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Population based algorithms for approximate optimal distributed control of wave equations

A. H. Borzabadi*, S. Mirassadi and M. Heidari

Abstract

In this paper, a novel hybrid iterative scheme to find approximate optimal distributed control governed by wave equations is considered. A partition of the time-control space is considered and the discrete form of the problem is converted to a quasi assignment problem. Then a population based algorithm, with a finite difference method, is applied to extract approximate optimal distributed control as a piecewise linear function. A convergence analysis is proposed for discretized form of the original problem. Numerical computations are given to show the proficiency of the proposed algorithm and the obtained results applying two popular evolutionary algorithms, genetic and particle swarm optimization algorithms.

Keywords: Optimal control problem; Evolutionary algorithm; Finite difference method; Wave equation

1 Introduction

In the past few decades, the science and engineering have witnessed a phenomenal growth in the field of optimal control problems (OCPs) governed by partial differential equations (PDEs), specially parabolic and hyperbolic equations. A large part of these improvements is due to the efforts of pioneer researchers such as J. L. Lions [15, 16, 17] and D. L. Russell [19, 20].

In particular, the controllability in wave equations are studied in [21, 22]. Kim and Erzberger [14] derived Riccati equation for optimum boundary con-

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trol of wave equation, with quadratic cost function. Applicability of Laplace transform for determination of time optimal control for hyperbolic class of problems has been shown in [10]. But solving optimal control problems governed by wave equations with analytical approaches, have some difficulties such as computing gradient, integrals in hyperbolic and parabolic equations and target functionals. For overcoming complexities due to the analytical approaches, the numerical approaches are created based on various techniques as regularization [9, 8, 11], measure theoretical concepts [1, 5, 6] and penalty method [12].

Recently, nature-inspired optimization methods have attracted more and more attention and these powerful tools have been applied for solving a wide range of OCPs [2, 3, 7].

In this paper, by combinating one of the population based algorithms (Evolutionary Algorithms (EAs)) and a numerical method for solving wave equations (finite difference method), an effective numerical scheme for finding approximate optimal control and state functions has been procreated for OCPs governed by wave equations with a distributed control and a non-classical boundary condition as follows:

$$\textit{minimize} \quad J(\nu(.,.)) = \int_0^T \int_0^L \Phi(t,x,\nu(x,t))\, dx\, dt \tag{1}$$

$$\textit{subject to} \quad u_{tt}(x,t) = u_{xx}(x,t) + \nu(x,t), \qquad (x,t) \in [0,L] \times [0,T] \tag{2}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in [0,L] \tag{3}$$

$$u(0,t) = \mu(t), \qquad u_x(L,t) - u_x(0,t) = \eta(t), \qquad t \in [0,T] \tag{4}$$

where $\varphi(x), \psi(x), \mu(t), \eta(t)$ are given functions and $\nu(x,t)$ is a bounded distributed control and gets its values in the interval $\mathcal{V} \subset \mathbb{R}$. The purpose is to find the approximate optimal control $\nu(x,t)$ and state $u(x,t)$ that minimize the functional (1) and satisfy the wave equation (2) with initial conditions (3), boundary conditions (4) and terminal conditions

$$u(x,T) = \omega(x), \quad u_t(x,T) = \zeta(x). \tag{5}$$

Here $\omega(x)$ and $\zeta(x)$ are target functions.

The paper is organized as follows. In Sec. 2, we describe the discretization of optimal distributed control problem governed by wave equation. The problem is considered as a quasi assignment problem. The convergence of this modification is proved in the third section. In Sec. 4, we present the algorithm for solving OCP (1)-(5). In Sec. 5, numerical results arising from applying and comparing the given algorithm using two EAs, i.e. particle swarm optimization(PSO) and genetic algorithm(GA), are presented.

2 Description of the method

To find the optimal solution we must examine the performance index in the set of all possibilities of control-state pairs. The set of admissible pairs consisting of pairs like (u, ν) satisfying in (2)-(4) is denoted by $\mathcal{P}$. In this section we consider a control space discretization based method considering equidistant partitions of $[0, T]$, $[0, L]$ and $\mathcal{V}$ as $\triangle_t = \{0 = t_0, t_1, \cdots, t_{n-1}, t_n = T\}$ $\triangle_x = \{0 = x_0, x_1, \cdots, x_{m-1}, x_m = L\}$ and $\triangle_\nu = \{v_0, v_1, \cdots, v_{l-1}, v_l\}$, respectively. Now the main problem can be considered as a quasi assignment problem, where a performance index can be assigned corresponding to each chosen partition and choosing the best performance index can lead to determine the near optimal control of the problem. A trivial way to determine the near optimal solution is to calculate all possible partitions and compare the corresponding trade offs. This trivial method of total enumeration needs $((m+1)(n+1))^{(l+1)}$ evaluation. A typical discretization is given in Figure

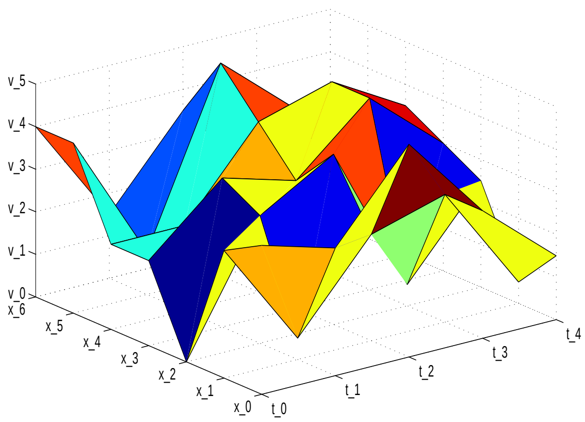

Figure 1: A typical control function in time-control space

1 with $n = 4$, $m = 6$ and $l = 5$. To avoid so many computations, we use the EAs for evaluating special partitions that guides us to the optimal one. For each partition of control we need its corresponding trajectory to evaluate the performance index. Trivially, the corresponding trajectory should be in discretized form.

For discretization of the wave equation (2)-(4), we use an approximate method like finite difference method as central difference approximation for the second partial derivative and forward difference approximation for the first partial derivative. By this method,

$$\frac{u_i^{j-1} - 2u_i^j + u_i^{j+1}}{k^2} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \bar{\nu}_i^j,$$
$$u_i^0 = \varphi_i,$$
$$\frac{u_i^1 - u_i^0}{k} = \psi_i,$$
$$u_0^j = \mu^j,$$
$$\frac{u_{m+1}^j - u_m^j}{h} - \frac{u_1^j - u_0^j}{h} = \eta^j,$$

where $u_i^j = u(x_i, t_j), \varphi_i = \varphi(x_i), \psi_i = \psi(x_i), \mu^j = \mu(t_j), \eta^j = \eta(t_j)$ and $\bar{\nu}_i^j = \bar{\nu}(x_i, t_j)$. Then we have

$$u_i^0 = \varphi_i, \; u_i^1 = k\psi_i + u_i^0, \; i = 0, 1, \cdots, m, \tag{6}$$
$$u_0^j = \mu_j, \; u_{m+1}^j = h\eta^j + u_m^j + u_1^j - u_0^j, \; j = 0, 1, \cdots, n, \tag{7}$$

and
$$u_i^{j+1} = \lambda^2 u_{i+1}^j + (2 - 2\lambda^2) u_i^j + \lambda^2 u_{i-1}^j - u_i^{j-1} + k^2 \bar{\nu}_i^j, \tag{8}$$
where $j = 1, 2, \cdots, n, \; i = 0, 1, \cdots, m$ and $\lambda = k/h$.

If $(u, \bar{\nu})$ be a pair of the trajectory and the control which satisfies in (10)-(11) and

$$\|u(x_i, t_n) - \omega(x_i)\| \leq \epsilon_1 \quad i = 0, 1, \cdots, m \tag{9}$$
$$\|u_t(x_i, t_n) - \zeta(x_i)\| \leq \epsilon_2 \quad i = 0, 1, \cdots, m \tag{10}$$

for given small numbers $\epsilon_1 > 0$ and $\epsilon_2 > 0$, then we can claim that, a good approximate pair for minimizing functional J in (1) has been found. Here $\| \cdot \|$ is the infinity norm.

Also in (1), the integral term can be estimated by a numerical method of integration, e.g. one of Newton-Cotes methods. After discretization of the OCP governed by wave equation, the problem is converted to optimization problem with two extra objective functions. We add the terms, $\sum_{i=1}^{m-1} \|u(x_i, t_n) - \omega(x_i)\|$ and $\sum_{i=1}^{m-1} \|u_t(x_i, t_n) - \zeta(x_i)\|$ to the original objective function and then, we apply EAs for this new criteria function. Therefore, by applying the above method, the OCP governed by wave equation is converted to constrained programming:

$$\text{(CP)}\min \; \sum_{i=1}^{m} \sum_{j=1}^{n} A_i B_j \Phi(t_j, x_i, \bar{\nu}_i^j) \tag{11}$$
$$+ M \sum_{i=1}^{m} \|u_i^n - \omega(x_i)\| + W \sum_{i=1}^{m} \|u_t(x_i, t_n) - \zeta(x_i)\|,$$

$$subject\ to\quad u_i^{j+1} = \lambda^2 u_{i+1}^j + (2-2\lambda^2)u_i^j + \lambda^2 u_{i-1}^j - u_i^{j-1} + k^2\bar{\nu}_i^j, \quad (12)$$

$$u_i^0 = \varphi_i,\ u_i^1 = k\psi_i + u_i^0,\ i = 0,1,\cdots,m, \quad (13)$$

$$u_0^j = \mu_j,\ u_{m+1}^j = h\eta^j + u_m^j + u_1^j - u_0^j,\ j = 0,1,\cdots,n, \quad (14)$$

where, A_i and B_j are the weights of a numerical method of integration, M and W are large positive numbers (as the parameters in penalty function approach).

3 Convergence

The solution of (CP) approximates the original problem by minimizing $J(u,\nu)$ over the subset $\mathcal{P}_N$ of $\mathcal{P}$ consists of all piecewise linear functions $u(.,.)$ and $\nu(.,.)$ with nodes at $u_i^j, \bar{\nu}_i^j, j = 0,1,\cdots,N, i = 0,1,\cdots,N$ which satisfies (11) and the objective function (11) for this nodes called J_N, here without loss of generality, we assume that $N = m = n$. Our first aim is to show that $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \cdots$ in an embedding fashion.

Lemma 1. *There exists an embedding that maps $\mathcal{P}_N$ to a subset of $\mathcal{P}_{N+1}$ for all $N = 1,2,\cdots$.*

Proof. For simplicity, we prove the case when $N = 1$. The proof for $N \geq 2$ is obtained analogously.
Let consider an arbitrary pair (u,ν) in $\mathcal{P}_1$ represented by $u_i^j, \bar{\nu}_i^j, j = 0,1, i = 0,1$. We have to find a corresponding pair $(\hat{u},\hat{\nu})$ in $\mathcal{P}_2$ with $\hat{u}_i^j, \hat{\bar{\nu}}_i^j, j = 0,1,2, i = 0,1,2$, as nodes that corresponds to (u,ν). We have from (11)

$$u_i^{j+1} = \lambda^2 u_{i+1}^j + (2-2\lambda^2)u_i^j + \lambda^2 u_{i-1}^j - u_i^{j-1} + k^2\bar{\nu}_i^j, \quad j = 0,1, \quad i = 0,1,$$

where $u_i^{j+1} = u(x_i, t_{j+1})$. On the other hand, a typical element $(\hat{u},\hat{\nu})$ in $\mathcal{P}_2$ satisfies

$$\hat{u}_i^{j+1} = \lambda^2 \hat{u}_{i+1}^j + (2-2\lambda^2)\hat{u}_i^j + \lambda^2 \hat{u}_{i-1}^j - \hat{u}_i^{j-1} + k^2\hat{\bar{\nu}}_i^j, \quad j = 0,1,2, \quad i = 0,1,2,$$

where $\hat{u}_i^{j+1} = \hat{u}(\hat{x}_i, \hat{t}_{j+1})$.
It is clear that here we have $\hat{x}_i = x_i, i = 0,1$ and $\hat{t}_j = t_j, j = 0,1$. Therefore we can choose $\hat{u}_i^j, \hat{\bar{\nu}}_i^j, j = 0,1,2, i = 0,1,2$ in such a way that

$$\hat{u}_i^j = u_i^j, \quad j = 0,1, \quad i = 0,1.$$

This shows that the constructed pair $(\hat{u},\hat{\nu})$ corresponds to (u,ν) and belongs to $\mathcal{P}_2$. $\square$

The above lemma has an important result in decreasing behavior of the optimal value of the objective function which leads to the following theorem.

Theorem 1. *If* $\mu_N = \inf_{\mathcal{P}_N} J_N$ *for* $N = 1, 2, \cdots$, *and* $\mu^* = \inf_{\mathcal{P}} J$ *exists, then* $\lim_{N\to\infty} \mu_N = \mu^*$.

Proof. By Lemma 1, we have $\mu_1 \geq \mu_2 \geq \cdots \geq \mu^*$. So, this decreasing and bounded sequence converges to a limit $\mu^0 \geq \mu^*$. It is enough to show that $\mu^0 = \mu^*$. If $\mu^0 > \mu^*$, then $\epsilon = \mu^0 - \mu^* > 0$ and by continuity of $J(u, \nu)$, we may find a pair $(u^j_{n_0}, \nu^j_{n_0})$, such that $|J(u^j_{n_0}, \nu^j_{n_0}) - \mu^*| < \epsilon$, then $J(u^j_{n_0}, \nu^j_{n_0}) < \mu^0$, and so $\mu_{n_0} < \mu^0$ which is incorrect and therefore $\mu^0 = \mu^*$. □

4 Algorithm of the approach

In this section, an algorithm on the basis of the previous discussions is presented. This algorithm is designed in two stages, initialization step and main steps, where the main steps contain the main structure of algorithm considering initialization step.

Initialization step:

Choose an equidistant partition for time interval $[0, T]$, with parameter discretization $k = t_{j+1} - t_j,\ j = 0, 1, \cdots, n-1$ and an equidistant partition for interval $[0, L]$, with parameter $h = x_{i+1} - x_i,\ i = 0, 1, \cdots, m-1$.

Main steps:

Step 1. Choose a population randomly.

Step 2. Compute $u^j_i,\ j = 0, 1, \cdots, n,\ i = 0, \cdots, m$, using (10)-(11).

Step 3. Fitness scores are assigned to each population using objective function of (CP).

Step 4. Apply the rules of EA for current population.

Step 5. Consider the new population as the current population.

Step 6. If the termination conditions are satisfied, stop; otherwise jump to Step 2.

5 Numerical results

In this section the proposed algorithm in the previous section is examined by one numerical example. We have applied PSO and GA as two of the most popular EAs.

Consider the following OCP:

$$
\begin{array}{ll}
\min & J = \int_0^1 \int_0^1 \nu^2(x,t)\ dt\ dx \\
\textit{subject to} & u_{tt}(x,t) = u_{xx}(x,t) + \nu(x,t), \qquad (x,t) \in (0,1)\times(0,1) \\
& u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad x \in (0,1) \\
& u(0,t) = 0, \qquad u_x(1,t) = u_x(0,t), \qquad t \in (0,1) \\
& u(x,T) = \sin(2\pi x), \qquad u_t(x,T) = \sin(4\pi x), \qquad t \in (0,1)
\end{array}
$$

For analytical solution of this example, see [13].

Our solutions by PSO and GA algorithms are shown in the following tables:

- where wave parameters, λ, be $0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ and the population sizes, (m), be 200 and the number of iterations,(k_{max}), be 500, results are shown in Table 1. 1.

- where the population size,(m), be $50, 100, 150, 200, 250, 300, 350$ and the number of iterations,(k_{max}), be 500 and parameter wave equation,λ, be 0.2, results are shown in Table 2. 2.

- where the number of iterations,(k_{max}), be $100, 200, 300, 400, 500, 600, 700$ and the population size,(m), be 200 and parameter wave equation,λ, be 0.2, results are shown in Table 3. 3.

Also comparison between $\omega(x)$ and $\zeta(x)$ with $u(x,T)$ and $u_t(x,T)$ are shown in Figures 2 and 3 ,respectively, when $m = 200, k_{max} = 500, \lambda = 0.2$.

Table 1: Comparison of the errors due to applying PSO and GA with increasing λ

parameter	$\|u_t(x,T) - \zeta(x)\|$	
λ	PSO	GA
0.1	$1.1662e-005$	$1.2375e-005$
0.2	$2.4418e-005$	$6.5369e-005$
0.3	$5.3563e-005$	$1.1038e-004$
0.4	$2.2052e-004$	$8.0804e-005$
0.5	$1.3987e-004$	$8.9201e-005$
0.6	$9.6621e-005$	$7.8682e-005$
0.7	$1.5991e-004$	$8.3949e-005$
0.8	$1.3152e-004$	$1.1785e-004$
0.9	$1.6093e-004$	$1.1817e-004$
1.0	$2.4100e-004$	$2.5027e-004$

Table 2: Comparison of the errors due to applying PSO and GA with increasing the number of population

number of population	$\|u_t(x,T)-\zeta(x)\|$	
	PSO	GA
50	$1.8470e-004$	$2.16684e-004$
100	$1.1513e-004$	$9.4944e-005$
150	$1.5074e-004$	$1.0690e-004$
200	$9.4195e-005$	$8.5366e-005$
250	$1.6513e-004$	$1.2451e-004$
300	$1.6264e-004$	$1.0645e-004$
350	$5.0556e-005$	$8.6299e-005$

Table 3: Comparison of the errors due to applying PSO and GA with increasing the number of iterations

number of iterations	$\|u_t(x,T)-\zeta(x)\|$	
	PSO	GA
100	$2.3324e-004$	$4.0122e-004$
200	$9.4547e-005$	$1.1504e-004$
300	$1.8058e-004$	$1.0747e-004$
400	$1.2749e-004$	$1.0134e-004$
500	$2.1751e-004$	$9.1401e-005$
600	$1.0138e-004$	$7.7138e-005$
700	$9.9944e-005$	$9.2749e-005$

6 Conclusion

In this paper, a hybrid approach for the resolution of OCPs governed by wave equations is presented. This approach is based on partitioning of the time-control space, finite difference method, penalty method and EAs. The derived results show the superiority of the approach.

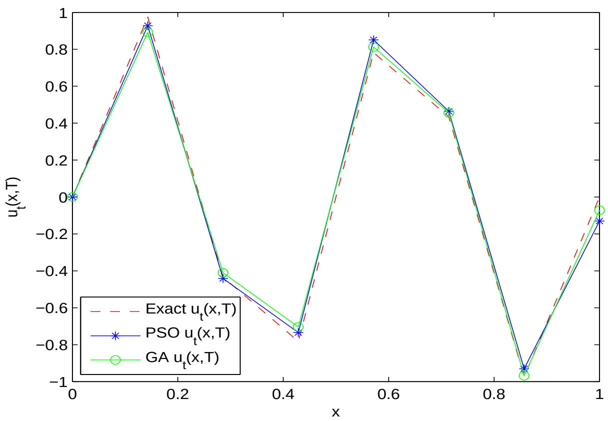

Figure 2: Diagram of $u(x,T)$ and $\omega(x)$

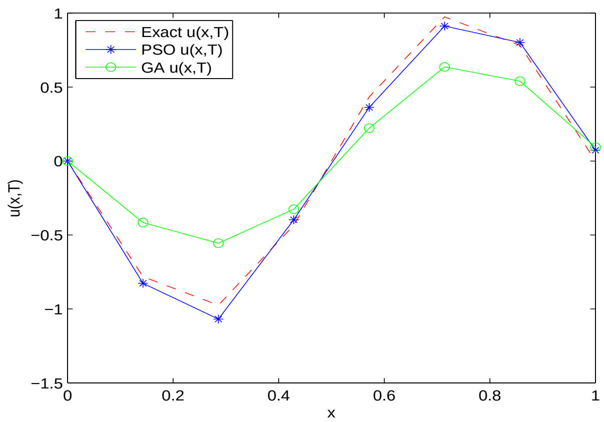

Figure 3: Diagram of $u_t(x,T)$ and $\zeta(x)$

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الگوریتمی بر پایه جمعیت برای تقریب کنترل بهینه توزیعی تحت معادلات موج

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چکیده : در این مقاله، یک روش تلفیقی تکراری نوین برای یافتن کنترل بهینه تقریبی توزیعی تحت معادله موج مورد بررسی قرار گرفته است. افرازی از فضای زمان–کنترل در نظر گرفته شده که مساله را گسسته می کند و سپس این شکل گسسته شده به یک مساله شبه تخصیص تبدیل شده است. آنگاه یک الگوریتم بر پایه جمعیت همراه با یک روش تفاضل متناهی برای استخراج کنترل بهینه تقریبی توزیعی به صورت یک تابع قطعه ای خطی به کار گرفته شده است. یک تحلیل همگرایی برای شکل گسسته مساله ابتدایی ارائه شده است. همچنین برای نشان دادن توانایی الگوریتم داده شده، نتایج عددی ارائه شده با نتایج حاصل از بکارگیری دو الگوریتم بر پایه جمعیت، الگوریتم ژنتیک و الگوریتم ازدحام ذرات مقایسه شده اند.

کلمات کلیدی : مسال کنترل بهینه؛ الگوریتم ارزیابی؛ روش تفاضلات متناهی؛ معادله موج.