

# Chebyshev Galerkin method for integro-differential equations of the second kind

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## Abstract

In this paper, we propose an efficient implementation of the Chebyshev Galerkin method for first order Volterra and Fredholm integro-differential equations of the second kind. Some numerical examples are presented to show the accuracy of the method.

**Keywords:** Volterra integro-differential equations; Galerkin method; Chebyshev polynomials.

## 1 Introduction

Integro-differential equations occur in various areas. These equations arise in mathematical modeling of many scientific phenomena, such as fluid dynamics, solid state physics, plasma physics, mathematical biology viscoelasticity [33], heat transfer [6], economics [26], chemostat [41], HIV models [4], biotissues [15], static analysis of wind towers or chimneys [35], and chemical kinetics [34]. Integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions may appear to any order [2, 40].

The concepts of integral equations have motivated a large amount of research work in recent years. Many numerical methods have been applied to solve these equations such as: El-gendi and Galerkin [11, 12, 27], Euler-Chebyshev [37], Variational iteration [39], Homotopy perturbation [10, 32], Chebyshev and Taylor collocation [1, 3, 8, 13, 20, 28], Chebyshev Wavelets [5],

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Spline collocation [7], finite element [9], sinc collocation [43], Bessel polynomials [42], Legendre polynomials [23], Bernstein polynomials [21] and Lagrange polynomials [31] and etc. [27, 36].

Galerkin method is a powerful tool for solving many kinds of equations in various fields of science and engineering. It is one of the most important weighted residual methods invented by the Russian mathematician Boris Grigoryevich Galerkin. Recently, various Galerkin algorithms have been applied in numerical solution of integral equations and integro differential equations. We can mention the following methods that are based on the Galerkin idea: Galerkin finite element [22], iterated Galerkin [38], Galerkin with hybrid functions [27], Crank–Nicolson least–squares Galerkin [18], Wavelet–Galerkin [16], Discrete Galerkin [30], Petrov–Galerkin [25], pseudo–spectral Legendre–Galerkin [14] and etc. There are many different families of orthogonal functions, which can be used. Chebyshev polynomials are considerably useful to solve integro-differential equations.

In this paper, the solution is approximated by a linear combination of the first  $N+1$  Chebyshev polynomials, with  $\{a_i\}_{i=0}^N$ , as coefficients. Approximate solution will be simplified as a polynomial in  $x$ . This approximation will be substituted in the equation. To determine  $a_j$ , one can consider inner product of both sides of the equation, by  $T_j(x)$ . This procedure reduces the problem to a system of equations. The generated system, which considering the type of the equation will be either linear or nonlinear, can be solved through various methods and the unknown coefficients can be found. Practically, all orthogonal polynomials, on a closed finite interval, can also be applied for approximating functions. But convergence of the partial sums of the first-kind Chebyshev expansion, of a continuous function on  $[-1, 1]$ , is faster than the partial sums of an expansion in any other orthogonal polynomials [3].

The outline of this paper is as follows: Section 2 presents the method for solving Volterra integro-differential equations. Numerical examples are given in Section 3. Finally, conclusion will be presented in Section 4.

## 2 Chebyshev Galerkin method

Consider the following Volterra integro-differential equation

$$u'(x) = f(x) + \lambda \int_a^x K(x,t)u(t) dt, \quad a \leq x \leq b, \quad (1)$$

$$u(a) = \alpha. \quad (2)$$

where  $u(x)$  is the unknown function,  $K(x,t)$  is a known continuous and square integrable function,  $f(x)$  is a known function, and  $\lambda$  is a real known parameter.

The method under study uses Chebyshev polynomials, well addressed in [29],

as a basis polynomial to approximate the solution on a closed finite interval. Assume that

$$u(x) \approx u_N(x) = \sum_{i=0}^N a_i T_i \left( \frac{2x - (b-a)}{b-a} \right), \quad (3)$$

where  $T_i \left( \frac{2x - (b-a)}{b-a} \right)$  is shifted Chebyshev polynomial at  $[a, b]$ . So we have

$$u'(x) \approx u'_N(x) = \sum_{i=0}^N \frac{2}{b-a} a_i T'_i \left( \frac{2x - (b-a)}{b-a} \right). \quad (4)$$

Substituting (3) and (4) into (1), results in

$$\begin{aligned} & \sum_{i=0}^N \frac{2}{b-a} a_i T'_i \left( \frac{2x - (b-a)}{b-a} \right) \\ &= f(x) + \lambda \sum_{i=0}^n a_i \int_a^x K(x, t) T_i \left( \frac{2t - (b-a)}{b-a} \right) dt, \quad a \leq x \leq b. \end{aligned} \quad (5)$$

To determine unknown coefficients  $a_i$ , we use the Galerkin idea by multiplying both sides of (5) by  $T_j \left( \frac{2x - (b-a)}{b-a} \right)$  and then integrating with respect to  $x$  from  $-1$  to  $1$ . So we have

$$\begin{aligned} & \sum_{i=0}^N \frac{2}{b-a} a_i \int_{-1}^1 T'_i \left( \frac{2x - (b-a)}{b-a} \right) T_j \left( \frac{2x - (b-a)}{b-a} \right) dx = \\ & \int_{-1}^1 f(x) T_j \left( \frac{2x - (b-a)}{b-a} \right) dx + \\ & \int_{-1}^1 \left( \lambda \sum_{i=0}^n a_i \int_a^x K(x, t) T_i \left( \frac{2t - (b-a)}{b-a} \right) dt \right) T_j \left( \frac{2x - (b-a)}{b-a} \right) dx, \end{aligned} \quad (6)$$

for  $j = 0, 1, \dots, N$ , or equivalently

$$\begin{aligned} & \sum_{i=0}^N \frac{2}{b-a} a_i \int_{-1}^1 T'_i \left( \frac{2x - (b-a)}{b-a} \right) T_j \left( \frac{2x - (b-a)}{b-a} \right) dx = \\ & \int_{-1}^1 f(x) T_j \left( \frac{2x - (b-a)}{b-a} \right) dx + \\ & \lambda \sum_{i=0}^n a_i \int_{-1}^1 \left( \int_a^x K(x, t) T_i \left( \frac{2t - (b-a)}{b-a} \right) dt \right) T_j \left( \frac{2x - (b-a)}{b-a} \right) dx. \end{aligned} \quad (7)$$

If needed the integrals can be calculated by numerical methods. This procedure generates a system of linear equations for the unknown  $\{a_i\}_{i=0}^N$ . Many researchers substitute initial condition

$$u(a) = \alpha \Rightarrow \sum_{i=0}^N a_i T_i \left( \frac{2a - (b-a)}{b-a} \right) = \sum_{i=0}^N a_i T_i(-1) = \alpha. \quad (8)$$

for the same number of equations in the foregoing linear system.

The unknown parameters are determined by solving the system of equations (7) and (8). Substituting these values in (3) gives the approximate solution of the integro-differential equation (1). Similarly one can apply this approach for a Fredholm integro-differential equation in the following general form:

$$\begin{aligned} u'(x) &= f(x) + \lambda \int_a^b K(x,t)u(t) dt, \quad a \leq x \leq b, \\ u(a) &= \alpha \end{aligned}$$

### 3 Numerical Examples

In this section, we intend to show the efficiency of the Galerkin method for solving Volterra integro-differential equations of the second kind by Chebyshev polynomials by presenting three illustrative examples. The absolute error for this formulation is defined by

$$E(x) = |u(x) - u_N(x)|.$$

**Example 1.** Consider the following Fredholm integro-differential equations of the second kind [17]

$$\begin{aligned} u'(x) &= u(x) - \frac{1}{2}x + \frac{1}{(x+1)} - \ln(x+1) + \frac{1}{(\ln 2)^2} \int_0^1 \frac{x}{t+1} u(t) dt, \quad (9) \\ u(0) &= 0, \end{aligned}$$

with the exact solution  $u(x) = \ln(x+1)$ .

To solve Equation (9) we approximate  $u(x)$  and  $u'(x)$  as follows:

$$\begin{aligned} u_4(x) &= \sum_{i=0}^4 a_i T_i \left( \frac{2x - (b-a)}{b-a} \right) = a_0 + a_1(2x-1) \\ &\quad + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + 18x - 1) \\ &\quad + a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1), \quad (10) \end{aligned}$$

and

$$u'_4(x) = \sum_{i=0}^4 \frac{2}{b-a} a_i T'_i \left( \frac{2x-(b-a)}{b-a} \right) = 2a_1 + a_2(16x-8) \\ + a_3(96x^2 - 96x + 18) + a_4(512x^3 - 768x^2 + 320x - 32). \quad (11)$$

Substituting (10) and (11) into (9), results in

$$2a_1 + a_2(16x-8) + a_3(96x^2 - 96x + 18) + a_4(512x^3 - 768x^2 + 320x - 32) \\ = a_0 + a_1(2x-1) + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + 18x - 1) \\ + a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) - \frac{1}{2}x + \frac{1}{(x+1)} - \ln(x+1) \\ + \frac{x}{(\ln 2)^2} \int_0^1 \frac{1}{t+1} (a_0 + a_1(2t-1) + a_2(8t^2 - 8t + 1) \\ + a_3(32t^3 - 48t^2 + 18t - 1) + a_4(128t^4 - 256t^3 + 160t^2 - 32t + 1)) dt, \quad (12)$$

By multiplying both sides of (12) by  $T_j \left( \frac{2x-(b-a)}{b-a} \right)$  and then integrating it with respect to  $x$  from  $-1$  to  $1$ , we obtain a system of linear equations which one of them is replaced by the equation

$$u(0) = 0 \Rightarrow \alpha_0 - \alpha_2 + \alpha_4 = 0 \quad (13)$$

Now the unknown coefficients  $\{a_i\}_{i=0}^4$  are determined by solving this system. Substituting these values in (3) gives the approximate solution of the integro-differential equation (1). The results have been shown in Table 1, for  $N = 4, 8, 12$ , and Error is plotted in Figure 1, for  $N = 12$ .

**Example 2.** Consider the following Volterra integro-differential equations of the second kind [40]

$$u'(x) = 1 - 2x \sin(x) + \int_0^x u(t) dt, \quad u(0) = 0.$$

The exact solution is  $y = x \cos(x)$ .

Table 2 shows the results for  $N = 4, 8, 12$ . Also Figure 2 shows absolute error for  $N = 12$ .

**Example 3.** Consider the following Volterra integro-differential equations [40]:

$$u'(x) = -1 + \frac{1}{2}x^2 - xe^x - \int_0^x tu(t) dt, \quad u(0) = 0.$$

The exact solution is  $y = 1 - e^x$ . Results have been shown in Table 3, for  $N = 4, 8, 12$ , and Error plotted in Figure 3, for  $N = 12$ .

Table 1: Absolute Error for Example 1

$x$	$N=4$		$N=8$		$N=12$	
	<i>Approx. solution</i>	<i>Abs. Error</i>	<i>Approx. solution</i>	<i>Abs. Error</i>	<i>Approx. solution</i>	<i>Abs. Error</i>
0.0	0	0	0	0	0	0
0.1	0.0949	4.233e-4	0.0953	4.556e-7	0.0953	4.998e-10
0.2	0.1817	6.317e-4	0.1823	7.690e-7	0.1823	7.938e-10
0.3	0.2615	8.544e-4	0.2624	1.082e-6	0.2624	1.146e-09
0.4	0.3353	1.161e-3	0.3365	1.416e-6	0.3365	1.508e-09
0.5	0.4039	1.543e-3	0.4055	1.860e-6	0.4055	1.964e-09
0.6	0.4680	1.971e-3	0.4700	2.361e-6	0.4700	2.481e-09
0.7	0.5282	2.430e-3	0.5306	2.909e-6	0.5306	3.088e-09
0.8	0.5848	2.951e-3	0.5878	3.614e-6	0.5878	3.821e-09
0.9	0.6382	3.626e-3	0.6418	4.438e-6	0.6419	4.687e-09
1.0	0.6885	4.629e-3	0.6931	5.611e-6	0.6931	5.930e-09

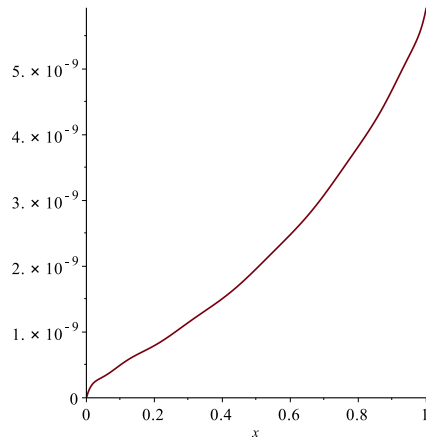


Figure 1: Absolute Error for Example 1

Table 2: Absolute Error for Example 2

$x$	$N=4$		$N=8$		$N=12$	
	<i>Approx. solution</i>	<i>Abs. Error</i>	<i>Approx. solution</i>	<i>Abs. Error</i>	<i>Approx. solution</i>	<i>Abs. Error</i>
0	0	0	0	0	0	0
0.1	0.0991	4.308e-4	0.0995	2.308e-9	0.0995	1.383e-15
0.2	0.1954	5.890e-4	0.1960	3.450e-9	0.1960	1.987e-15
0.3	0.2859	6.954e-4	0.2866	4.423e-9	0.2866	2.570e-15
0.4	0.3676	8.387e-4	0.3684	5.136e-9	0.3684	3.046e-15
0.5	0.4378	1.023e-3	0.4388	6.131e-9	0.4388	3.586e-15
0.6	0.4940	1.214e-3	0.4952	7.178e-9	0.4952	4.160e-15
0.7	0.5340	1.381e-3	0.5354	8.072e-9	0.5354	4.747e-15
0.8	0.5558	1.537e-3	0.5574	9.368e-9	0.5574	5.509e-15
0.9	0.5577	1.776e-3	0.5594	1.094e-8	0.5594	6.371e-15
1.0	0.5380	2.300e-3	0.5403	1.382e-8	0.5403	8.087e-15

Table 3: Absolute Error for Example 3

$x$	$N=4$		$N=8$		$N=12$	
	<i>Approx. solution</i>	<i>Abs. Error</i>	<i>Approx. solution</i>	<i>Abs. Error</i>	<i>Approx. solution</i>	<i>Abs. Error</i>
0	0	0	0	0	0	0
0.1	-0.1050	1.750e-4	-0.1052	5.037e-10	-0.1052	2.053e-16
0.2	-0.2212	2.406e-4	-0.2214	7.428e-10	-0.2214	2.930e-16
0.3	-0.3496	2.785e-4	-0.3499	9.426e-10	-0.3499	3.730e-16
0.4	-0.4915	3.251e-4	-0.4918	1.069e-9	-0.4918	4.332e-16
0.5	-0.6483	3.850e-4	-0.6487	1.240e-9	-0.6487	4.955e-16
0.6	-0.8217	4.448e-4	-0.8221	1.407e-9	-0.8221	5.561e-16
0.7	-1.0130	4.891e-4	-1.0140	1.516e-9	-1.0140	6.081e-16
0.8	-1.2250	5.177e-4	-1.2260	1.679e-9	-1.2260	6.757e-16
0.9	-1.4590	5.648e-4	-1.4600	1.874e-9	-1.4600	7.438e-16
1.0	-1.7180	7.204e-4	-1.7180	2.311e-9	-1.7180	9.235e-16

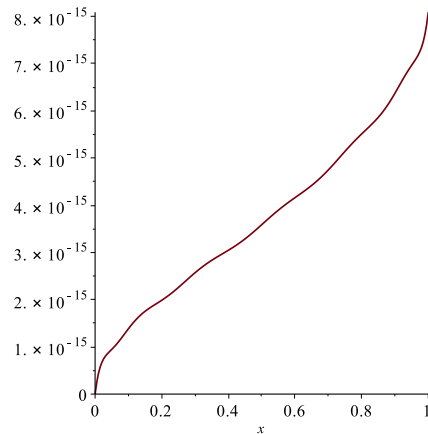


Figure 2: Absolute Error for Example 2

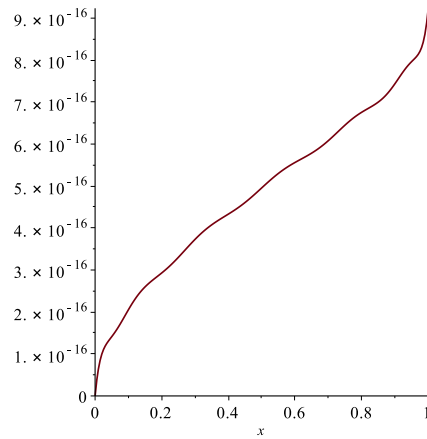


Figure 3: Absolute Error for Example 3

## 4 Conclusion

This article deals with the numerical solution of the first order Volterra integro-differential equations of the second kind, using Galerkin method by Chebyshev Polynomials. This technique is tested on three examples and the results are satisfactory. In addition this method is portable to high order Volterra integro-differential equations of the second kind and easy to program.

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## روش گالرکین چیشف برای معادلات انتگرال-دیفرانسیل از مرتبه دوم

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**چکیده:** در این مقاله، یک اجرای موثر از روش چیشف گالرکین، برای معادلات انتگرال-دیفرانسیل مرتبه اول ولترا و فردهلم نوع دوم پیشنهاد می‌شود. چندین مثال عددی برای نشان دادن دقت روش ارائه شده است.

**کلمات کلیدی:** معادلات انتگرال-دیفرانسیل ولترا؛ روش گالرکین؛ چندجمله‌ای‌های چیشف.