right)^{2k} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{2k+1} = \left(\frac{c^-}{c^+}\right)^{2k} \begin{bmatrix} \frac{\kappa^-}{\kappa^+} & 0 \\ 0 & \frac{\rho^+}{\rho^-} \end{bmatrix}.$$

3.2 A general piecewise smooth coefficient

For a general piecewise smooth coefficient, by using again the condition $[U] = 0$ and imposing the relations $[U_t] = 0, \dots, [U_{ttttt}] = 0$, we obtain

$$\begin{aligned}
U_x^+ &= Q_1 U_x^-, \\
U_{xx}^+ &= Q_2 U_x^- + Q_3 U_{xx}^-, \\
U_{xxx}^+ &= Q_4 U_x^- + Q_5 U_{xx}^- + Q_6 U_{xxx}^-, \\
U_{xxxx}^+ &= Q_7 U_x^- + Q_8 U_{xx}^- + Q_9 U_{xxx}^- + Q_{10} U_{xxxx}^-, \\
U_{xxxxx}^+ &= Q_{11} U_x^- + Q_{12} U_{xx}^- + Q_{13} U_{xxx}^- + Q_{14} U_{xxxx}^- + Q_{15} U_{xxxxx}^-, \\
U_{xxxxxx}^+ &= Q_{16} U_x^- + Q_{17} U_{xx}^- + Q_{18} U_{xxx}^- + Q_{19} U_{xxxx}^- + Q_{20} U_{xxxxx}^- \\
&\quad + Q_{21} U_{xxxxxx}^-,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
Q_1 &= -GA^-, \\
Q_2 &= G^2 \left(-B_+^{(1)} GA^- + B_-^{(1)} \right), \\
Q_3 &= G^2 B_-^{(2)}, \\
Q_4 &= G^3 \left(-C_+^{(1)} Q_1 + C_-^{(1)} - C_+^{(2)} Q_2 \right), \\
Q_5 &= G^3 \left(-C_+^{(2)} Q_3 + C_-^{(2)} \right), \\
Q_6 &= G^3 C_-^{(3)}, \\
Q_7 &= G^4 \left(-D_+^{(1)} Q_1 + D_-^{(1)} - D_+^{(3)} Q_4 - D_+^{(2)} Q_2 \right), \\
Q_8 &= G^4 \left(-D_+^{(2)} Q_3 - D_+^{(3)} Q_5 + D_-^{(2)} \right), \\
Q_9 &= G^4 \left(-D_+^{(3)} Q_6 + D_-^{(3)} \right), \\
Q_{10} &= G^4 D_-^{(4)}, \\
Q_{11} &= G^5 \left(-E_+^{(1)} Q_1 - E_+^{(2)} Q_2 - E_+^{(3)} Q_4 - E_+^{(4)} Q_7 + E_-^{(1)} \right), \\
Q_{12} &= G^5 \left(-E_+^{(2)} Q_3 - E_+^{(3)} Q_5 - E_+^{(4)} Q_8 + E_-^{(2)} \right), \\
Q_{13} &= G^5 \left(-E_+^{(3)} Q_6 - E_+^{(4)} Q_9 + E_-^{(3)} \right), \\
Q_{14} &= G^5 \left(-E_+^{(4)} Q_{10} + E_-^{(4)} \right), \\
Q_{15} &= G^5 E_-^{(5)}, \\
Q_{16} &= G^6 \left(-F_+^{(1)} Q_1 - F_+^{(2)} Q_2 - F_+^{(3)} Q_4 - F_+^{(4)} Q_7 - F_+^{(5)} Q_{11} + F_-^{(1)} \right) \\
Q_{17} &= G^6 \left(-F_+^{(2)} Q_3 - F_+^{(3)} Q_5 - F_+^{(4)} Q_8 - F_+^{(5)} Q_{12} + F_-^{(2)} \right) \\
Q_{18} &= G^6 \left(-F_+^{(3)} Q_6 - F_+^{(4)} Q_9 - F_+^{(5)} Q_{13} + F_-^{(3)} \right) \\
Q_{19} &= G^6 \left(-F_+^{(4)} Q_{10} - F_+^{(5)} Q_{14} + F_-^{(4)} \right) \\
Q_{20} &= G^6 \left(-F_+^{(5)} Q_{15} + F_-^{(5)} \right) \\
Q_{21} &= G^6 F_-^{(6)}
\end{aligned} \tag{12}$$

where $G = (-A^+)^{-1}$ and the other matrices have already been introduced in previous sections.

4 Approximation at the interface for acoustic equation

We first investigate the approximation of the solution at the irregular points for the case of piecewise constant coefficients. At these points we impose the same method as (6) and let the coefficients to be determined appropriately.

Then we obtain seven unknown 2×2 matrices to maintain the sixth order accuracy of the method. The details are presented at x_J and a similar argument is also applied to the other irregular points. The symbol J indicates a fixed number corresponding to the interval (x_J, x_{J+1}) that contains the interface α .

Theorem 4.1. *If the coefficients of (6) satisfy the following linear system of equations*

$$\sum_{l=1}^4 \alpha_{il} \Gamma_{J,l} + \sum_{l=5}^7 \alpha_{il} \Gamma_{J,l} D_i = F_i^-, \quad (i = 0, 1, \dots, 6) \quad (13)$$

where,

$$\alpha_{il} = \left(\frac{r_l}{h}\right)^i, \quad (i = 0, 1, \dots, 6, \quad l = 1, 2, \dots, 7) \quad (14)$$

$$F_i^- = (\alpha_{14} - \lambda A^-)^i - \alpha_{14}^i, \quad i = 0, 1, \dots, 6. \quad (15)$$

then, the method (6) is of order 6 at the irregular point x_J .

Proof. To prove this result we consider the local truncation error at x_J up to sixth order

$$\begin{aligned} L = & \frac{1}{k} \sum_{l=1}^7 \Gamma_{j,l} U_{j+l-4} + (AU_x - \frac{1}{2}kA^2U_{xx} + \frac{1}{6}k^2A^3U_{xxx} - \frac{1}{24}k^3A^4U_{xxxx} \\ & + \frac{1}{120}k^4A^5U_{xxxxx} - \frac{1}{720}k^5A^6U_{xxxxxx})_J + O(k^6). \end{aligned} \quad (16)$$

Using the relation $A^2 = c^2I$ we get

$$\begin{aligned} L = & \frac{1}{k} \sum_{l=1}^7 \Gamma_{j,l} U_{j+l-4} + (AU_x - \frac{1}{2}kc^2U_{xx} + \frac{1}{6}k^2c^2AU_{xxx} - \frac{1}{24}k^3c^4U_{xxxx} \\ & + \frac{1}{120}k^4c^4AU_{xxxxx} - \frac{1}{720}k^5c^6U_{xxxxxx})_J + O(k^6). \end{aligned} \quad (17)$$

Now to proceed with the proof we need to expand each term of (17) up to sixth order about $x = \alpha$. To this end, we distinguish two sets of points in first summation

$$\begin{aligned} U_{j+l-4} = & U^- + r_l U_x^- + \frac{1}{2}r_l^2 U_{xx}^- + \frac{1}{6}r_l^3 U_{xxx}^- + \frac{1}{24}r_l^4 U_{xxxx}^- \\ & + \frac{1}{120}r_l^5 U_{xxxxx}^- + \frac{1}{720}r_l^6 U_{xxxxxx}^-, \quad 1 \leq l \leq 4, \end{aligned} \quad (18)$$

$$\begin{aligned}
U_{j+l-4} = & D_0 U^- + r_l D_1 U_x^- + \frac{1}{2} r_l^2 D_2 U_{xx}^- + \frac{1}{6} r_l^3 D_3 U_{xxx}^- + \frac{1}{24} r_l^4 D_4 U_{xxxx}^- \\
& + \frac{1}{120} r_l^5 D_5 U_{xxxxx}^- + \frac{1}{720} r_l^6 D_6 U_{xxxxxx}^- \quad 5 \leq l \leq 7, \quad (19)
\end{aligned}$$

where $U^- = \lim_{x \rightarrow \alpha^-} U(x, t)$, and

$$r_l = x_{J-4+l} - \alpha, \quad (l = 1, 2, \dots, 7).$$

Note that we have used the jump conditions (10) in (19). If we substitute (18), (19) and similar expansions for other terms into (17) we obtain L as a function of U^- , U_x^- , U_{xx}^- , U_{xxx}^- , U_{xxxx}^- , U_{xxxxx}^- and U_{xxxxxx}^- . Therefore, to achieve sixth order accuracy we have to force the coefficients of these terms to be zero. These systems of matrix equations are exactly the same as (13), (14) and (15). \square

In theorem 4.1, the unknown 2×2 matrices $\Gamma_{J,l}$, $l = 1, 2, \dots, 7$ will be obtained by solving the linear system (13). These linear systems can be easily converted to some lower order linear systems. As the matrices D_j , $j = 1, \dots, 7$, are diagonal it is possible to decouple these systems to four 7×7 linear systems; e.g., the first 7×7 linear system determines the scalar unknowns $(\Gamma_{J,l})_{11}$, $l = 1, 2, \dots, 7$. In fact, because of this property, there are only two different coefficient matrices in these four systems of linear equations. These properties are valid at all irregular points. It should be mentioned that in the tested numerical problems we did not get any ill-conditioning warning due to the coefficient matrices. On the other irregular points similar relations can be derived. So, at the grid point $J-1$ one obtains,

$$\sum_{l=1}^5 \alpha_{il} \Gamma_{J-1,l} + \sum_{l=6}^7 \alpha_{il} \Gamma_{J-1,l} D_i = F_i^-, \quad (i = 0, 1, \dots, 6)$$

where,

$$\alpha_{il} = \left(\frac{r_l}{h} - 1\right)^i, \quad (i = 0, 1, \dots, 6, \quad l = 1, 2, \dots, 7).$$

At the grid point $J-2$ we have

$$\sum_{l=1}^6 \alpha_{il} \Gamma_{J-2,l} + \alpha_{i7} \Gamma_{J-2,l} D_i = F_i^-, \quad (i = 0, 1, \dots, 6)$$

where,

$$\alpha_{il} = \left(\frac{r_l}{h} - 2\right)^i, \quad (i = 0, 1, \dots, 6, \quad l = 1, 2, \dots, 7).$$

At the grid point $J+1$ we have

$$\sum_{l=1}^3 \alpha_{il} \Gamma_{J+1,l} D_i^{-1} + \sum_{l=4}^7 \alpha_{il} \Gamma_{J+1,l} = F_i^+, \quad (i = 0, 1, \dots, 6)$$

where,

$$\alpha_{il} = \left(\frac{r_l}{h} + 1\right)^i, \quad (i = 0, 1, \dots, 6, \quad l = 1, 2, \dots, 7).$$

At the grid point $J + 2$ we obtain

$$\sum_{l=1}^2 \alpha_{il} \Gamma_{J+2,l} D_i^{-1} + \sum_{l=3}^7 \alpha_{il} \Gamma_{J+2,l} = F_i^+, \quad (i = 0, 1, \dots, 6)$$

where,

$$\alpha_{il} = \left(\frac{r_l}{h} + 2\right)^i, \quad (i = 0, 1, \dots, 6, \quad l = 1, 2, \dots, 7).$$

At the grid point $J + 3$ we have

$$\alpha_{i1} \Gamma_{J+3,1} D_i^{-1} + \sum_{l=2}^7 \alpha_{il} \Gamma_{J+3,l} = F_i^+, \quad (i = 0, 1, \dots, 6)$$

where,

$$\alpha_{il} = \left(\frac{r_l}{h} + 3\right)^i, \quad (i = 0, 1, \dots, 6, \quad l = 1, 2, \dots, 7).$$

The given formulation of the immersed interface method demonstrates the possibility of the direct extension of these relations to higher orders. The closed form formulas for right hand side matrices (15) are valid for lower and higher order formulations. For higher order methods α_{14} should only be replaced with a new one; for example, this element for fourth order method is α_{13} . The proof of the following theorem is similar to Theorem 4.1 and so we omit it.

Theorem 4.2. *If the coefficient matrices of (6) satisfy the following system of matrix equations*

$$\sum_{l=1}^m \alpha_{il} \Gamma_{j,l} D_i^{-1} + \sum_{l=m+1}^{M+1} \alpha_{il} \Gamma_{j,l} = F_i^*, \quad (i = 0, 1, \dots, M - 1)$$

where for $j \leq J$, $F_i^* = F_i^- D^{-1}$ and for $j \geq J + 1$, $F_i^* = F_i^+$. Then, the method (6) gives a M th order approximation of the solution of (1) at irregular grid x_j .

Now we extend Theorem 4.1 to the case where the coefficients are piecewise smooth. The local truncation error for a general piecewise smooth co-

efficient at x_J can be represented as follows

$$L = \frac{1}{k} \sum_{l=1}^7 \Gamma_{j,l} U_{j+l-4} - \left(T^{(1)} U_x + k T^{(2)} U_{xx} + k^2 T^{(3)} U_{xxx} + k^3 T^{(4)} U_{xxxx} + k^4 T^{(5)} U_{xxxxx} + k^5 T^{(6)} U_{xxxxxx} \right)_J + O(k^6).$$

where

$$\begin{aligned} T^{(1)} &= -A + \frac{1}{2} k B^{(1)} + \frac{1}{6} k^2 C^{(1)} + \frac{1}{24} k^3 D^{(1)} + \frac{1}{120} k^4 E^{(1)} + \frac{1}{720} k^5 F^{(1)}, \\ T^{(2)} &= \frac{1}{2} B^{(2)} + \frac{1}{6} k C^{(2)} + \frac{1}{24} k^2 D^{(2)} + \frac{1}{120} k^3 E^{(2)} + \frac{1}{720} k^4 F^{(2)}, \\ T^{(3)} &= \frac{1}{6} C^{(3)} + \frac{1}{24} k D^{(3)} + \frac{1}{120} k^2 E^{(3)} + \frac{1}{720} k^3 F^{(3)}, \\ T^{(4)} &= \frac{1}{24} D^{(4)} + \frac{1}{120} k E^{(4)} + \frac{1}{720} k^2 F^{(4)}, \\ T^{(5)} &= \frac{1}{120} E^{(5)} + \frac{1}{720} k F^{(5)}, \\ T^{(6)} &= \frac{1}{720} F^{(6)}. \end{aligned}$$

To obtain a sixth order method it is required that the matrices $\Gamma_{J,l}$ satisfy the following linear matrix system

$$\sum_{l=1}^4 \alpha_{i,l} \Gamma_{J,l} + \sum_{l=5}^7 \alpha_{i,l} \Gamma_{J,l} Q^{(i,l)} = R_i, \quad i = 1, 2, \dots, 7. \quad (20)$$

where

$$\begin{aligned} Q^{(1,l)} &= I, \\ Q^{(2,l)} &= Q_1 + \frac{1}{2} r_l Q_2 + \frac{1}{6} r_l^2 Q_4 + \frac{1}{24} r_l^3 Q_7 + \frac{1}{120} r_l^4 Q_{11} + \frac{1}{720} r_l^5 Q_{16}, \\ Q^{(3,l)} &= Q_3 + \frac{1}{3} r_l Q_5 + \frac{1}{12} r_l^2 Q_8 + \frac{1}{60} r_l^3 Q_{12} + \frac{1}{360} r_l^4 Q_{17}, \\ Q^{(4,l)} &= Q_6 + \frac{1}{4} r_l Q_9 + \frac{1}{20} r_l^2 Q_{13} + \frac{1}{120} r_l^3 Q_{18}, \\ Q^{(5,l)} &= Q_{10} + \frac{1}{5} r_l Q_{14} + \frac{1}{30} r_l^2 Q_{19}, \\ Q^{(6,l)} &= Q_{15} + \frac{1}{6} r_l Q_{20}, \\ Q^{(7,l)} &= Q_{21}. \end{aligned}$$

and

$$\begin{aligned}
R_1 &= 0, \\
R_2 &= \nu T^{(1)}, \\
R_3 &= 2\nu\alpha_{1,4}T^{(1)} + 2\nu^2T^{(2)}, \\
R_4 &= 3\nu\alpha_{2,4}T^{(1)} + 6\nu^2\alpha_{1,4}T^{(2)} + 6\nu^3T^{(3)}, \\
R_5 &= 4\nu\alpha_{3,4}T^{(1)} + 12\nu^2\alpha_{2,4}T^{(2)} + 24\nu^3\alpha_{1,4}T^{(3)} + 24\nu^4T^{(4)}, \\
R_6 &= 5\nu\alpha_{4,4}T^{(1)} + 20\nu^2\alpha_{4,3}kT^{(2)} + 60\nu^3\alpha_{4,2}T^{(3)} + 120\nu^3\alpha_{4,1}T^{(4)} + 120\nu^4T^{(5)}, \\
R_7 &= 6\nu\alpha_{5,4}T^{(1)} + 30\nu^2\alpha_{4,4}T^{(2)} + 120\nu^3\alpha_{3,4}T^{(3)} + 360\nu^4\alpha_{2,4}T^{(4)} \\
&\quad + 720\nu^5\alpha_{1,4}T^{(5)} + 720\nu^6T^{(6)}.
\end{aligned}$$

We note that the matrices $Q^{(i,l)}$ are diagonal 2×2 matrices and it is possible to solve the linear systems (20) in a similar way as discussed before.

At the irregular point x_{J+1} we have

$$\sum_{l=1}^3 \alpha_{i,l} \Gamma_{J,l} \hat{Q}^{(i,l)} + \sum_{l=4}^7 \alpha_{i,l} \Gamma_{J,l} = R_i, \quad i = 1, 2, \dots, 7.$$

where

$$\begin{aligned}
\hat{Q}^{(1,l)} &= I, \\
\hat{Q}^{(2,l)} &= Q_1^{-1} + \frac{1}{2}r_l Q_2^{-1} + \frac{1}{6}r_l^2 Q_4^{-1} + \frac{1}{24}r_l^3 Q_7^{-1} + \frac{1}{120}r_l^4 Q_{11}^{-1} + \frac{1}{720}r_l^5 Q_{16}^{-1}, \\
\hat{Q}^{(3,l)} &= Q_3^{-1} + \frac{1}{3}r_l Q_5^{-1} + \frac{1}{12}r_l^2 Q_8^{-1} + \frac{1}{60}r_l^3 Q_{12}^{-1} + \frac{1}{360}r_l^4 Q_{17}^{-1}, \\
\hat{Q}^{(4,l)} &= Q_6^{-1} + \frac{1}{4}r_l^4 Q_9^{-1} + \frac{1}{20}r_l^2 Q_{13}^{-1} + \frac{1}{120}r_l^3 Q_{18}^{-1}, \\
\hat{Q}^{(5,l)} &= Q_{10}^{-1} + \frac{1}{5}r_l Q_{14}^{-1} + \frac{1}{30}r_l^2 Q_{19}^{-1}, \\
\hat{Q}^{(6,l)} &= Q_{15}^{-1} + \frac{1}{6}r_l Q_{20}^{-1}, \\
\hat{Q}^{(7,l)} &= Q_{21}^{-1}.
\end{aligned}$$

Similar formulae can be deduced for other irregular points.

5 Numerical Results

In this section, some test problems are given to show the efficiency of the derived high order method. The simulation results are given for different cases. The test problems Test 1 and Test 2 illustrate general behavior of the solution of the acoustic wave equation at the interface. Also, the numerical results of the test problem Test 4 (in two cases 4-1, 4-2) are given for the acoustic wave equation with piecewise smooth coefficients. The parameter

values, the function $f(x)$, and the CFL number are clearly specified in each test problem. The numerical order of accuracy and L_1 and L_∞ errors are reported for the test problem Test 3, see Table 1, which verifies numerically the long time behavior of this approximation. The test problem Test 5 shows the numerical stability of the method. The computational cost for the calculation of the coefficient matrices at irregular points is independent of N , the number of spatial grid points in the discretization. The coefficient matrices can be computed in a couple of milliseconds on a desktop computer.

The numerical results are given for the following acoustic wave equation problem [3]:

If $x < \alpha$:

$$\begin{aligned} u(x, t) &= \frac{1}{\rho_l c_l} \left(f\left(t - \frac{x - \alpha}{c_l}\right) + \frac{\rho_l c_l - \rho_r c_r}{\rho_l c_l + \rho_r c_r} f\left(t + \frac{x - \alpha}{c_l}\right) \right), \\ p(x, t) &= f\left(t - \frac{x - \alpha}{c_l}\right) - \frac{\rho_l c_l - \rho_r c_r}{\rho_l c_l + \rho_r c_r} f\left(t + \frac{x - \alpha}{c_l}\right), \end{aligned} \quad (21)$$

and if $\alpha \leq x$:

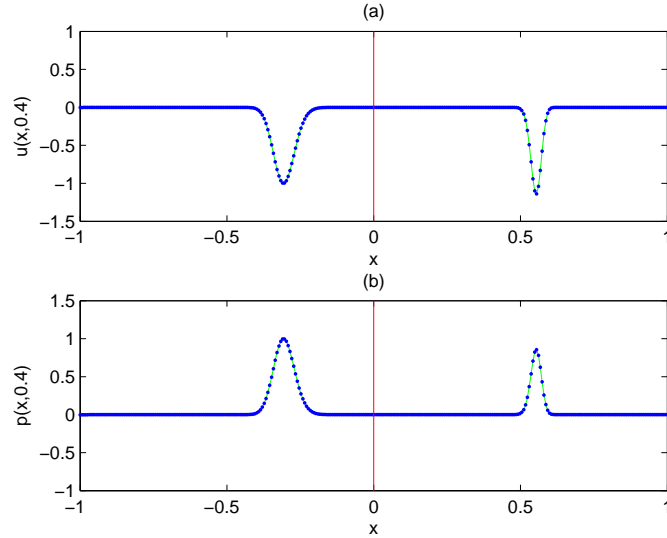
$$\begin{aligned} u(x, t) &= \frac{2}{\rho_l c_l + \rho_r c_r} f\left(t - \frac{x - \alpha}{c_r}\right), \\ p(x, t) &= \frac{2\rho_r c_r}{\rho_l c_l + \rho_r c_r} f\left(t - \frac{x - \alpha}{c_r}\right), \end{aligned} \quad (22)$$

where, $f(x)$ is a smooth function.

Test 1: In this test, we consider $f(x) = e^{-200(x^2-1/2)^2}$ and the parameters are chosen as $\alpha = 0$, $\rho_l = -1$, $\rho_r = -1.5$, $c_l = 1$, $c_r = 0.5$. Figures 1 and 2 illustrate numerical and exact solutions for u and p with $N = 320$ and $c_{\max}\lambda = 0.8$. There are two gaussian pulses going to the right and a pulse hits the interface and then transmits with a generated reflecting pulse. The CFL number in this case is about one and there is no spurious oscillation.

Test 2: In this test, problem we consider a rather high frequency function $f(x) = \sin(30x)$ with parameters $\alpha = 0$, $\rho_l = 0.5$, $\rho_r = 1.0$, $c_l = 0.8$, $c_r = 1.0$. Figures 3 and 4 illustrate numerical and exact solutions for u and p with $N = 320$ and $c_{\max}\lambda = 0.8$. After hitting the interface the magnitudes of velocity and pressure are changed.

Test 3: To illustrate the long time behavior of this method, we consider the jump $f(x) = \sin(\pi x)$, as initial data for acoustic equations with $\alpha = \frac{\pi}{2}$, and the same parameters as in Test 2. The numerical results are reported in Table 1 for several values of N at $t = 50\pi$. The numerical order of accuracy and L_1 and L_∞ errors are presented in this Table in which LaxW-IIM denotes the Lax-Wendroff immersed interface method.

Figure 1: Test 1: Numerical(.) and exact(-) solutions for u and p at $t = 0.4$

It is well known that a typical high order method provides first order results for interface problems[14, 15]. In Table 2, we show the result of eliminating of jump conditions and using the high order numerical method (6) without interface treatment. We can clearly see that the results are at most first order.

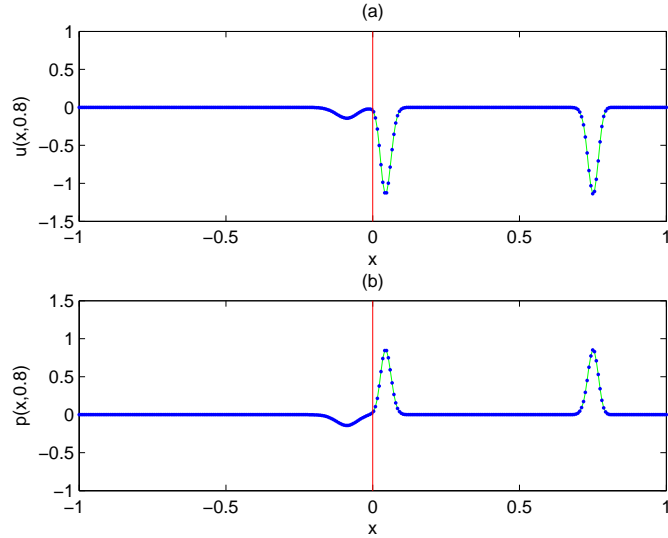
Table 1: (Test 3) The L_1 , L_∞ errors and the numerical order of accuracy for LaxW-IIM method over the whole interval and the same quantities at irregular points for $f(x) = \sin(\pi x)$ at $t = 50\pi$ are reported

N	LaxW-IIM				Irregular points			
	L_1 error	order	L_∞ error	order	L_1 error	order	L_∞ error	order
15	8.37E-001		1.56E-001		2.44E-001		1.06E-001	
30	4.03E-002	4.17	3.32E-003	4.89	6.92E-003	4.89	1.81E-003	5.58
60	1.11E-003	5.06	4.14E-005	5.46	1.43E-004	5.46	3.05E-005	5.75
120	3.05E-005	5.12	5.26E-007	5.85	2.37E-006	5.85	4.42E-007	6.04
240	8.90E-007	5.07	7.18E-009	5.98	3.67E-008	5.98	6.54E-009	6.04

Test 4: In this test, problem we consider two variable coefficient problems.

Let us define a general form of the variable coefficient,

$$(\rho(x), c(x)) = \begin{cases} (\rho^- + f_1(x), c^- + g_1(x)), & x < \alpha, \\ (\rho^+ + f_2(x), c^+ + g_2(x)), & x > \alpha. \end{cases} \quad (23)$$

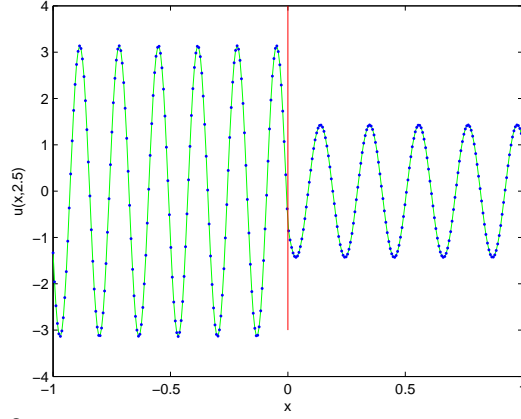
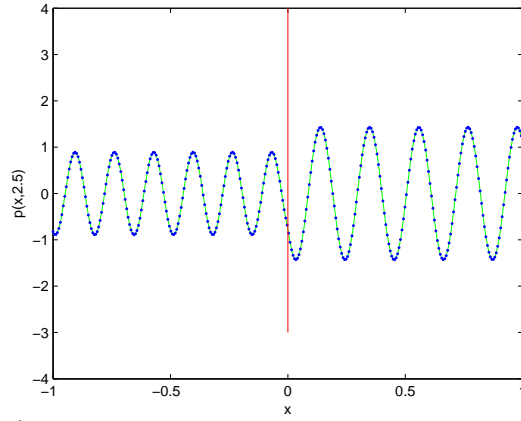
Figure 2: Test 1: Numerical(.) and exact(-) solutions for u and p at $t = 0.8$ Table 2: (Test 3) The L_1 , L_∞ errors and the numerical order of accuracy for method in equation (6) over the whole interval and the same quantities at irregular points for $f(x) = \sin(\pi x)$ at $t = 50\pi$, without interface treatment, are reported

N	LaxW-IIM				Irregular points			
	L_1 error	order	L_∞ error	order	L_1 error	order	L_∞ error	order
15	7.91E+000		2.35E+000		5.17E+000		2.35E+000	
30	3.84E+000	0.99	8.60E-001	1.57	1.65E+000	1.57	8.16E-001	1.45
60	4.19E+000	0.12	3.56E-001	1.14	7.33E-001	1.14	3.56E-001	1.17
120	4.29E+000	0.03	1.84E-001	0.92	3.84E-001	0.92	1.84E-001	0.94
240	4.33E+000	0.01	9.24E-002	0.97	1.95E-001	0.97	9.24E-002	0.99

where f_1, f_2, g_1 and g_2 are arbitrary and smooth functions which vanish at $x = \alpha$ and ρ^-, ρ^+, c^- and c^+ are constants.

To illustrate the behavior of the numerical solution near the interface, we consider the following two sets of functions and we show the numerical quality of solution for the first set through some figures and for the second set of functions Table 4 is given in which the order of accuracy is reported numerically.

1. The first coefficient set:

Figure 3: Test 2: Numerical(.) and exact(-) solutions for u at $t = 2.5$ Figure 4: Test 2: Numerical(.) and exact(-) solutions for p at $t = 2.5$

$$\begin{aligned}
 f_1(x) &= (x + 1) \sin(x - \alpha), \\
 g_1(x) &= (x + 1)(x - \alpha), \\
 f_2(x) &= (x - 1) \sin(x - \alpha), \\
 g_2(x) &= (x - 1)(x - \alpha).
 \end{aligned} \tag{24}$$

2. The second coefficient set:

$$\begin{aligned}
 f_1(x) &= 0, \\
 g_1(x) &= 0, \\
 f_2(x) &= 0, \\
 g_2(x) &= -6(e^{\frac{\alpha}{c^+}} - 1).
 \end{aligned} \tag{25}$$

Table 3: Jumps in the coefficients of Test 4-1

k	$f_1^{(k)}(0^-)$	$f_2^{(k)}(0^+)$	$g_1^{(k)}(0^-)$	$g_2^{(k)}(0^+)$	$\rho^{(k)}(0^-)$	$\rho^{(k)}(0^+)$	$c^{(k)}(0^-)$	$c^{(k)}(0^+)$
0	0	0	0	0	0.5	1	0.8	1
1	1	-1	1	-1	1	-1	1	-1
2	2	2	2	2	2	2	2	2
3	-1	1	0	0	-1	1	0	0
4	-4	-4	0	0	-4	-4	0	0
5	1	-1	0	0	1	-1	0	0

The parameters for the first set are the same as in Test 2 and for the second one we consider $c^- = 1, c^+ = \frac{1+\sqrt{5}}{2}, \rho^- = 1, \rho^+ = -6$. The details of the jump discontinuities of the first coefficient set (24) are reported in Table 3. From Table 3 it is clearly seen that the first order derivatives of ρ and c are discontinuous and since c is a polynomial, its higher order derivatives become zero while the higher derivatives of ρ are discontinuous.

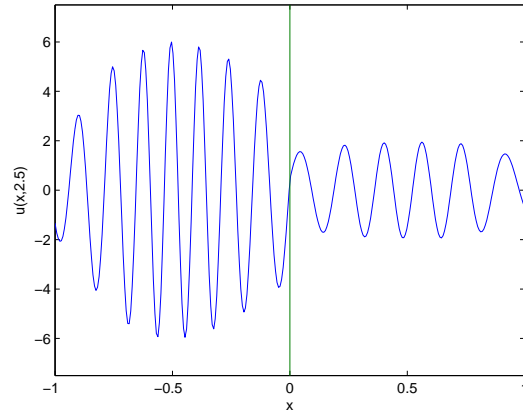
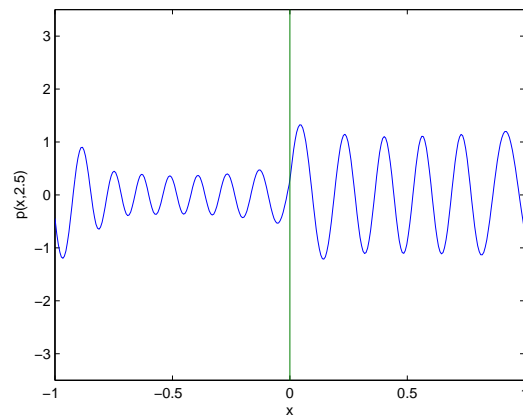
Figures 5 and 6 for (24) illustrate the quality of the numerical solutions for u and p with $N = 320$ and $c_{\max}\lambda = 0.8$. This behavior confirms that the method has been able to successfully capture the solution near the interface without any spurious oscillations.

The numerical order of accuracy and errors for (25), the second coefficient set, has been shown in Table 4 for the final time $t = 0.5$ with the following initial data,

$$u(x, 0) = \begin{cases} 2e^{-x} - e^x, & x \leq 0, \\ e^{0.8x}, & x > 0 \end{cases}, \quad p(x, 0) = \begin{cases} 2e^{-x} + e^x, & x \leq 0, \\ 3e^{2x}, & x > 0. \end{cases} \quad (26)$$

It should be mentioned that the results of Table 4 have been obtained by implementing our formulations with exact coefficients, confirming that the true order of accuracy of the presented method for this type of coefficients is also 6. Since the computation of jump conditions for the case of piecewise constant coefficients is simple, in practice one might prefer to use some approximation of the exact coefficients in the implementation.

Since the obtained approximation is close to the exact coefficient, the order of accuracy of the numerical results obtained by the method of this paper should be closer to the true 6th order. We have examined the presented method using best uniform approximation of degree zero for the coefficients near the interface and we obtained an order of accuracy of at least 2. We expect to get a solution of higher order of accuracy if the best uniform polynomial approximation of a higher degree is used.

Figure 5: Test 4-1: Quality of the numerical solution for u at $t = 2.5$ Figure 6: Test 4-1: Quality of the numerical solution for p at $t = 2.5$

eliminate this index for simplicity. For linear stability the eigenvalues of G should lie in the unit circle in the complex plane. We numerically locate the eigenvalues of G . This matrix depends on $A(x)$ and λ . Therefor we report the stability results for several values of these parameters.

Table 5: The norm of eigenvalues of influence matrix for different values of parameters. The letter p denotes the periodic boundary conditions

λ	c	ρ	$\rho(G)$	$\rho(G_p)$
1.000	1.000	-1.000	0.543	1.500
2.000	0.500	-1.500	0.554	0.658
1.250	0.800	0.500	0.837	1.187
1.000	1.000	1.000	0.558	1.500

We remark that the boundary conditions have important role in the stability of the problem. In the case of frozen coefficients, i.e. $A(x) = A^+$ or $A(x) = A^-$, the results are shown in the Table 5. In this case we can choose λ large enough for different values of $A(x)$. While, a comparison between different rows of this table shows that in general the eigenvalues are nondecreasing with variation of parameters. Therefor, for nonsmooth coefficients that the situation is more complicated, the inequality $\max_x \{|c(x)|, 1\} \frac{k}{h} < 1$ is a reasonable criteria and numerical tests confirm that this criteria in our test problems.

Test 5: In this test we consider an initial condition $u(x_j, 0) = R_j e^{-6(\frac{x_j - \alpha}{5h})^2}$ and $v(x, 0) = 2u(x, 0)$, where R_j are uniformly distributed random numbers in the interval $[0, 1]$. This example is a variant of a similar one dimensional case in [7]. The parameters are $c_l = 1.0$, $c_r = 0.5$, $\rho_l = 2.5$, $\rho_r = 10.0$, $N = 1000$ and $c_{\max}\lambda = 0.99$. The results are given in Figure 7, which is a typical test among many other tests. There are no noise generation visible near the interface and the norms of the solutions do not grow with $\frac{k}{h}$.

5.2 Two dimensional problems

Implementation of high order interface method for two dimensional acoustic wave equations requires high order jump conditions on the interface. In most applications the standard jump conditions are available in the literature. Such jump conditions are usually given in the normal and tangential directions to the interface. Therefor, we need to define a local coordinate in a typical point on the interface to obtain the required approximations at the interface(see Figure 8). This is done after transformation of the equation to

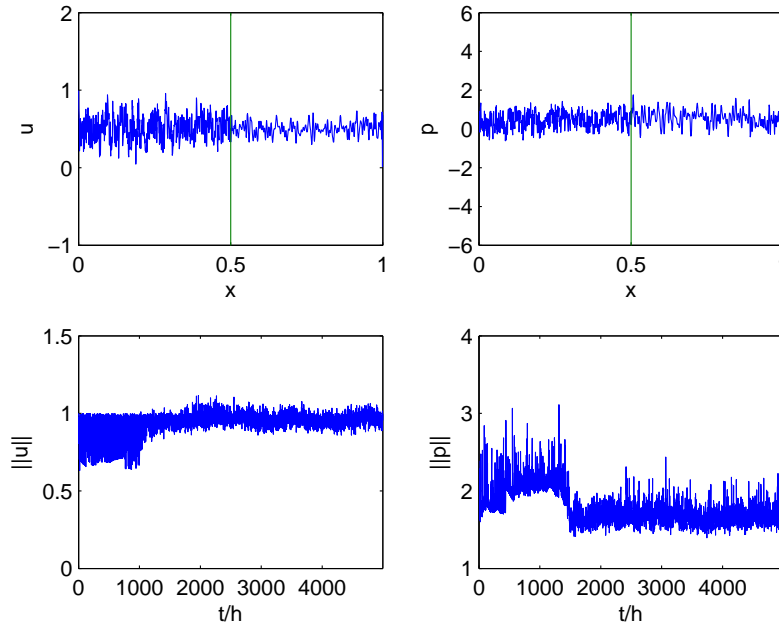


Figure 7: Test 5: The norm and the solution after a long time

the new local coordinate system in $\xi - \eta$ plane. The same formulation in one dimensional case will direct us to the set of equations to be solved for the 2D and 3D cases.

6 Conclusions and discussions

In this paper, we have presented a sixth order immersed interface method for acoustic wave equation with discontinuous coefficient. The effect of piecewise constant and a more general piecewise smooth coefficients on the derived formulations has been investigated. We have also provided different numerical tests which confirm the efficiency of the method and justify their order of accuracy and numerical stability. It should be mentioned that, using jump conditions do not impose a considerable computational cost in the calculations and one should only solve some low order linear systems to obtain the coefficients. In fact, the special treatment of the interface is a preprocessing stage in the implementation of immersed interface method and without loss of overall speed of computation it is also applicable in the parallel computers. In the numerical results, we applied the Lax-Wendroff method for time discretization. However, the weighted essentially nonoscillatory (WENO) and

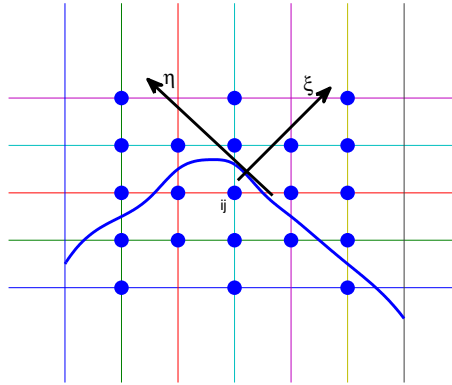


Figure 8: Local coordinates in a two dimensional grid with a curve interface

the total variation diminishing (TVD) methods [6, 9] reduce the possible oscillations in the solution. These methods recently have been added to the CLAWPACK software [9] for numerical solution of conservation laws.

References

1. Driscoll, T. A. and Fornberg, B. *A block pseudospectral method for maxwells equations I. one-dimensional case*, Journal of Comput. Phys. 140 (1998) 47-65.
2. Farzi, J. and Hosseini, S. M. *A high order method for the solution of a one-way wave equation in heterogeneous media*, Far East J. Appl. Math. 36 (3) (2009) 317-330.
3. Gustafsson, B. and Wahlund, P. *Time compact high order difference methods for wave propagation in discontinuous media*, SIAM J. Sci. Comput. 26 (2004) 272-293.
4. Gustafsson, B. and Mossberg, E. *High order difference methods for wave propagation*, SIAM J. Sci. Comput. 26 (2004) 259-271.
5. Gustafsson, B., Kreiss, H. O. and Olinger, J. *Time dependent problems and difference methods*, first ed., John Wiley and Sons, (1996).
6. Jiang, G. S. and Shu, C. W. *Efficient implementation of weighted ENO schemes*, J. Comput. Phys., 126 (1996) 202-228.
7. Larsson, J. and Gustafsson, B. *Stability criteria for hybrid difference methods*, J. comp. phys. 227 (2008) 2886-2898.

8. LeVeque, R. J. *Wave propagation algorithms for multidimensional hyperbolic systems*, J. comp. phys. 131 (1997) 327-353.
9. LeVeque, R. J. *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, Cambridge, 2004.
10. Li, Z. and Ito, K. *The immersed interface method: numerical solutions of PDEs involving interfaces and irregular domains*, SIAM, 2006.
11. Peskin, C. S. *Numerical analysis of blood flow in the heart*, J. Comput. Phys., 25 (1977) 220-252.
12. Piraux, J. and Lombard, B. *A new interface method for hyperbolic problems with discontinuous coefficients. one-dimensional acoustic example*, J. Comput. Phys., 168 (2001) 227-248.
13. Qiu, J. and Shu, C. W. *Finite difference WENO schemes with Lax-Wendroff type time discretizations*, SIAM J. Sci. Comput. 24 (2003) 2185-2198.
14. Sei, A. and Symes, W. W. *Error analysis of numerical schemes for the wave equation in heterogeneous media*, Appl. Numer. Math. 15 (1994) 465-480.
15. Symes, W. W. and Vdovina, T. *Interface error analysis for numerical wave propagation*, Comput. Geosci. 13 (2009) 363-371.
16. Trefethen, L. N. *Group velocity in finite difference schemes*, SIAM. Rev. 24 (1982) 113-136.
17. Wiegmann, A. and Bube, K. P. *The explicit-jump immersed interface method: finite difference methods for PDE with piecewise smooth solutions*, SIAM J. Numer. Anal., 37 (2000) 827-862.
18. Zhang, C. and LeVeque, R. J. *Immersed interface methods for wave equations with discontinuous coefficients*, Wave Motion, 25 (1997) 237-263.
19. Zhao, S. and Wei, G. W. *High-order FDTD methods via derivative matching for Maxwell's equations with material interfaces*, J. Comput. Phys., 200 (2004) 60-103.