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Research Article

Numerical solution of nonlinear fractional Riccati differential equations using compact finite difference method

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Abstract

This paper aims to apply and investigate the compact finite difference methods for solving integer-order and fractional-order Riccati differential equations. The fractional derivative in the fractional case is described in the Caputo sense. In solving the Riccati equation, we first approximate first-order derivatives using the approach of compact finite difference. In this way, the system of nonlinear equations is obtained, which solves the Riccati equation. In addition, we examine the convergence analysis of the proposed approach for the fractional and nonfractional cases and prove that the methods are convergent under some suitable conditions. Examples are also given to illustrate the efficiency of our method compared to other methods.

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Keywords: Fractional Riccati equation; Caputo fractional derivative; Compact finite difference methods.

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1 Introduction

In recent years, there has been a growing interest in fractional computation [14, 26, 31, 33, 34]. Fractional differential equations have become increasingly important as they have applications in various fields of science and engineering [13]. Numerous phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance, and other sciences can be described successfully by models using mathematical tools of fractional calculation, that is, the theory of fractional-order derivatives and integrals. Much important work on theoretical analysis [38, 10] has been carried out, but the analytical solutions of most fractional differential equations cannot be achieved explicitly. Numerical solution strategies based on convergence and stability analysis were used by many authors [12, 11, 13, 16, 20, 35, 36, 39, 41, 22]. Liu has carried out extensive research on the finite difference method of fractional differential equations [22, 23, 24]. The two most frequently used are the Riemann–Liouville and Caputo type. The difference between the two definitions is in the order of evaluation [29].

In this paper, we consider the following Riccati equation:

$$\begin{cases} u'(x) = p(x) + q(x)u(x) + r(x)u^2(x), & 0 < x < T, \\ u(0) = 0. \end{cases} \tag{1}$$

Also, we consider the following fractional Riccati equation:

$$D^{\alpha}u(x) = p(x) + q(x)u(x) + r(x)u^2(x), \qquad 0 < \alpha \leq 1, \qquad 0 < x < T, \tag{2}$$

along with the initial condition

$$u(0) = 0, \tag{3}$$

where $x \in \mathbb{R}$ and $p(x)$, $q(x)$, and $r(x)$ are known functions. Moreover, D^{α} is the Caputo derivative operator of the fractional-order α, which is defined as below:

$$D^{\alpha}u(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} u'(s)\, ds. \tag{4}$$

In the past, two scholars, Bernoulli (1654-1705) and Riccati (1676-1754) introduced and assessed a particular case of differential equations (2). The Riccati differential equations (RDEs) and fractional Riccati differential equations (FRDEs) are used in many physical phenomena. Such applications can include control systems, robust stabilization, diffusion problems, network synthesis, optimal filtering, stochastic theory, controls, financial mathematics, optimal control, river flows, robust stabilization, financial mathematics

dynamic games, linear systems with Markovian jumps, stochastic control, econometric models, and invariant embedding [32, 28, 4, 19, 15, 9, 21, 5, 30]. Many researchers have used numerical approaches to solve the RDEs and FRDEs. Some standard procedures can be referenced, including the differential transform method [7], series solutions Adomian's decomposition method [1], Homotopy perturbation method [1], variational iteration method [18], Homotopy analysis method [37], piecewise spectral-collocation method [6], and so on [8, 27, 25, 3].
This paper aims to obtain numerical solutions to (1)–(3) using a high-order compact finite difference approach.

Several researchers have employed the compact finite difference method to solve fractional differential equations. Du, Cao, and Sun [14] have used the compact finite difference method to solve the fractional diffusion-wave equation. Gao and Sun [17] have also employed the compact finite difference method to solve the fractional sub-diffusion equation. They have also proved the stability and convergence of their method. Cui [13] solved the one-dimensional fractional diffusion equation via a high-order compact finite difference scheme and obtained a fully discrete implicit system by Grunwald–Letnikov's discretization of the Riemann–Liouville derivative.

The present study is organized as follows: In Sections 2, 3, and 4, the compact finite difference methods are reviewed and applied to solve (1)–(3). Also, their convergence is discussed. In Section 5, the numerical results obtained by the proposed methods are presented. We also compare the results of our approach and those of the proposed methods in [2]. The conclusion and the advantages of the proposed technique are presented in Section 6.

2 Compact finite difference scheme

In this work, our primary goal is to apply the compact finite difference method to solve (1)–(3). For this, we first subdivide the range $0 \leq x \leq T$ to N equal partitions with step length h as follows:

$$x_0 = 0, \qquad x_i = ih, \qquad i = 0, 1, \dots, N, \qquad h = \frac{T}{N}. \tag{5}$$

Set

$$u_i \approx u(x_i), \qquad u'_i \approx u'(x_i).$$

For the first derivatives, the following compact finite difference scheme was given in [40]:

$$\begin{cases} 4u'_1+u'_2=\frac{1}{h}(\frac{-11}{12}u_0-4u_1+6u_2-\frac{4}{3}u_3+\frac{1}{4}u_4), \\ u'_{i-1}+4u'_i+u'_{i+1}=\frac{3}{h}(-u_{i-1}+u_{i+1}), \qquad i=1,\dots,N-1, \\ u'_{N-2}+4u'_{N-1}=\frac{1}{h}(-\frac{1}{4}u_{N-4}+\frac{4}{3}u_{N-3}-6u_{N-2}+4u_{N-1}+\frac{11}{12}u_N). \end{cases} \tag{6}$$

All above relations have the accuracy of $O(h^4)$. The matrix form for (23) is

$$A_1u'=\frac{1}{h}B_1u, \tag{7}$$

where

$$A_1=\begin{pmatrix} 0 & 4 & 1 & 0 & \dots 0 \\ 1 & 4 & 1 & 0 & \dots 0 \\ 0 & \ddots & \ddots & \ddots & \ddots 0 \\ \vdots & \dots & 0 & 1 & 4 & 1 \\ 0 & \dots & 0 & 1 & 4 & 0 \end{pmatrix}_{(N+1)\times(N+1)},$$

$$B_1=\begin{pmatrix} -\frac{11}{12} & -4 & 6 & \frac{4}{3} & \frac{1}{4} & 0 & \dots & 0 \\ -3 & 0 & 3 & 0 & 0 & 0 & \dots & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & \dots & 0 & -\frac{1}{4} & \frac{4}{3} & -6 & 4 & \frac{11}{12} \end{pmatrix}_{(N+1)\times(N+1)}.$$

Also, $u=[u_0,u_1,\dots,u_N]^T$ and $u'=[u'_0,u'_1,\dots,u'_N]^T$.

Lemma 1. The coefficient matrix A_1 is invertible.

Proof. Let us expand A_1 along the first column. Then

$$det(A_1)=-det\begin{pmatrix} 4 & 1 & 0 & \dots 0 \\ 1 & 4 & 1 & \dots 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 \dots & 1 & 4 & 1 \\ 0 \dots & 1 & 4 & 0 \end{pmatrix}_{N\times N}.$$

Now, by expanding along the last column, we have

$$det(A_1)=(-1)^N det\begin{pmatrix} 4 & 1 & 0 & \dots 0 \\ 1 & 4 & 1 & \dots 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 \dots & 1 & 4 & 1 \\ 0 \dots & 0 & 1 & 4 \end{pmatrix}_{(N-1)\times(N-1)} \neq 0.$$

□

According to Lemma 1, from (24), we have $u' = \frac{1}{h}A_1^{-1}B_1u$. By defining $C = A_1^{-1}B_1$, the following relation holds for u':

$$u' = \frac{1}{h}Cu, \tag{8}$$

and in the component form, we have

$$u_i' = \frac{1}{h}\sum_{j=0}^{N} c_{i+1,j+1}u_j, \qquad i = 0, \dots, N. \tag{9}$$

Lemma 2. The coefficient matrix B_1 is invertible.

Proof. Let us expand B_1 along the first row. Then

$$det(B_1) = -det\begin{pmatrix} -3 & 0 & 3 & 0 & 0 & 0 & \dots & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & \dots & 0 & -\frac{1}{4} & \frac{4}{3} & -6 & 4 & \frac{11}{12} \end{pmatrix}_{N\times N}.$$

Now, by expanding along the last row, we have

$$det(B_1) = (-1)^N det\begin{pmatrix} -3 & 0 & 3 & 0 & 0 & 0 & \dots & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & 0 & 0 & 0 & -3 & 0 & 3 \end{pmatrix}_{(N-1)\times(N-1)} \neq 0.$$

□

According to Lemma 2, it follows that the matrix C is invertible.

3 Compact finite difference scheme for Riccati problem in $\alpha = 1$ case and its convergence

This section uses the compact finite difference scheme for the nonfractional Riccati problem and investigates its convergence. Consider the subsequent classical Riccati initial value problem

$$u'(x) = p(x) + q(x)u(x) + r(x)u^2(x), \qquad 0 < x < T. \tag{10}$$

Its initial condition is

$$u(0) = 0. \tag{11}$$

Using (10), we have

$$u_0' = p(x_0). \tag{12}$$

So, using (9), equation (12) can be written as

$$\frac{1}{h}\sum_{j=0}^{N} c_{1,j+1} u_j = p(x_0). \tag{13}$$

For $x = x_i$, one can write (10) as

$$u'(x_i) = p(x_i) + q(x_i)u(x_i) + r(x_i)u^2(x_i), \qquad i = 1, \dots, N. \tag{14}$$

Thus from (9)

$$\frac{1}{h}\sum_{j=0}^{N} c_{i+1,j+1} u_j - p(x_i) - q(x_i)u_i - r(x_i)u_i^2 = 0, \qquad i = 1, \dots, N. \tag{15}$$

Equations (13) and (15) form a system including $N+1$ equations and $N+1$ unknowns $u_0, u_1, \dots, u_N$, that can be solved by Maple software.

Now, the convergence analysis of the proposed method for (10) along with initial conditions (11) is investigated.

Theorem 1. Let $U = [u(x_0), u(x_1), \dots, u(x_N)]^T$ be the vector of exact solution to (1) along with its initial condition, and let $u = [u_0, u_1, \dots, u_N]^T$ be the numerical solution at the same points obtained by (13) and (15). Then

$$\|E\| \le O(h^2), \tag{16}$$

provided $h\|C^{-1}\|\|M\| \le 1$, where $E = [e_0, e_1, \dots, e_N]^T$ and $e_i = u(x_i) - u_i$, $i = 0, \dots, N$ ($\|\cdot\|$ is the infinity norm).

Proof. According to (13) and (15), for a numerical solution, we have

$$\begin{cases} \frac{1}{h}\sum_{j=0}^{N} c_{1,j+1} u_j = p(x_0), \\ \frac{1}{h}\sum_{j=0}^{N} c_{i+1,j+1} u_j - p(x_i) - q(x_i)u_i - r(x_i)u_i^2 = 0, \qquad i = 1, \dots, N, \end{cases} \tag{17}$$

and for an exact solution, we have

$$\begin{cases} \frac{1}{h}\sum_{j=0}^{N} c_{1,j+1} u(x_j) = p(x_0) + O(h^4), \\ \frac{1}{h}\sum_{j=0}^{N} c_{i+1,j+1} u(x_j) - p(x_i) - q(x_i)u(x_i) - r(x_i)u^2(x_i) = O(h^4), \quad i = 1, \dots, N. \end{cases} \tag{18}$$

By subtracting (17) and (18), one concludes that

$$\begin{cases} \frac{1}{h}\sum_{j=0}^{N} c_{1,j+1}(u(x_j)-u_j) = O(h^4), \\ \frac{1}{h}\sum_{j=0}^{N} c_{i+1,j+1}(u(x_j)-u_j) - q(x_i)(u(x_i)-u_i) \\ \quad -r(x_i)(u^2(x_i)-u_i^2) = O(h^4), \qquad i=1,\ldots,N. \end{cases} \tag{19}$$

Using the Taylor expansion, we have

$$u^2(x_i) - u_i^2 = \frac{\partial u^2}{\partial u}|_{x=x_i}(u(x_i)-u_i) + O(h^2), \qquad i=1,\ldots,N. \tag{20}$$

In relation (19), we have

$$\begin{cases} \frac{1}{h}\sum_{j=0}^{N} c_{1,j+1}(u(x_j)-u_j) = O(h^4), \\ \frac{1}{h}\sum_{j=0}^{N} c_{i+1,j+1}(u(x_j)-u_j) - q(x_i)(u(x_i)-u_i) \\ \quad -2r(x_i)u(x_i)(u(x_i)-u_i) = O(h^4)+O(h^2), \qquad i=1,\ldots,N. \end{cases} \tag{21}$$

Thus, one concludes that

$$\begin{cases} \sum_{j=0}^{N} c_{1,j+1}e_j = O(h^5), \\ \sum_{j=0}^{N} c_{i+1,j+1}e_j - hq(x_i)e_i - 2hr(x_i)u(x_i)e_i = O(h^3), \quad i=1,\ldots,N, \end{cases} \tag{22}$$

where $e_j = u(x_j) - u_j$, $j = 0,\ldots,N$, and $u_i \approx u(x_i)$. Therefore, (22) can be written as

$$\begin{cases} c_{11}e_0 + c_{12}e_1 + c_{13}e_2 + \cdots + c_{1,N+1}e_N = O(h^5), \\ c_{21}e_0 + c_{22}e_1 + c_{23}e_2 + \cdots + c_{2,N+1}e_N - hq(x_1)e_1 - 2hr(x_1)u(x_1)e_1 = O(h^3), \\ c_{31}e_0 + c_{32}e_1 + c_{33}e_2 + \cdots + c_{3,N+1}e_N - hq(x_2)e_2 - 2hr(x_2)u(x_2)e_2 = O(h^3), \\ \vdots \\ c_{N+1,1}e_0 + c_{N+1,2}e_1 + c_{N+1,3}e_2 + \cdots + c_{N+1,N+1}e_N \\ -hq(x_N)e_N - 2hr(x_N)u(x_N)e_N = O(h^3). \end{cases} \tag{23}$$

The matrix form of the above equations is as follows:

$$[C - hQ - hRJ]E = T, \tag{24}$$

where $Q = diag(0, q(x_1),\ldots,q(x_N))$, $R = diag(0, r(x_1),\ldots,r(x_N))$, $J = diag(0, 2u(x_1),\ldots,2u(x_N))$, and

$$T=\begin{pmatrix} O(h^5)\\ O(h^3)\\ O(h^3)\\ \vdots\\ O(h^3)\end{pmatrix}_{(N+1)\times 1} \qquad C=\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1,N+1}\\ c_{21} & c_{22} & \cdots & c_{2,N+1}\\ \vdots & \vdots & \ddots & \vdots\\ c_{N+1,1} & c_{N+1,2} & \cdots & c_{N+1,N+1}\end{pmatrix}_{(N+1)\times(N+1)}. \tag{25}$$

By replacing $M = Q + RJ$ in relation (24), we have $[C - hM]E = T$. Because C is invertible, we can write

$$(I - hC^{-1}M)E = C^{-1}T. \tag{26}$$

Now, if we assume $h\|C^{-1}\|\|M\| \leq 1$, then we conclude the matrix $I - hC^{-1}M$ is invertible. By the geometric series theorem, we have

$$\|(I - hC^{-1}M)^{-1}\| \leq \frac{1}{1 - h\|C^{-1}\|\|M\|}. \tag{27}$$

From (26), we have $E = (I - hC^{-1}M)^{-1}C^{-1}T$. Thus $\|E\| \leq \|(I - hC^{-1}M)^{-1}\|\|C^{-1}\|\|T\|$.

Now from relation (27), we can write $\|E\| \leq \frac{1}{1-h\|C^{-1}\|\|M\|}\|C^{-1}\|\|T\|$. Because $\|T\| \equiv O(h^3)$, we can derive $\|E\| \leq \frac{O(h^3)}{O(h)} \equiv O(h^2)$.

□

4 Implement the compact finite difference scheme for the fractional Riccati problem and its convergence

In this section, we introduce a compact finite difference scheme for the fractional Riccati problem of order $0 < \alpha < 1$. According to (2), we rewrite the Caputo derivative in $x = x_i$, $i = 1, \ldots, N$, as

$$D^{\alpha}u(x_i) = \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{i-1}\int_{x_k}^{x_{k+1}} \frac{u'(s)}{(x_i - s)^{\alpha}}ds. \tag{28}$$

Now, the above equation can be written as

$$
\begin{aligned}
D^{\alpha}u(x_i) &\approx \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{i-1}\int_{x_k}^{x_{k+1}} u_i'(x_i-s)^{-\alpha}ds \\
&= \frac{1}{\Gamma(1-\alpha)}u_i'\sum_{k=0}^{i-1}\int_{x_k}^{x_{k+1}} (x_i-s)^{-\alpha}ds \qquad (29)\\
&= \frac{1}{\Gamma(1-\alpha)}u_i'\sum_{k=0}^{i-1}[\frac{(x_i-x_k)^{1-\alpha}-(x_i-x_{k+1})^{1-\alpha}}{1-\alpha}].
\end{aligned}
$$

Substituting $x_i = ih$ in (29), we have

$$
\begin{aligned}
D^{\alpha}u(x_i) &\approx \frac{1}{\Gamma(1-\alpha)}u_i'\sum_{k=0}^{i-1}[\frac{h^{1-\alpha}((i-k)^{1-\alpha}-(i-k-1)^{1-\alpha})}{1-\alpha}] \\
&= \frac{u_i'}{h^{\alpha-1}\Gamma(2-\alpha)}\sum_{k=0}^{i-1}a_{i-k},
\end{aligned} \qquad (30)
$$

where $a_{i-k} = (i-k)^{1-\alpha} - (i-k-1)^{1-\alpha}$, $i = 1,\dots,N$ and $k = 0,\dots,i-1$.

Thus, the solution to (2) can be approximated using the following equations:

$$
\frac{u_i'}{h^{\alpha-1}\Gamma(2-\alpha)}\sum_{k=0}^{i-1}a_{i-k} = p(x_i)+q(x_i)u_i+r(x_i)u_i^2, \quad 0<\alpha<1,\ i=1,\dots,N, \qquad (31)
$$

where $u_i' = \frac{1}{h}\sum_{j=0}^{N} c_{i+1,j+1}u_j$, $i = 1,\dots,N$. In the matrix form, (31) is equivalent to

$$
Fu' = \rho(G + Qu + Ru^2), \qquad (32)
$$

where $\rho = h^{\alpha-1}\Gamma(2-\alpha)$, $Q = diag(q(x_1),\dots,q(x_N))$, $R = diag(r(x_1),\dots,r(x_N))$, $u = [u_1,\dots,u_N]^T$, $u' = [u_1',\dots,u_N']^T$, $G = \begin{pmatrix} p(x_1)\\ p(x_2)\\ \vdots \\ p(x_N)\end{pmatrix}$, and

$$
F = \begin{pmatrix}
a_1 & 0 & 0 & 0 & \dots & 0\\
0 & a_1+a_2 & 0 & 0 & \dots & 0\\
0 & 0 & a_1+a_2+a_3 & 0 & \dots & 0\\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & \dots & 0 & a_1+a_2+\dots+a_{N-1} & 0\\
0 & 0 & \dots & 0 & 0 & a_1+a_2+\dots+a_N
\end{pmatrix}. \qquad (33)
$$

For $i = 1, \dots, N$, (31) can be used to form a system including N equations and N unknowns $u_1, \dots, u_N$, that can be solved by Maple software.

Now, we discuss the issue of convergence. For convergence analysis of the fractional case, we need the following Lemma.

Lemma 3. [35] Suppose $u \in C^2[0, x_i]$. Then

$$\begin{aligned} &|\int_0^{x_i} \frac{u'(s)}{(x_i - s)^\alpha} ds - \sum_{k=0}^{i-1} u'_i \int_{x_k}^{x_{k+1}} (x_i - s)^{-\alpha} ds| \\ &\quad \leq \frac{1}{1-\alpha}[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1 + 2^{-\alpha})] \max_{0 \leq s \leq x_i} |u''(s)| h^{2-\alpha}. \end{aligned} \tag{34}$$

From (30), we have

$$D^\alpha u(x_i) = \frac{1}{h^{\alpha-1}\Gamma(2-\alpha)} \sum_{k=0}^{i-1} a_{i-k} u'(x_i) + R_i, \qquad i = 1, \dots, N, \tag{35}$$

where according to Lemma 3

$$R_i \leq \frac{1}{1-\alpha}[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1 + 2^{-\alpha})] \max_{0 \leq s \leq x_i} |u''(s)| h^{2-\alpha}. \tag{36}$$

For $x = x_i$, by replacing (35) into (2), we have

$$\sum_{k=0}^{i-1} a_{i-k} u'(x_i) = \rho(p(x_i) + q(x_i)u(x_i) + r(x_i)u^2(x_i)) + \tilde{R}_i, \qquad i = 1, \dots, N, \tag{37}$$

where $\rho = h^{\alpha-1}\Gamma(2-\alpha)$ and $\tilde{R}_i = h^{\alpha-1}\Gamma(2-\alpha)R_i, i = 1, \dots, N$.

In the matrix form, (37) is equivalent to

$$FU' = \rho(G + QU + RU^2) + \tilde{R}, \tag{38}$$

where F is the matrix defined in relation (33), $U' = [u'(x_1), \dots, u'(x_N)]^T$, $U = [u(x_1), \dots, u(x_N)]^T$, and $\tilde{R} = h^{\alpha-1}\Gamma(2-\alpha)[R_1, \dots, R_N]^T$.

Theorem 2. Let $U = [u(x_1), \dots, u(x_N)]^T$ be the vector of exact solution to (2) along with its initial condition at points $x_0, x_1, \dots, x_N$, and let $u = [u_1, \dots, u_N]^T$ be the numerical solution obtained by (31). Then

$$\|E\| \leq O(h^{2-\alpha}), \tag{39}$$

provided $h\|C^{-1}\|\|M + N\| \leq 1$, where $E = U - u$ and

$$J = \begin{pmatrix} 2u(x_1) & 0 & \dots & 0 \\ 0 & 2u(x_2) & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 2u(x_N) \end{pmatrix}.$$

Proof. According to (38) and (32), for the exact and numerical solutions, we have

$$\begin{cases} FU' = \rho(G + QU + RU^2) + \tilde{R}, \\ Fu' = \rho(G + Qu + Ru^2). \end{cases} \tag{40}$$

By using (40), one concludes that

$$F(U' - u') = \rho(Q(U - u) + R(U^2 - u^2)) + \tilde{R}. \tag{41}$$

Therefore, by replacing $u' = \frac{1}{h}Cu$ from (8) and $U' = \frac{1}{h}CU + T_1$ into (41), we have

$$\frac{1}{h}C(U - u) - \rho F^{-1}Q(U - u) - \rho F^{-1}R(U^2 - u^2) = F^{-1}\tilde{R} + T_1, \tag{42}$$

where $T_1 \equiv O(h^4)$ is the local truncation error of system (23).

Moreover, $U^2 - u^2$ can be written as

$$U^2 - u^2 = \begin{pmatrix} u^2(x_1) - u_1^2 \\ u^2(x_2) - u_2^2 \\ \vdots \\ u^2(x_N) - u_N^2 \end{pmatrix} = JE + T_2, \tag{43}$$

where

$$T_2 = \begin{pmatrix} O(h^2) \\ O(h^2) \\ \vdots \\ O(h^2) \end{pmatrix}.$$

Therefore, by replacing (43) into (42), we have

$$\frac{1}{h}CE - \rho F^{-1}QE - \rho F^{-1}RJE = F^{-1}\tilde{R} + T_1 + \rho F^{-1}RT_2. \tag{44}$$

By inserting relations $M = \rho F^{-1}Q$ and $N = \rho F^{-1}R$ into (44), it can be written as

$$(C - hM - hN)E = h(F^{-1}\tilde{R} + T_1 + NT_2), \tag{45}$$

$$(I - hC^{-1}(M + N))E = hC^{-1}(F^{-1}\tilde{R} + T_1 + NT_2). \tag{46}$$

Now, if $h\|C^{-1}\|\|M+N\| \leq 1$, then $(I - hC^{-1}(M+N)$ is invertible and

$$E = h(I - hC^{-1}(M+N))^{-1}C^{-1}(F^{-1}\tilde{R} + T_1 + NT_2),$$

$$\|E\| \leq h\|(I - hC^{-1}(M+N))^{-1}\|\|C^{-1}\|(\|F^{-1}\|\|\tilde{R}\| + \|T_1\| + \|N\|\|T_2\|).$$

It follows that

$$\|E\| \leq \frac{h\|C^{-1}\|(\|F^{-1}\|\|\tilde{R}\| + \|T_1\| + \|N\|\|T_2\|)}{1 - h\|C^{-1}\|\|M+N\|}. \tag{47}$$

Therefore, using the relations $\tilde{R} = h^{\alpha-1}\Gamma(2-\alpha)R$ and (36), we have $\|\tilde{R}\| \equiv O(h)$, so

$$\|E\| \leq \frac{O(h^2)}{O(h^\alpha)} + \frac{O(h^5)}{O(h^\alpha)} + \frac{O(h^3)}{O(h^\alpha)} = O(h^{2-\alpha}) + O(h^{5-\alpha}) + O(h^{3-\alpha}) \equiv O(h^{2-\alpha}). \tag{48}$$

□

5 Numerical results

This section applies our compact finite difference schemes to two examples to illustrate their effectiveness. Maple software is used for obtaining numerical results.

Example 1. Consider the following fractional RDE as the first example:

$$\begin{cases} D^\alpha u(x) = 1 - u^2(x), & 0 < \alpha \leq 1, \quad 0 < x < T, \\ u(0) = u_0 = 0. \end{cases} \tag{49}$$

The exact solution is $u(x) = \frac{\exp(2x)-1}{\exp(2x)+1}$ for $\alpha = 1$; see [2].

In Figure 1, a comparison between the exact solution for $\alpha = 1$ and the numerical solution for $\alpha = 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 0.999, 1$, and $T = 1$ is shown. Table 1 presents numerical solutions at some points of $[0, 1]$ and for different values of α, at $T = 1$. Table 2 presents a comparison between the exact solution for $\alpha = 1$ and the numerical solution for $T = 10$. Also, Figure 2 shows a comparison between the exact solution for $\alpha = 1$ and the numerical solution for $T = 10$.

We have calculated the rate of convergence of our methods (denoted by ROC) with the following formula:

$$ROC = \log_2\left(\frac{Error^{2h}}{Error^h}\right). \tag{50}$$

Table 3 shows the obtained maximum errors and ROC for $\alpha = 1$, $T = 1$, and $N = 5, 10, 20, 40, 80, 160$. Also, Figure 3 shows the numerical and exact solutions for $\alpha = 1$, $T = 1$, and $N = 10$. The numerical rate of convergence is highly consistent with our theoretical analysis results.

In Table 4, we compare the approximate solution and exact solution of the present method with the trigonometric transform method (TTM) [2] at points $0.2, 0.4, 0.6, 0.8, 1$, for $\alpha = 1$. Also, in Table 5, we compare the error of solutions of the present method with TTM [2] for $\alpha = 1$.

Table 1: Exact solutions and numerical solutions of Example 1 for $N = 10$, $T = 1$, and $\alpha = 0.3, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 0.999, 1$

α	0.3	0.6	0.7	0.8	0.9	0.95	0.99	0.999	1	Exact
x										
0.1	5.38×10^{-1}	2.66×10^{-1}	2.06×10^{-1}	1.60×10^{-1}	1.25×10^{-1}	1.11×10^{-1}	1.01×10^{-1}	9.98×10^{-2}	9.96×10^{-2}	9.96×10^{-2}
0.2	7.43×10^{-1}	4.34×10^{-1}	3.53×10^{-1}	2.88×10^{-1}	2.37×10^{-1}	2.16×10^{-1}	2.00×10^{-1}	1.97×10^{-1}	1.97×10^{-1}	1.97×10^{-1}
0.3	8.34×10^{-1}	5.51×10^{-1}	4.66×10^{-1}	3.95×10^{-1}	3.37×10^{-1}	3.12×10^{-1}	2.95×10^{-1}	2.91×10^{-1}	2.91×10^{-1}	2.91×10^{-1}
0.4	8.83×10^{-1}	6.38×10^{-1}	5.57×10^{-1}	4.86×10^{-1}	4.27×10^{-1}	4.02×10^{-1}	3.84×10^{-1}	3.80×10^{-1}	3.79×10^{-1}	3.79×10^{-1}
0.5	9.14×10^{-1}	7.05×10^{-1}	6.31×10^{-1}	5.64×10^{-1}	5.07×10^{-1}	4.83×10^{-1}	4.66×10^{-1}	4.62×10^{-1}	4.62×10^{-1}	4.62×10^{-1}
0.6	9.34×10^{-1}	7.57×10^{-1}	6.91×10^{-1}	6.30×10^{-1}	5.79×10^{-1}	5.56×10^{-1}	5.40×10^{-1}	5.37×10^{-1}	5.37×10^{-1}	5.37×10^{-1}
0.7	9.48×10^{-1}	7.99×10^{-1}	7.41×10^{-1}	6.87×10^{-1}	6.41×10^{-1}	6.21×10^{-1}	6.07×10^{-1}	6.04×10^{-1}	6.04×10^{-1}	6.04×10^{-1}
0.8	9.58×10^{-1}	8.32×10^{-1}	7.82×10^{-1}	7.35×10^{-1}	6.95×10^{-1}	6.78×10^{-1}	6.66×10^{-1}	6.64×10^{-1}	6.64×10^{-1}	6.64×10^{-1}
0.9	9.65×10^{-1}	8.59×10^{-1}	8.16×10^{-1}	7.76×10^{-1}	7.42×10^{-1}	7.28×10^{-1}	7.18×10^{-1}	7.16×10^{-1}	7.16×10^{-1}	7.16×10^{-1}
1.0	9.71×10^{-1}	8.81×10^{-1}	8.44×10^{-1}	8.10×10^{-1}	7.82×10^{-1}	7.70×10^{-1}	7.63×10^{-1}	7.61×10^{-1}	7.61×10^{-1}	7.61×10^{-1}

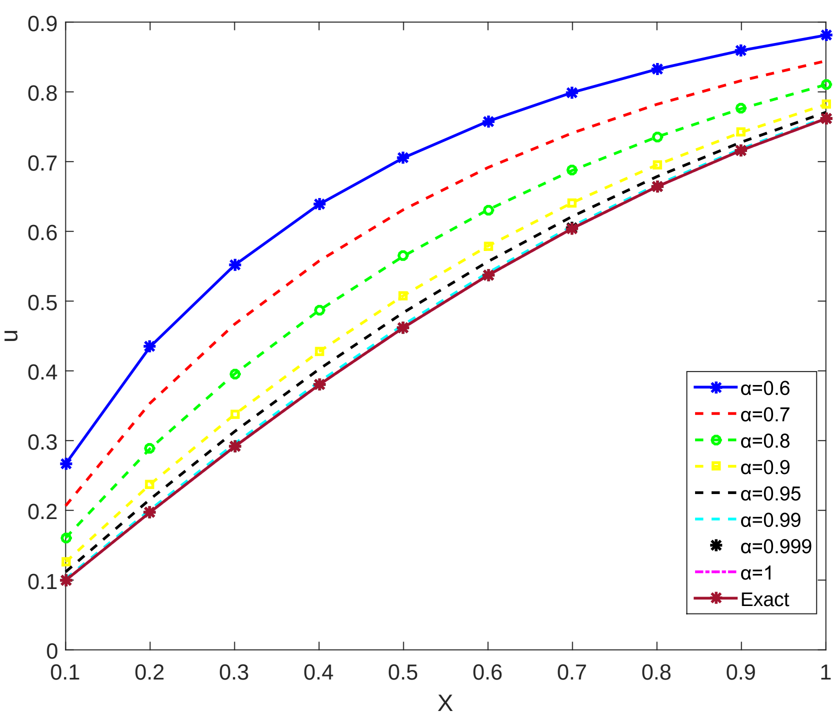

Figure 1: Comparison between the exact solution of Example 1 for $\alpha = 1$ and numerical solutions for $\alpha = 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 0.999, 1$ and $T = 1$

Table 2: Comparison between the exact solution and numerical solutions of Example 1 for $\alpha = 1$, $T = 10$, and $N = 100$

x	Numerical solution	Exact solution	Error
1	0.7615917576	0.7615941559	2.3983554×10^{-6}
2	0.9640223166	0.9640275800	5.2634336×10^{-6}
3	0.9950446865	0.9950547536	1.0067173×10^{-5}
4	0.9993096449	0.9993292997	1.9654751×10^{-5}
5	0.9998709681	0.9999092042	3.8236123×10^{-5}
6	0.9999133772	0.9999877116	7.4334413×10^{-5}
7	0.9998538332	0.9999983369	1.4450373×10^{-4}
8	0.9997188603	0.9999997749	2.8091453×10^{-4}
9	0.9994538522	0.9999999695	5.4611733×10^{-4}

Table 3: Maximum absolute errors and ROC of Example 1 for $\alpha = 1$, $T = 1$, and $N = 5, 10, 20, 40, 80, 160$

N	Maximum Absolute Error	ROC
5	7.95×10^{-4}	–
10	1.91×10^{-5}	5.38
20	9.47×10^{-7}	4.33
40	3.43×10^{-8}	4.79
80	1.16×10^{-9}	4.88
160	3.99×10^{-11}	4.87

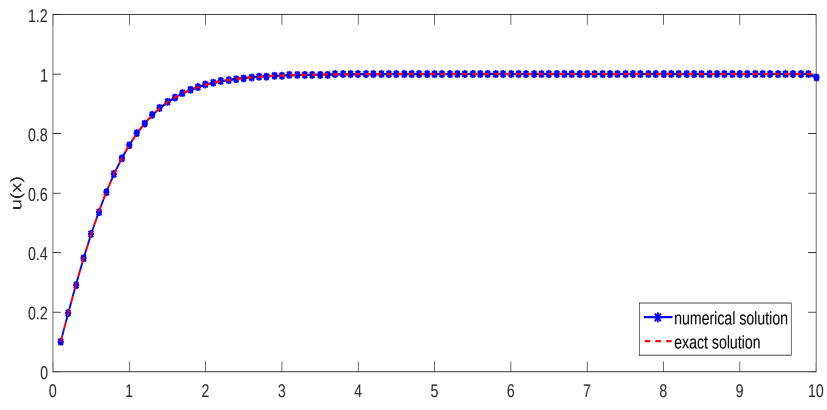

Figure 2: Comparison between the exact solution and numerical solutions of Example 1 for $\alpha = 1$, $T = 10$, and $N = 100$

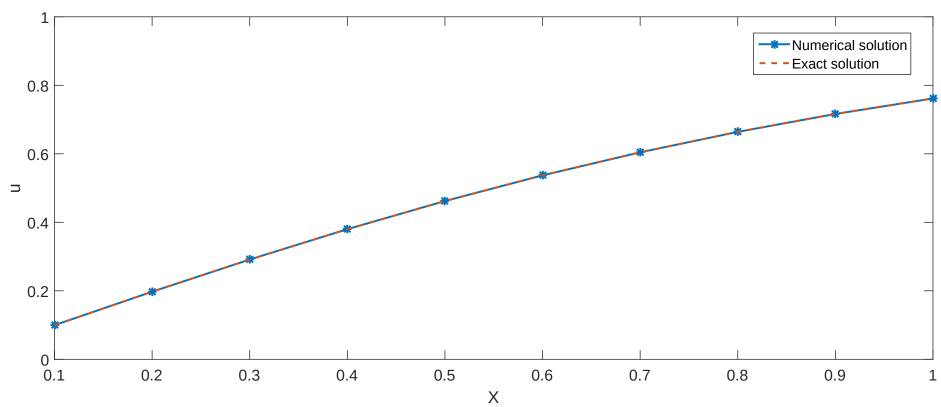

Figure 3: Comparison between the exact solution and numerical solutions of Example 1 for $\alpha = 1$, $T = 1$, and $N = 10$

Example 2. Let the following FRDE be the second example

$$D^{\alpha}u(x) = 1 + 2u(x) - u^2(x), \qquad 0 < \alpha \leq 1, \qquad 0 < x < T, \tag{51}$$

with the initial condition

$$u_0 = u(0) = 0. \tag{52}$$

The exact solution for $\alpha = 1$ is $u(x) = 1 + \sqrt{2}\tanh(\sqrt{2}x + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1}))$; see [2].

Table 4: Comparison between the approximation solution and exact solution of the presented method with TTM [2] for $\alpha = 1$, $T = 1$, and $N = 10$ for Example 1

x	TTM [2]	proposed method	Exact
0.0	0.0	0.0	0.0
0.2	0.197773	0.197378	0.197374
0.4	0.380422	0.379951	0.379949
0.6	0.537449	0.537051	0.537050
0.8	0.664285	0.664036	0.664037
1.0	0.761671	0.761572	0.761594

Table 5: Comparison between the absolute error of solution by our method with TTM [2] for $\alpha = 1$ and $T = 1$, for Example 1

x	Error of proposed method	Error of TTM [2]
0.0	0.0	0.0
0.2	1.4598×10^{-6}	7.2107×10^{-4}
0.4	1.5961×10^{-6}	1.7216×10^{-3}
0.6	4.6060×10^{-7}	2.7186×10^{-3}
0.8	1.1006×10^{-6}	3.3906×10^{-3}
1.0	1.9061×10^{-5}	3.6117×10^{-3}

In Figure 4, a comparison between the exact solution for $\alpha = 1$ and the numerical solution for $\alpha = 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 0.999, 1$ and $T = 1$ is shown. Also, Table 6 presents numerical solutions at some points of $[0, 1]$ and for different values of α at $T = 1$.

Table 7 shows the obtained maximum errors and ROC for $\alpha = 1$, $T = 1$, and $N = 5, 10, 20, 40, 80, 160$ Also, Figure 5 shows the numerical and exact solutions for $\alpha = 1$, $T = 1$, and $N = 10$. The numerical rate of convergence is highly consistent with our theoretical analysis results.

Table 8 represents the present method and the achieved results of particle swarm optimization (PSO) [2], modified homotopy perturbation method (MHPM) [2], Chebyshev wavelets (CW) [2], fractional variational iteration method (FVI) [2], Legendre wavelets method (LWM) [2], and Pad´e-variational iteration method (PVI) [2].

Table 9 presents a comparison between the exact solution for $\alpha = 1$ and the numerical solution for $T = 8$. Also, Figure 6 shows a comparison between the exact solution for $\alpha = 1$ and the numerical solution for $T = 8$ and $N = 80$.

Table 6: Exact solutions and Numerical solutions of Example 2 for $N = 10$, $T = 1$, and $\alpha = 0.3, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 0.999, 1$

α	0.3	0.6	0.7	0.8	0.9	0.95	0.99	0.999	1	Exact
x										
0.1	1.38	3.79×10^{-1}	2.65×10^{-1}	1.92×10^{-1}	1.44×10^{-1}	1.25×10^{-1}	1.13×10^{-1}	1.10×10^{-1}	1.10×10^{-1}	1.10×10^{-1}
0.2	1.92	7.21×10^{-1}	5.25×10^{-1}	3.94×10^{-1}	3.04×10^{-1}	2.70×10^{-1}	2.47×10^{-1}	2.42×10^{-1}	2.41×10^{-1}	2.41×10^{-1}
0.3	2.14	1.02	7.80×10^{-1}	6.05×10^{-1}	4.82×10^{-1}	4.35×10^{-1}	4.02×10^{-1}	3.95×10^{-1}	3.95×10^{-1}	3.95×10^{-1}
0.4	2.24	1.29	1.02	8.22×10^{-1}	6.74×10^{-1}	6.16×10^{-1}	5.76×10^{-1}	5.68×10^{-1}	5.67×10^{-1}	5.67×10^{-1}
0.5	2.30	1.51	1.25	1.03	8.75×10^{-1}	8.10×10^{-1}	7.66×10^{-1}	7.56×10^{-1}	7.55×10^{-1}	7.55×10^{-1}
0.6	2.34	1.70	1.45	1.24	1.07	1.01	9.64×10^{-1}	9.54×10^{-1}	9.53×10^{-1}	9.53×10^{-1}
0.7	2.36	1.84	1.62	1.43	1.27	1.20	1.16	1.15	1.15	1.15
0.8	2.37	1.96	1.77	1.59	1.45	1.39	1.35	1.34	1.34	1.34
0.9	2.38	2.05	1.89	1.74	1.61	1.56	1.53	1.52	1.52	1.52
1.0	2.39	2.12	1.99	1.87	1.76	1.72	1.69	1.69	1.68	1.68

Table 7: Maximum absolute errors and ROC of Example 2 for $\alpha = 1$, $T = 1$, and $N = 5, 10, 20, 40, 80, 160$

N	Maximum Absolute Error	ROC
5	6.35×10^{-3}	–
10	3.63×10^{-5}	7.45
20	3.63×10^{-6}	3.32
40	1.73×10^{-7}	4.39
80	7.05×10^{-9}	4.62
160	2.98×10^{-10}	4.57

Table 8: Comparison of the numerical solutions of the equation in Example 2 with $\alpha = 1$ and $T = 1$

x	SJOM [2]	MHPM [2]	PSO [2]	CW [2]	FVI [2]	PVI [2]	LWM [2]	Our Method	Exact
0.6	1.007291	1.370240	1.296320	1.349150	1.331462	1.873658	1.296302	0.953552	0.953567
0.7	1.253674	1.367499	1.416139	1.481449	1.497600	2.112944	1.416311	1.152926	1.152950
0.8	1.467499	1.794879	1.506936	1.599235	1.630234	2.260134	1.506913	1.346363	1.346365
0.9	1.629901	1.962239	1.569252	1.705303	1.724439	2.339134	1.569221	1.526897	1.526913
1.0	1.787222	2.087384	1.605580	1.801763	1.776542	2.379356	1.605571	1.689487	1.689500

Table 9: Comparison between the exact solution and numerical solutions of Example 2 for $\alpha = 1$, $T = 8$, and $N = 80$

x	Numerical solution	Exact solution	Error
0.8	1.346362994	1.346363655	6.6128045×10^{-7}
1.6	2.246290755	2.246285959	4.7957279×10^{-6}
2.4	2.395782816	2.395756424	2.6391922×10^{-5}
3.2	2.412338083	2.412281528	5.6554231×10^{-5}
4.0	2.414131848	2.414012382	1.1946588×10^{-4}
4.8	2.414445422	2.414192625	2.527976×10^{-4}
5.6	2.414746423	2.414211383	5.3504015×10^{-4}
6.4	2.415345681	2.414213335	1.1323455×10^{-3}
7.2	2.416609669	2.414213538	2.3961302×10^{-3}
8.0	2.418416749	2.414213559	4.2031900×10^{-3}

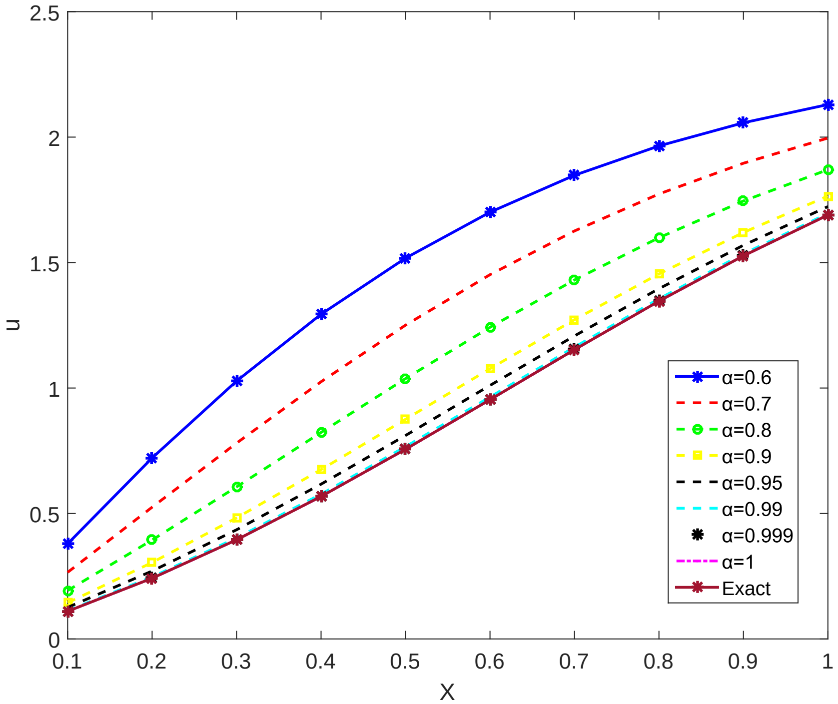

Figure 4: Comparison between exact solution of Example 2 for $\alpha = 1$ and numerical solutions for $\alpha = 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 0.999, 1$ and $T = 1$

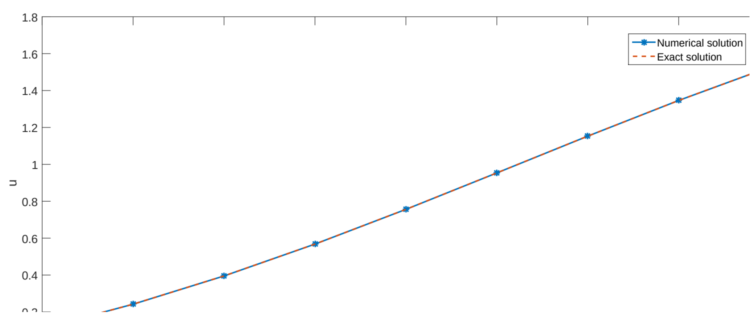

Figure 5: Comparison between the exact solution and numerical solution of Example 2 for $\alpha = 1$, $T = 1$, and $N = 10$

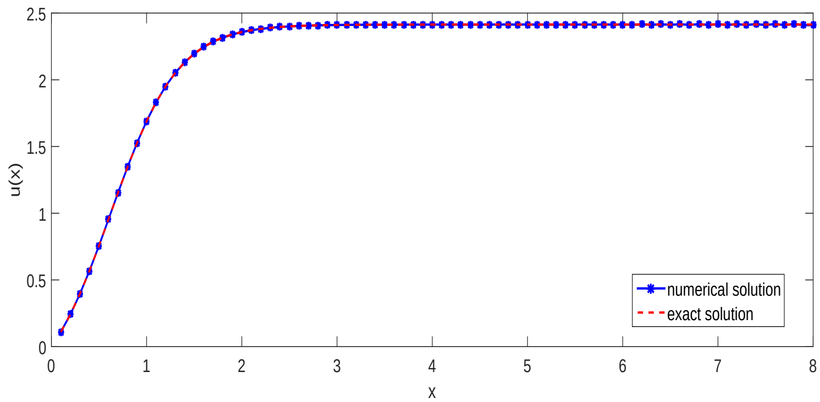

Figure 6: Comparison between the exact solution and numerical solutions of Example 2 for $\alpha = 1$, $T = 8$, and $N = 80$

6 Conclusions

This paper proposed a high-order compact finite difference method for the Riccati problem. The convergence analysis has been discussed. The numerical results presented in Tables 1–9 showed that the method is effective and that the numerical experiment is very consistent with our theoretical analysis results.

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