



2D-fractional Muntz–Legendre polynomials for solving the fractional partial differential equations

E. Hengamian Asl, J. Saberi-Nadjafi* and M. Gachpazan

Abstract

We present a numerical method for solving linear and nonlinear fractional partial differential equations (FPDEs) with variable coefficients. The main aim of the proposed method is to introduce an orthogonal basis of two-dimensional fractional Muntz–Legendre polynomials. By using these polynomials, we approximate the unknown functions. Furthermore, an operational matrix of fractional derivative in the Caputo sense is provided for computing the fractional derivatives. The proposed approximation together with the Tau method reduces the solution of the FPDEs to the solution of a system of algebraic equations. Finally, to show the validity and accuracy of the presented method, we give some numerical examples.

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*Corresponding author

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Elias Hengamian Asl

Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran. e-mail: hengamianasl61@gmail.com

Jafar Saberi-Nadjafi

Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran. e-mail: najafi141@gmail.com

Mortaza Gachpazan

Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran. e-mail: gachpazan@um.ac.ir

1 Introduction

It is well known that fractional ordinary differential equations and fractional partial differential equations are important parts of the numerical analysis, statistical mechanics, and physics in recent years (see [5, 11]).

Recently, several numerical techniques have been proposed by researchers for solving some classes of the fractional ordinary differential equations and the fractional partial differential equations (FPDEs). For example, Nagy [9] used the Sinc-Chebyshev collocation method for fractional nonlinear Klein–Gordon equations. Yin et al. [18] applied two-dimensional fractional-order Legendre functions for solving the numerical solution of the fractional partial differential equations. Chen, Sun, and Liu [3] used a generalized fractional-order Legendre function for solving FPDEs with variable coefficients. For more numerical methods, the interested reader may refer to [6, 4, 2, 8, 1, 13].

In the present paper, we apply two-dimensional fractional Muntz–Legendre polynomials (2D-FMLPs) to find a numerical solution of linear and nonlinear FPDEs with variable coefficients.

This article is organized as follows. Review of Caputo fractional derivative is briefly provided in Section 2, and also we review one-dimensional fractional Muntz–Legendre polynomials in this section. In Section 3, two-dimensional FMLPs are introduced. In Section 4, we introduce an operational matrix of fractional derivative in the Caputo sense. In Section 5, we approximate the solution of FPDEs, using the Tau method based on 2D-FMLPs. Section 6 presents some examples of FPDEs to show the efficiency and the accuracy of the proposed method. Finally, a conclusion is expressed in Section 7.

2 Preliminaries

In this section, we review definitions and preliminary qualities of the fractional calculus, which are used in this paper.

Definition 1. The Caputo fractional derivative of order α is defined as

$$D^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha < n, \quad n \in \mathbb{N}, \\ \frac{d^n u(t)}{dt^n}, & \alpha = n, \quad t > 0. \end{cases}$$

It can easily be shown that

$$D^\alpha C = 0,$$

when C is a constant and

$$D^\alpha t^v = \begin{cases} 0, & v \in \mathbb{N}_0 \text{ and } v < [\alpha], \\ \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)} t^{v-\alpha}, & v \in \mathbb{N}_0 \text{ and } v \geq [\alpha] \text{ or } v \notin \mathbb{N}_0 \text{ and } v > [\alpha], \end{cases} \quad (1)$$

where $[\alpha]$ is the integer part of the positive real number α and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Moreover, we have

$$D^\alpha \left(\sum_{i=1}^n a_i u_i(t) \right) = \sum_{i=1}^n a_i D^\alpha u_i(t),$$

where $a_i, i = 1, \dots, n$ are constants.

The fraction Muntz–Legendre polynomials $L_i(t; \alpha)$ on the interval $[0, 1]$ are given by the following formula [4]:

$$L_i(t; \alpha) = \sum_{k=0}^i C_{i,k} t^{k\alpha}, \quad C_{i,k} = \frac{(-1)^{i-k}}{\alpha^i k! (i-k)!} \prod_{v=0}^{i-1} ((k+v)\alpha + 1). \quad (2)$$

Remark 1. According to (2), the analytic form of $L_i(t; \alpha)$ can be written as follows:

$$L_i(t; \alpha) = \sum_{k=0}^i b_{k,i} t^{k\alpha}, \quad (3)$$

where

$$b_{k,i} = \frac{(-1)^{i-k} \Gamma(\frac{1}{\alpha} + k + i)}{k! (i-k)! \Gamma(\frac{1}{\alpha} + k)}. \quad (4)$$

Also, we have

$$L_i(t; \alpha) = P_i^{(0, \frac{1}{\alpha}-1)}(2t^\alpha - 1), \quad \alpha > 0, \quad (5)$$

where $P_i^{(\alpha, \beta)}$ are the Jacobi polynomial with parameters $\alpha, \beta > -1$; see [4, 16]. Using (5) and the recurrence relation between the Jacobi polynomials [16], we can obtain the following recurrence formula:

$$L_{i+1}(t; \alpha) = a_i^\alpha L_i(t; \alpha) - b_i^\alpha L_{i-1}(t; \alpha), \quad i = 1, 2, \dots,$$

where

$$a_i^\alpha = \frac{(2i + \frac{1}{\alpha})[(2i + \frac{1}{\alpha} - 1)(2i + \frac{1}{\alpha} + 1)(2t^\alpha - 1) - (\frac{1}{\alpha} - 1)^2]}{2(i+1)(i + \frac{1}{\alpha})(2i + \frac{1}{\alpha} - 1)},$$

$$b_i^\alpha = \frac{i(i + \frac{1}{\alpha} - 1)(2i + \frac{1}{\alpha} + 1)}{(i+1)(i + \frac{1}{\alpha})(2i + \frac{1}{\alpha} - 1)},$$

$$L_0(t; \alpha) = 1, \quad L_1(t; \alpha) = (\frac{1}{\alpha} + 1)t^\alpha - \frac{1}{\alpha}.$$

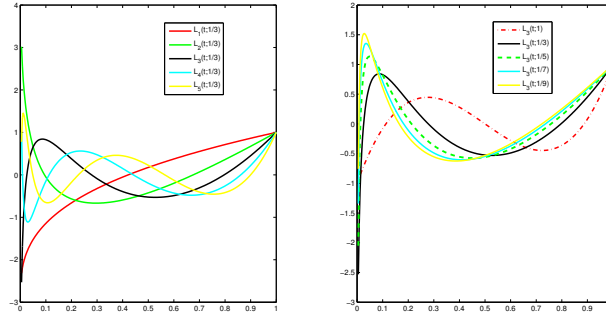


Figure 1: The graphs of $L_i(t; \alpha)$ for various values of i and α

Note that $L_i(1; \alpha) = 1$. Also, for $\alpha = 1$, we obtain shift Legendre polynomials [15] as follows:

$$L_i(t; 1) = P_i^{(0,0)}(2t - 1), \quad t \in [0, 1].$$

For $\alpha \neq 1$, we have

$$L_i(t; \alpha) = P_i^{(0, \frac{1}{\alpha} - 1)}(2t^\alpha - 1) \neq P_i^{(0,0)}(2t^\alpha - 1),$$

where $P_i^{(0,0)}(2t^\alpha - 1)$ are fractional shift Legendre polynomials; see [6].

Remark 2. The FMLPs are orthogonal on the interval $[0, 1]$ with the orthogonality relation:

$$\int_0^1 L_i(t; \alpha) L_j(t; \alpha) dt = \frac{1}{2i\alpha + 1} \delta_{ij}, \quad (6)$$

where δ_{ij} is the Kronecker function.

The graphs of FMLPs are shown in Figure 1.

Definition 2. An arbitrary function $u(t)$ that is integrable in $[0, 1]$, can be expanded as follows:

$$u(t) = \sum_{i=0}^{\infty} a_i L_i(t; \alpha), \quad (7)$$

where

$$a_i = (2i\alpha + 1) \int_0^1 u(t) L_i(t; \alpha) dt, \quad i = 0, 1, \dots$$

In practice, only the first $(m + 1)$ -terms of FMLPs are considered. Then it can be written by the infinite series of (7) as

$$u(t) \simeq u_m(t) = \sum_{i=0}^m a_i L_i(t; \alpha) = A^T \phi(t; \alpha), \quad (8)$$

where

$$A = [a_0, a_1, \dots, a_m]^T, \\ \phi(t; \alpha) = [L_0(t; \alpha), L_1(t; \alpha), \dots, L_m(t; \alpha)]^T.$$

The following theorem shows that the approximation FMLPs converges to $u(t)$.

Theorem 1.[Error bound] Suppose $D^{k\alpha}u(t) \in C[0, 1]$ for $k = 0, 1, \dots, m$. If $u_m(t)$ in (8) is the best approximation to $u(t)$ from $M_{m,\alpha} = \text{span}\{L_0(t; \alpha), L_1(t; \alpha), \dots, L_m(t; \alpha)\}$, then

$$\|u(t) - u_m(t)\|_\omega \leq \frac{M_\alpha}{\Gamma(m\alpha + 1)\sqrt{2m\alpha + 1}},$$

where $M_\alpha \geq |D^{m\alpha}u(t)|$, $t \in [0, 1]$.

Proof. By applying generalized Taylor's formula (see [3, 10]), we have

$$u(t) = \sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} D^{k\alpha}u(0^+) + \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} D^{m\alpha}u(\xi), \quad 0 < \xi < t, \quad t \in [0, 1].$$

Also,

$$\left| u(t) - \sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} D^{k\alpha}u(0^+) \right| \leq \frac{M_\alpha t^{m\alpha}}{\Gamma(m\alpha + 1)}.$$

On the other hand, $u_m(t) = A^T \phi(t; \alpha)$ is the best approximation to $u(t)$, and $\sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} D^{k\alpha}u(0^+) \in M_{m,\alpha}$. So, it can be written as

$$\begin{aligned} \|u(t) - u_m(t)\|_\omega^2 &\leq \left\| u(t) - \sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} D^{k\alpha}u(0^+) \right\|_\omega^2 \\ &\leq \frac{M_\alpha^2}{\Gamma(m\alpha + 1)^2} \int_0^1 t^{2m\alpha} dt = \frac{M_\alpha^2}{\Gamma(m\alpha + 1)^2(2m\alpha + 1)}. \end{aligned}$$

Therefore,

$$\|u(t) - u_m(t)\|_\omega \leq \frac{M_\alpha}{\Gamma(m\alpha + 1)\sqrt{2m\alpha + 1}}.$$

Hence, the proof is complete. \square

3 Two-dimensional fractional Muntz–Legendre polynomials

In this section, we define the fractional Muntz–Legendre polynomials in the domain $\Omega = [0, 1] \times [0, 1]$. To this end, we can define these polynomials as follows.

Definition 3. Let $\{L_n(t; \alpha)\}_{n=0}^{\infty}$ be the one-dimensional fractional Muntz–Legendre polynomials on $[0, 1]$. We call $\{L_i(x; \alpha)L_j(t; \beta)\}_{i,j=0}^{\infty}$ the 2D-FMLPs (two-dimensional fractional Muntz–Legendre polynomials) on $\Omega = [0, 1] \times [0, 1]$.

Theorem 2. 2D-FMLPs are orthogonal on Ω .

Proof. For $i \neq p$ and $j \neq q$, we have

$$\begin{aligned} & \int_0^1 \int_0^1 L_i(x; \alpha)L_j(t; \beta)L_p(x; \alpha)L_q(t; \beta) dx dt \\ &= \left(\int_0^1 L_i(x; \alpha)L_p(x; \alpha) dx \right) \left(\int_0^1 L_j(t; \beta)L_q(t; \beta) dt \right) = 0, \end{aligned}$$

and for $i = p$ and $j = q$, we have

$$\begin{aligned} \int_0^1 \int_0^1 [L_i(x; \alpha)]^2 [L_j(t; \beta)]^2 dx dt &= \left(\int_0^1 [L_i(x; \alpha)]^2 dx \right) \left(\int_0^1 [L_j(t; \beta)]^2 dt \right) \\ &= \frac{1}{2i\alpha + 1} \cdot \frac{1}{2j\beta + 1}. \end{aligned}$$

□

Definition 4. An arbitrary function $u(x, t)$ that is integrable in $\Omega = [0, 1] \times [0, 1]$ can be expanded as follows:

$$u(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} L_i(x; \alpha) L_j(t; \beta), \quad (9)$$

where

$$a_{ij} = (2i\alpha + 1)(2j\beta + 1) \int_0^1 \int_0^1 u(x, t) L_i(x; \alpha) L_j(t; \beta) dx dt.$$

In practice, only the first $(m+1)(n+1)$ -terms of 2D-FMLPs are considered. Then it can be written by the infinite series of (9) as

$$u(x, t) \simeq u_{m,n}(x, t) = \sum_{i=0}^m \sum_{j=0}^n a_{ij} L_i(x; \alpha) L_j(t; \beta) = \phi^T(x; \alpha) A \phi(t; \beta), \quad (10)$$

where

$$\begin{aligned}\phi(x; \alpha) &= [L_0(x; \alpha), L_1(x; \alpha), \dots, L_m(x; \alpha)]^T, \\ \phi(t; \beta) &= [L_0(t; \beta), L_1(t; \beta), \dots, L_n(t; \beta)]^T,\end{aligned}$$

and A is an $(m+1) \times (n+1)$ matrix as follows:

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m0} & a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

To calculate the coefficients a_{ij} in (10), we use the Tau method, which is discussed in Section 6.

Remark 3. The product matrix of FMLPs can be obtained as follows:

$$\phi(x; \alpha)\phi^T(x; \alpha)A \simeq \tilde{A}\phi(x; \alpha),$$

where \tilde{A} is an $(m+1) \times (m+1)$ product matrix for the matrix A . The elements of \tilde{A} are as follows,

$$\tilde{A}_{i,j} = (2j\alpha + 1) \sum_{k=0}^m a_k g_{ijk}, \quad i, j = 0, \dots, m,$$

where

$$g_{ijk} = \int_0^1 L_i^\alpha(x) L_j^\alpha(x) L_k^\alpha(x) dx.$$

The following theorems state the existence of uniqueness and convergence analyses.

Theorem 3. If the double series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} L_i(x; \alpha) L_j(t; \beta)$$

converges uniformly to $u(x, t)$ on the $\Omega = [0, 1] \times [0, 1]$, then

$$a_{ij} = (2i\alpha + 1)(2j\beta + 1) \int_0^1 \int_0^1 u(x, t) L_i(x; \alpha) L_j(t; \beta) dx dt.$$

Proof. Set

$$u(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} L_i(x; \alpha) L_j(t; \beta),$$

and let m, n be fixed. We can write

$$\begin{aligned} & \int_0^1 \int_0^1 u(x, t) L_m(x; \alpha) L_n(t; \beta) dx dt \\ &= \int_0^1 \int_0^1 \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} L_i(x; \alpha) L_j(t; \beta) \right) L_m(x; \alpha) L_n(t; \beta) dx dt \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \left(\int_0^1 L_i(x; \alpha) L_m(x; \alpha) dx \right) \left(\int_0^1 L_j(t; \beta) L_n(t; \beta) dt \right) \\ &= a_{ij} \left(\int_0^1 L_i^2(x; \alpha) dx \right) \left(\int_0^1 L_j^2(t; \beta) dt \right) \\ &= a_{ij} \left(\frac{1}{2i\alpha + 1} \right) \left(\frac{1}{2j\beta + 1} \right). \end{aligned}$$

□

Lemma 1. If $u(x, t)$ is a continuous function on $[0, 1] \times [0, 1]$ and the double series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} L_i(x; \alpha) L_j(t; \beta) \quad (11)$$

converges uniformly to $u(x, t)$, then the series (11) is the FMLPs expansion of $u(x, t)$.

Proof. The proof of this lemma is similar to that of [8, Lemma 1]. □

Lemma 2. If two continuous functions defined on $[0, 1] \times [0, 1]$ have the identical FMLPs expansions, then these two functions are identical.

Proof. The lemma is proved similar to Lemma 2 in [8]. □

Theorem 4. If the 2D-FMLPs expansion of a continuous function $u(x, t)$ converges uniformly, then every 2D-FMLPs expansion converges to the function $u(x, t)$.

Proof. It is a result of Lemmas 1 and 2. □

4 FMLPs operational matrix of fractional derivatives

In this section, we introduce an operational matrix of fractional derivative. First, we present the following lemma.

Lemma 3. Let $L_i(t; \alpha)$ be an FMLP. Then the Caputo fractional derivative of $L_i(t; \alpha)$ of order $\gamma > 0$ can be obtained by the following relation:

$$D^\gamma L_i(t; \alpha) = \sum_{k=0}^i b'_{k,i} \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha - \gamma + 1)} t^{k\alpha - \gamma},$$

where

$$b'_{k,i} = \begin{cases} 0, & k\alpha \in \mathbb{N}_0 \text{ and } k\alpha < \gamma, \\ b_{k,i}, & k\alpha \in \mathbb{N}_0 \text{ and } k\alpha \geq \gamma \text{ or } k\alpha \notin \mathbb{N}_0 \text{ and } k\alpha \geq [\gamma]. \end{cases}$$

and $b_{k,i}$ is given in (4).

Proof. By using relations (1) and (3), the proof of the lemma is clear. \square

Now, the next theorem gives the operational matrix of fractional derivative of FMLPs.

Theorem 5. If $\phi(x; \alpha)$ is a fractional Muntz–Legendre vector given in (10), then

$$D^\gamma \phi(x; \alpha) = D^{(\gamma)} \phi(x; \alpha),$$

where $D^{(\gamma)}$ is the $(m+1) \times (n+1)$ operational matrix of fractional derivative of order $\gamma > 0$. The elements of $D^{(\gamma)}$ are defined as follows:

$$D_{ij}^{(\gamma)} = (2j\alpha + 1) \sum_{k=0}^i \sum_{s=0}^j b_{s,j} b'_{k,i} \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha - \gamma + 1)} \cdot \frac{1}{(k+s)\alpha - \gamma + 1},$$

where

$$b'_{k,i} = \begin{cases} 0, & k\alpha \in \mathbb{N}_0 \text{ and } k\alpha < \gamma, \\ b_{k,i}, & k\alpha \in \mathbb{N}_0 \text{ and } k\alpha \geq \gamma \text{ or } k\alpha \notin \mathbb{N}_0 \text{ and } k\alpha \geq [\gamma]. \end{cases}$$

Proof. According to Lemma 3, we have

$$D^\gamma L_i(t; \alpha) = \sum_{k=0}^i b'_{k,i} \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha + 1 - \gamma)} t^{k\alpha - \gamma}. \quad (12)$$

On the other hand, the approximation of $t^{k\alpha - \gamma}$ by using (8) gives the following relation:

$$t^{k\alpha - \gamma} \simeq \sum_{j=0}^m a_{k,j} L_j(t; \alpha), \quad (13)$$

where

$$\begin{aligned}
a_{k,j} &= (2j\alpha + 1) \int_0^1 t^{k\alpha - \gamma} L_j(x; \alpha) dt \\
&= (2j\alpha + 1) \sum_{s=0}^j b_{s,j} \int_0^1 t^{k\alpha - \gamma + s\alpha} dt \\
&= (2j\alpha + 1) \sum_{s=0}^j b_{s,j} \frac{1}{k\alpha - \gamma + s\alpha + 1}. \tag{14}
\end{aligned}$$

Now, according to Equations (12)–(14), we can write

$$D^\gamma L_i(t; \alpha) = (2j\alpha + 1) \sum_{j=0}^m \sum_{k=0}^i \sum_{s=0}^j b_{s,j} b'_{k,i} \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha + 1 - \gamma)} \cdot \frac{1}{(k\alpha - \gamma + s\alpha + 1)} L_j(t; \alpha).$$

Thus,

$$D_{ij}^\gamma = (2j\alpha + 1) \sum_{k=0}^i \sum_{s=0}^j b_{s,j} b'_{k,i} \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha - \gamma + 1)} \cdot \frac{1}{(k + s)\alpha - \gamma + 1},$$

and this completes the proof. \square

5 Approximate solution to the FPDEs

In this section, we approximate the solution of the linear and nonlinear FPDEs by applying the operational matrix of fractional derivatives of 2D-FMLPs.

5.1 Numerical solution of linear FPDEs with variable coefficients

To study the linear FPDEs with variable coefficients, the following form is considered:

$$a(x) \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} + b(t) \frac{\partial^\beta u(x, t)}{\partial t^\beta} + c(x) \frac{\partial u(x, t)}{\partial x} + d(x) u(x, t) = f(x, t), \tag{15}$$

$$u(0, t) = g_0(t), \quad u(x, 0) = h_0(x), \tag{16}$$

where $(x, t) \in \Omega = (0, 1] \times (0, 1]$, $0 < \alpha, \beta \leq 1$ and α, β are the order of the fractional derivatives in the Caputo sense. In (15) and (16), the continuous functions a , b , c , d , g_0 , h_0 , and f are known and the function $u(x, t)$ is unknown.

To solve problem (15)–(16), we approximate the known functions, the unknown function $u(x, t)$, and the differentials $\frac{\partial u(x, t)}{\partial x}$, $\frac{\partial^\alpha u(x, t)}{\partial x^\alpha}$, $\frac{\partial^\beta u(x, t)}{\partial t^\beta}$ by applying (10) and Theorem 5 as follows:

$$u(x, t) \simeq \phi^T(x; \alpha)U\phi(t; \beta), \quad f(x, t) \simeq \phi^T(x; \alpha)F\phi(t; \beta), \quad (17)$$

$$a(x) \simeq A^T\phi(x; \alpha), \quad b(t) \simeq \phi^T(t; \beta)B, \quad (18)$$

$$c(x) \simeq C^T\phi(x; \alpha), \quad d(x) \simeq E^T\phi(x; \alpha), \quad (19)$$

$$\frac{\partial u(x, t)}{\partial x} \simeq \phi^T(x; \alpha)(D^{(1)})^T U\phi(t; \beta), \quad (20)$$

$$\frac{\partial^\alpha u(x, t)}{\partial x^\alpha} \simeq \phi^T(x; \alpha)(D^{(\alpha)})^T U\phi(t; \beta), \quad (21)$$

$$\frac{\partial^\beta u(x, t)}{\partial t^\beta} \simeq \phi^T(x; \alpha)UD^{(\beta)}\phi(t; \beta). \quad (22)$$

Substituting (17)–(22) in (15), we have

$$\begin{aligned} & \phi^T(x; \alpha)\tilde{A}^T(D^\alpha)^T U\phi(t; \beta) + \phi^T(x; \alpha)UD^{(\beta)}\tilde{B}\phi(t; \beta) \\ & + \phi^T(x; \alpha)\tilde{C}^T(D^{(1)})^T U\phi(t; \beta) + \phi^T(x; \alpha)\tilde{E}^T U\phi(t; \beta) \simeq \phi^T(x; \alpha)F\phi(t; \beta) \end{aligned}$$

or

$$\phi^T(x; \alpha)[\tilde{A}^T(D^\alpha)^T U + UD^{(\beta)}\tilde{B} + \tilde{C}^T(D^{(1)})^T U + \tilde{E}^T U]\phi(t; \beta) \quad (23)$$

$$\simeq \phi^T(x; \alpha)F\phi(t; \beta). \quad (24)$$

In addition, for the initial and boundary conditions (16), we have

$$u(0, t) \simeq \phi^T(0; \alpha)U\phi(t; \beta) \simeq G_0^T\phi(t; \beta), \quad g_0(t) \simeq G_0^T\phi(t; \beta), \quad (25)$$

$$u(x, 0) \simeq \phi^T(x; \alpha)U\phi(0; \beta) \simeq \phi^T(x; \alpha)H_0, \quad h_0(x) \simeq \phi^T(x; \alpha)H_0. \quad (26)$$

Now, by using the Tau method in [14], we could generate $m \times n$ equations from (24) as follows:

$$\begin{aligned} & \int_0^1 \int_0^1 L_i^\alpha(x)\phi^T(x; \alpha)[\tilde{A}^T(D^\alpha)^T U + UD^{(\beta)}\tilde{B} + \tilde{C}^T(D^{(1)})^T U + \tilde{E}^T U]\phi(t; \beta)L_j^\beta(t)dxdt \\ & \simeq \int_0^1 \int_0^1 L_i^\alpha(x)\phi^T(x; \alpha)F\phi(t; \beta)L_j^\beta(t)dxdt. \quad (27) \end{aligned}$$

By using the orthogonality property (6), we have

$$\int_0^1 \int_0^1 L_i^\alpha(x) \phi^T(x; \alpha) \phi(t; \beta) L_j^\beta(t) dx dt = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{(2i\alpha+1)(2j\beta+1)} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

for $i = 0, 1, \dots, m-1$ and $j = 0, 1, \dots, n-1$. Therefore, we rewrite (27) as follows:

$$\tilde{A}^T (D^\alpha)^T U + U D^{(\beta)} \tilde{B} + \tilde{C}^T (D^{(1)})^T U + \tilde{E}^T U \simeq F. \quad (28)$$

Also, we generate $2n + m + 1$ equations from conditions (25) and (26) as follows:

$$\phi^T(0; \alpha) U \simeq G_0^T, \quad j = 0, 1, \dots, n, \quad (29)$$

$$U \phi(0; \beta) \simeq H_0, \quad i = 0, 1, \dots, m-2. \quad (30)$$

Now, we have a system of the $mn + m + n + 1$ algebraic equations of Equations (28)–(30) with the $mn + m + n + 1$ elements of the unknown matrix U . After solving this matrix system, by using the fixed point method, we can find the approximate solution $u_{m,n}(x, t) = \phi^T(x; \alpha) U \phi(t; \beta)$.

5.2 Numerical solution of nonlinear FPDEs

To study the various types of nonlinear FPDEs, the following forms are considered:

(a) Time-fractional convection-diffusion equation:

A class of time-fractional convection-diffusion equation may be written as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx} - cu_x + N(u) + f(x, t), \quad (31)$$

$$u(0, t) = g_0(t), \quad u(1, t) = g_1(t), \quad u(x, 0) = h_0(x), \quad 0 < x < 1, t > 0, \quad (32)$$

where $N(u)$ is a nonlinear operator. In this paper, we choose one or more terms of the nonlinear operator $N(u)$ in the form of u^k , $k \in \mathbb{N}$, uu_x , or uu_{xx} .

(b) Time- fractional-order Klein–Gordon equation:

A class of time-fractional order Klein–Gordon equation reads as follows:

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} + \frac{\partial^2 u(x, t)}{\partial x^2} + au(x, t) + bu^2(x, t) + cu^3(x, t) &= f(x, t), \\ u(0, t) = g_0(t), \quad u(1, t) = g_1(t), \quad u(x, 0) &= h_0(x). \end{aligned}$$

(c) Time- fractional-order gas dynamics equation:

A class of time-fractional order gas dynamics equation reads as follows:

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} + u(x, t) \frac{\partial u(x, t)}{\partial x} - u(x, t)(1 - u(x, t)) &= f(x, t), \\ u(0, t) = g_0(t), \quad u(x, 0) = h_0(x). \end{aligned}$$

To solve problem **(a)**, **(b)**, or **(c)**, we approximate the unknown function $u(x, t)$ as follows:

$$\begin{aligned} u(x, t) &\simeq u_{m,n}(x, t) = \sum_{i=0}^m \sum_{j=0}^n u_{ij} L_i(x) L_j(t; \alpha) \\ &= U^T \Upsilon(x, t) = U^T (\phi(x) \otimes \phi(t; \alpha)), \end{aligned}$$

where \otimes denotes the Kronecker product and $L_i(x) = L_i(x; 1)$ are the Muntz–Legendre polynomials with $\alpha = 1$. Also $\phi(x)$, $\phi(t; \alpha)$, U , and $\Upsilon(x, t)$ read as

$$\begin{aligned} \phi(x) &= \phi(x; 1) = [L_0(x), L_1(x), \dots, L_m(x)]^T, \\ \phi(t; \alpha) &= [L_0(t; \alpha), L_1(t; \alpha), \dots, L_n(t; \alpha)]^T, \end{aligned}$$

$$U = \begin{bmatrix} u_{00} \\ u_{01} \\ \vdots \\ u_{0n} \\ u_{10} \\ \vdots \\ u_{i,j-1} \\ u_{i,j} \\ u_{i,j+1} \\ \vdots \\ u_{m,n-1} \\ u_{mn} \end{bmatrix}, \quad \Upsilon(x, t) = \begin{bmatrix} L_0(x)L_0(t; \alpha) \\ L_0(x)L_1(t; \alpha) \\ \vdots \\ L_0(x)L_m(t; \alpha) \\ L_1(x)L_0(t; \alpha) \\ \vdots \\ L_i(x)L_{j-1}(t; \alpha) \\ L_i(x)L_j(t; \alpha) \\ L_i(x)L_{j+1}(t; \alpha) \\ \vdots \\ L_m(x)L_{n-1}(t; \alpha) \\ L_m(x)L_n(t; \alpha) \end{bmatrix}.$$

Furthermore, the first and second order derivative operator matrices of the vector $\Upsilon(x, t)$ can be obtained by

$$\begin{aligned}\frac{\partial}{\partial x}\Upsilon(x, t) &\simeq \mathbf{D}_x\Upsilon(x, t), & \mathbf{D}_x &= D_x^{(1)} \otimes I_{n+1}, \\ \frac{\partial^2}{\partial x^2}\Upsilon(x, t) &\simeq \mathbf{D}_x^2\Upsilon(x, t), & \mathbf{D}_x^2 &= D_x^{(2)} \otimes I_{n+1},\end{aligned}$$

where $D^{(1)}$ and $D^{(2)}$ are the first and second order derivative matrices of the vector $\phi(x)$, respectively.

The Caputo fractional derivatives operator of order $\alpha > 0$ of the vector $\Upsilon(x, t)$ can be obtained by

$$D_t^\alpha\Upsilon(x, t) \simeq \mathbf{D}_t^\alpha\Upsilon(x, t), \quad \mathbf{D}_t^\alpha = I_{m+1} \otimes D_t^{(\alpha)},$$

where $D_t^{(\alpha)}$ is the Caputo fractional derivatives matrix of vector $\phi(t; \alpha)$, which is obtained in Theorem 5.

Remark 4. The product matrix of FMLPs vectors can be obtained as follows:

$$\Upsilon(x, t)\Upsilon^T(x, t)U \simeq \tilde{U}\Upsilon(x, t),$$

where \tilde{U} is an $(m+1)(n+1) \times (m+1)(n+1)$ product matrix for the vector U as follows:

$$\tilde{U} = [\tilde{U}^{0,0}, \tilde{U}^{0,1}, \dots, \tilde{U}^{0,n}, \tilde{U}^{1,0}, \dots, \tilde{U}^{1,n}, \dots, \tilde{U}^{m,0}, \dots, \tilde{U}^{m,n}]^T,$$

where $\tilde{U}^{p,q}$, $p = 0, 1, \dots, m$, $q = 0, 1, \dots, n$, is an $1 \times (m+1)(n+1)$ matrix as

$$\tilde{U}^{p,q} = [\tilde{U}_{0,0}^{p,q}, \tilde{U}_{0,1}^{p,q}, \dots, \tilde{U}_{0,n}^{p,q}, \tilde{U}_{1,0}^{p,q}, \dots, \tilde{U}_{1,n}^{p,q}, \dots, \tilde{U}_{m,0}^{p,q}, \dots, \tilde{U}_{m,n}^{p,q}],$$

and each element of $\tilde{U}^{p,q}$ is obtained as

$$\tilde{U}_{k,l}^{p,q} = (2k+1)(2l+1) \sum_{i=0}^m \sum_{j=0}^n u_{ij} g_{ijpqkl}, \quad p, k = 0, 1, \dots, m, \quad q, l = 0, 1, \dots, n,$$

where g_{ijpqkl} is given by

$$g_{ijpqkl} = \int_0^1 \int_0^1 L_i(x)L_j(t; \alpha)L_p(x)L_q(t; \alpha)L_k(x)L_l(t; \alpha)dxdt.$$

Remark 5. Assume that $f(x, t) \simeq F^T\Upsilon(x, t)$ and $g(x, t) \simeq G^T\Upsilon(x, t)$, where F and G are an $(m+1)(n+1)$ -vector as

$$\begin{aligned}F &= [f_{0,0}, f_{0,1}, \dots, f_{0,n}, f_{1,0}, \dots, f_{i,j-1}, f_{i,j}, f_{i,j+1}, \dots, f_{m,n-1}, f_{m,n}]^T, \\ G &= [g_{0,0}, g_{0,1}, \dots, g_{0,n}, g_{1,0}, \dots, g_{i,j-1}, g_{i,j}, g_{i,j+1}, \dots, g_{m,n-1}, g_{m,n}]^T,\end{aligned}$$

where, from Definition 4, each element of F and G is obtained as

$$\begin{aligned} f_{i,j} &= (2i+1)(2j\alpha+1) \int_0^1 \int_0^1 f(x,t) L_i(x) L_j(t; \alpha) dx dt, \\ g_{i,j} &= (2i+1)(2j\alpha+1) \int_0^1 \int_0^1 g(x,t) L_i(x) L_j(t; \alpha) dx dt. \end{aligned}$$

We can write

$$f(x,t)g(x,t) \simeq A^T \Upsilon(x,t) \Upsilon^T(x,t) B \simeq A^T \tilde{B} \Upsilon(x,t) \simeq B^T \tilde{A} \Upsilon(x,t).$$

Lemma 4. The nonlinear operator $N(u(x,t)) = u^k(x,t)$, where $u(x,t) \simeq U^T \Upsilon(x,t)$, can be approximated by

$$u^k(x,t) \simeq U^T \tilde{U}^{k-1} \Upsilon(x,t).$$

Proof. From Remark 5, we have

$$\begin{aligned} u^2(x,t) &= u(x,t)u(x,t) \simeq U^T \tilde{U} \Upsilon(x,t), \\ u^k(x,t) &= u^2(x,t)u^{k-2}(x,t) \simeq U^T \tilde{U} \Upsilon(x,t) u(x,t) u^{k-3}(x,t) \\ &\simeq U^T \tilde{U} \Upsilon(x,t) \Upsilon^T(x,t) U u^{k-3}(x,t) \\ &\simeq U^T \tilde{U}^2 \Upsilon(x,t) u^{k-3}(x,t) \simeq \dots \simeq U^T \tilde{U}^{k-1} \Upsilon(x,t). \end{aligned}$$

□

Remark 6. The nonlinear operator $N_1(u, u_x) = uu_x$ and $N_2(u, u_{xx}) = uu_{xx}$, can be approximated by

$$\begin{aligned} N_1(u, u_x) &= uu_x \simeq U^T \mathbf{D}_x \tilde{U} \Upsilon(x,t), \\ N_2(u, u_{xx}) &= uu_{xx} \simeq U^T \mathbf{D}_x^2 \tilde{U} \Upsilon(x,t). \end{aligned}$$

Now, to solve one of the problems (a), (b), or (c), for instance, problem (a), with the nonlinear operator $N(u) = uu_x + uu_{xx} + u^2$, by applying the above approximations, we rewrite (31) as

$$[U^T (\mathbf{D}_t^\alpha - \mathbf{D}_x^2 + c\mathbf{D}_x) - \tilde{N}] \Upsilon(x,t) = F^T \Upsilon(x,t), \quad (33)$$

where

$$N(u) \simeq \tilde{N} \Upsilon(x,t) = U^T [\mathbf{D}_x \tilde{U} + \mathbf{D}_x^2 \tilde{U} + \tilde{U}] \Upsilon(x,t), \quad f(x,t) \simeq F^T \Upsilon(x,t).$$

As in the Tau method in [14], we should generate $(m-1)n$ equations from (33) as follows:

$$[U^T (\mathbf{D}_t^\alpha - \mathbf{D}_x^2 + c\mathbf{D}_x) - \tilde{N}] \int_0^1 \int_0^1 \Upsilon(x,t) L_i(x) L_j(t; \alpha) dx dt \quad (34)$$

$$= F^T \int_0^1 \int_0^1 \Upsilon(x, t) L_i(x) L_j(t; \alpha) dx dt,$$

for $i = 0, 1, \dots, m-2$ and $j = 0, 1, \dots, n-1$. On the other hand, by using the orthogonality property (6), for $i = 0, 1, \dots, m-2$ and $j = 0, 1, \dots, n-1$, we have

$$\int_0^1 \int_0^1 \Upsilon(x, t) L_i(x) L_j(t; \alpha) dx dt = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{(2i+1)(2j\alpha+1)} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, we rewrite $(m-1)n$ equations (34) as follows:

$$[U^T (\mathbf{D}_t^\alpha - \mathbf{D}_x^2 + c\mathbf{D}_x) - \tilde{N}] = F^T. \quad (35)$$

Also, we generate $2n + m + 1$ equations from conditions (32) as follows:

$$U^T (\phi(0) \otimes I_{n+1}) = G_0^T, \quad g_0(t) = G_0^T \phi(t; \alpha), \quad j = 0, 1, \dots, n, \quad (36)$$

$$U^T (\phi(1) \otimes I_{n+1}) = G_1^T, \quad g_1(t) = G_1^T \phi(t; \alpha), \quad j = 0, 1, \dots, n, \quad (37)$$

$$U^T (I_{m+1} \otimes \phi(0; \alpha)) = H_0^T, \quad h_0(x) = H_0^T \phi(x), \quad i = 0, 1, \dots, m-2. \quad (38)$$

Now, we have a system of the $mn + m + n + 1$ nonlinear algebraic equations of (35)–(38) with the $mn + m + n + 1$ elements of the unknown matrix U . After solving this linear system, by using the fixed point method, we can find the approximate solution $u_{m,n}(x, t) = U^T \Upsilon(x, t)$.

6 Numerical illustration

In this section, we present some examples of linear and nonlinear of FPDEs to show the efficiency of the proposed method. The results will be compared with the exact solutions and other methods. The accuracy of the presented

method is estimated by the absolute error $E_{m,n}$ and the maximum absolute error $\epsilon_{m,n}$, which are given as follows:

$$(E_{m,n})_{i,j} = |u(\theta_i, \beta_j) - u_{m,n}(\theta_i, \beta_j)|,$$

$$\epsilon_{m,n} = \max\{|u(\theta_i, \beta_j) - u_{m,n}(\theta_i, \beta_j)|\},$$

where θ_i and β_j are defined as,

$$(\theta_i, \beta_j) = (x_i^{\frac{1}{\alpha}}, t_j^{\frac{1}{\beta}}), \quad i = 0, 1, \dots, m, \quad j = 0, 1, \dots, n,$$

where x_i and t_j are Chebyshev–Gauss–Lobatto points with the following relations:

$$x_i = \frac{1}{2} - \frac{1}{2} \cos \frac{\pi i}{m}, \quad t_j = \frac{1}{2} - \frac{1}{2} \cos \frac{\pi j}{n}.$$

Example 1. Consider the linear FPDE with variable coefficients

$$x \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + \frac{\partial^\beta u(x,t)}{\partial t^\beta} = \frac{1}{\Gamma(2-\alpha)} x^{2-\alpha} t + \frac{1}{\Gamma(2-\beta)} x t^{1-\beta}, \quad (39)$$

$$u(0, t) = u(x, 0) = 0, \quad (x, t) \in (0, 1] \times (0, 1], \quad 0 < \alpha, \beta \leq 1. \quad (40)$$

The exact solution is $u(x, t) = xt$. Applying the presented method to solve (39) and (40) with $m = n = 2$, for $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$, we get

$$U = \begin{bmatrix} 2.5000e-01 & 2.0000e-01 & 5.0000e-02 \\ 2.0000e-01 & 1.6000e-01 & 4.0000e-02 \\ 5.0000e-02 & 4.0000e-02 & 1.0000e-02 \end{bmatrix} \simeq \begin{bmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{20} \\ \frac{1}{5} & \frac{4}{25} & \frac{1}{25} \\ \frac{1}{20} & \frac{1}{25} & \frac{1}{100} \end{bmatrix}.$$

Therefore, the approximation solution is

$$\begin{aligned} u_{2,2}(x, t) &= \phi^T(x; \frac{1}{2}) U \phi(t; \frac{1}{2}) \\ &= [1 \ 3\sqrt{x} - 2 \ 10x - 12\sqrt{x} + 3] \begin{bmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{20} \\ \frac{1}{5} & \frac{4}{25} & \frac{1}{25} \\ \frac{1}{20} & \frac{1}{25} & \frac{1}{100} \end{bmatrix} \begin{bmatrix} 1 \\ 3\sqrt{t} - 2 \\ 10t - 12\sqrt{t} + 3 \end{bmatrix} = xt, \end{aligned}$$

which is actually the exact solution. The results are shown in Tables 1 and 2. Table 1 shows the comparison of the presented method with other methods in [3, 14]. From Table 1, it is seen that we obtained a lesser error. Table 2 shows the maximum absolute error $\epsilon_{m,n}$ for various values of m, n, α , and β . It can be seen that the approximate solution is obtained from the proposed method is in a good agreement with the exact solution.

Example 2. Consider the linear FPDE with variable coefficients

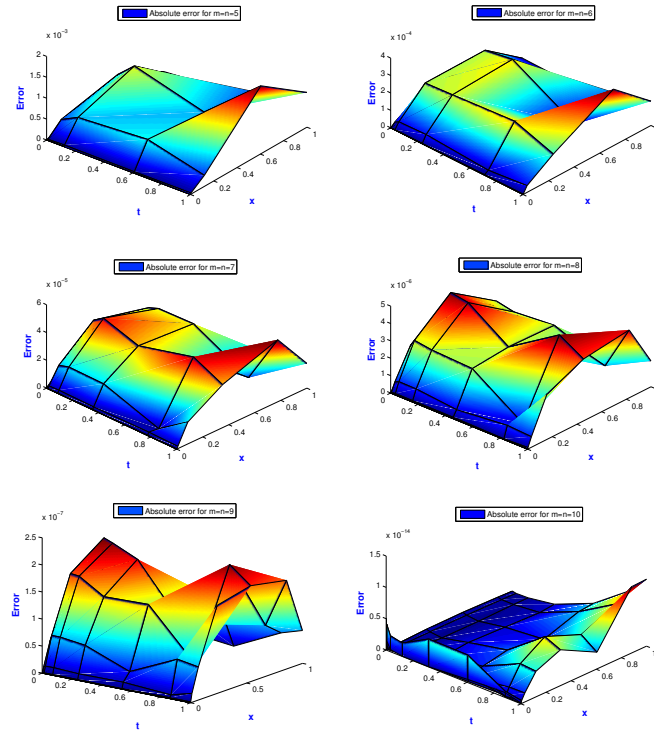


Figure 2: The absolute error function with $\alpha = \beta = \frac{1}{5}$ and $m = n = 5, 6, 7, 8, 9, 10$ for Example 2

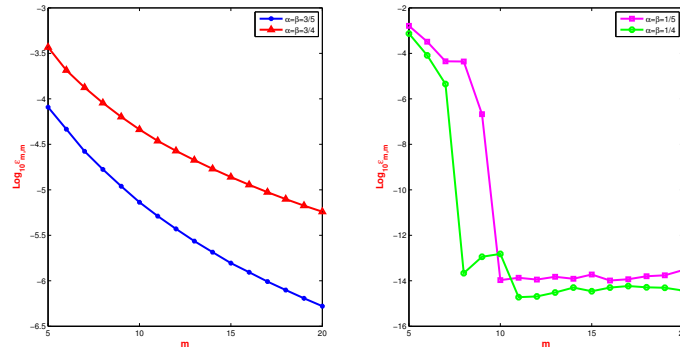


Figure 3: The logarithmic graphs of $\epsilon_{m,m}$ for Example 2

Table 1: The absolute error for example 1

| (x, t) | the presented method | method in [3] | method in [14] |
|------------|---|---|--|
| | $\alpha = \beta = \frac{1}{2}$ $m = n = 2$ | $\alpha = \beta = \frac{1}{2}$ $m = n = 2$ | $\alpha = \beta = \frac{1}{2}$ $\alpha = \beta = 1$ |
| (0.1, 0.2) | $3.4694e - 18$ | $3.47e - 17$ | $1.20e - 04$ |
| (0.5, 0.2) | 0 | 0 | $7.61e - 03$ |
| (1.1, 0.2) | $2.7756e - 17$ | $5.55e - 17$ | $2.11e - 02$ |
| (1.5, 0.2) | $5.5511e - 17$ | $1.39e - 16$ | $3.27e - 02$ |
| (1.9, 0.2) | 0 | $1.67e - 16$ | $4.65e - 02$ |

Table 2: Maximum absolute error for Example 1

| $\varepsilon_{m,n}$ | $\alpha = \beta = \frac{1}{5}$ | $\alpha = \beta = \frac{1}{3}$ | $\alpha = \beta = 1$ |
|---------------------|--------------------------------|--------------------------------|----------------------|
| $\varepsilon_{5,5}$ | $4.3696e - 14$ | $3.8858e - 15$ | $3.1086e - 15$ |
| $\varepsilon_{7,7}$ | $7.1054e - 15$ | $1.4433e - 15$ | $1.1102e - 15$ |

$$\sin x \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} + (1 + \cos t) \frac{\partial^\beta u(x, t)}{\partial t^\beta} + \sqrt{x} u(x, t) = g(x, t),$$

$$u(0, t) = u(x, 0) = 0, \quad (x, t) \in (0, 1] \times (0, 1], \quad 0 < \alpha, \beta < 1,$$

where

$$g(x, t) = t^2 \sin x \left(\frac{x^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} \right) + (1-x)(x(1+\cos x) \frac{2t^{2-\beta}}{\Gamma(3-\beta)} + x^{\frac{3}{2}} t^2).$$

The exact solution is $u(x, t) = t^2(x - x^2)$. The results are shown in Table 3 and Figures 2 and 3. Figure 2 shows the absolute error function $E_{m,n}$ for $\alpha = \beta = \frac{1}{5}$ and various values of m and n . Figure 3 shows the logarithmic graphs of $\varepsilon_{m,n}$ ($\log_{10} \varepsilon_{m,n}$) for various values of α . It is seen that the absolute error converges to zero by increasing m and n ($m = n$).

Table 3: Maximum absolute error for Example 2

| $\varepsilon_{m,n}$ | $\alpha = \beta = \frac{1}{5}$ | $\alpha = \beta = \frac{1}{4}$ | $\alpha = \beta = \frac{3}{5}$ | $\alpha = \beta = \frac{3}{4}$ |
|-----------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| $\varepsilon_{5,5}$ | $1.6256e - 03$ | $7.4183e - 04$ | $8.1055e - 05$ | $3.6924e - 04$ |
| $\varepsilon_{6,6}$ | $3.2465e - 04$ | $8.1570e - 05$ | $4.6443e - 05$ | $2.0615e - 04$ |
| $\varepsilon_{7,7}$ | $4.4705e - 05$ | $4.4833e - 06$ | $2.6524e - 05$ | $1.3334e - 04$ |
| $\varepsilon_{8,8}$ | $4.3807e - 05$ | $2.1630e - 14$ | $1.6770e - 05$ | $9.0206e - 05$ |
| $\varepsilon_{9,9}$ | $2.1148e - 07$ | $1.1367e - 13$ | $1.0939e - 05$ | $6.3418e - 05$ |
| $\varepsilon_{10,10}$ | $1.0694e - 14$ | $1.5205e - 13$ | $7.3022e - 06$ | $4.6034e - 05$ |

Example 3. Consider the following time-fractional order Klein–Gordon equation

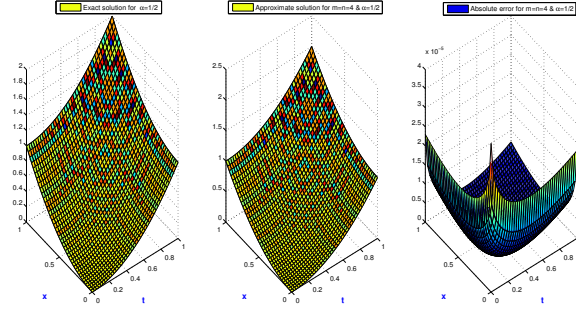


Figure 4: Comparison between the exact solution and the approximate solution with absolute error function $E_{4,4}(x, t)$ for $\alpha = 1/2$ for Example 3

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \frac{\partial^2 u(x, t)}{\partial x^2} - u(x, t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - x^2 - t^2 - 2,$$

$$u(0, t) = t^2, u(x, 0) = x^2, u(1, t) = 1 + t^2, \quad (x, t) \in (0, 1) \times (0, 1], 0 < \alpha < 1.$$

The exact solution is $u(x, t) = x^2 + t^2$. Applying the presented method to solve this problem for $\alpha = \frac{1}{2}$ with $m = 4, n = 4$, we get

$$U = \begin{bmatrix} 6.6667e-01 & 3.8095e-01 & 2.1429e-01 & 6.3492e-02 & 7.9365e-03 \\ 3.8095e-01 & 1.3292e-13 & -9.7353e-14 & 3.7575e-14 & 5.2979e-15 \\ 2.1429e-01 & 1.0596e-14 & -7.8481e-15 & -2.3027e-14 & -3.2506e-15 \\ 6.3492e-02 & -3.3587e-14 & 2.6891e-14 & -1.3973e-14 & -1.8461e-15 \\ 7.9365e-03 & -1.0251e-14 & 8.1934e-15 & 1.3784e-15 & 5.2590e-17 \end{bmatrix}.$$

Therefore, the approximation solution is obtained as

$$u_{4,4}(x, t) = \phi^T(x; \frac{1}{2}) U \phi(t; \frac{1}{2})$$

$$= \begin{bmatrix} 1 \\ 3x^{\frac{1}{2}} - 2 \\ 10x - 12x^{\frac{1}{2}} + 3 \\ 30x^{\frac{1}{2}} + 35x^{\frac{3}{2}} - 60x - 4 \\ 210x - 280x^{\frac{3}{2}} - 60x^{\frac{1}{2}} + 126x^2 + 5 \end{bmatrix}^T \times U \times \begin{bmatrix} 1 \\ 3t^{\frac{1}{2}} - 2 \\ 10t - 12t^{\frac{1}{2}} + 3 \\ 30t^{\frac{1}{2}} + 35t^{\frac{3}{2}} - 60t - 4 \\ 210t - 280t^{\frac{3}{2}} - 60t^{\frac{1}{2}} + 126t^2 + 5 \end{bmatrix}$$

$$= 0.000045x - 0.00006x^{\frac{1}{2}} - (3t^{\frac{1}{2}} - 2)((1.2094e - 13)x^{\frac{1}{2}} - (1.6947e - 12)x^{\frac{3}{2}} \\ + (3.1530e - 14)x + (1.2916e - 12)x^2 - 3.8095e - 01) + x^2 + ((8.9304e - 14)x \\ - (7.9339e - 14)x^{\frac{3}{2}} - (3.6375e - 15)x^{\frac{1}{2}} + (6.6263e - 15)x^2 + 7.9365e - 03)(210t \\ - 280t^{\frac{3}{2}} - 60t^{\frac{1}{2}} + 126t^2 + 5) + (30t^{\frac{1}{2}} + 35t^{\frac{3}{2}} - 60t - 4)((8.9757e - 13)x \\ - (8.7501e - 13)x^{\frac{3}{2}} - (1.1285e - 13)x^{\frac{1}{2}} + (1.7368e - 13)x^2 + 6.3492e - 02) \\ + (10t - 12t^{\frac{1}{2}} + 3)((1.1724e - 13)x^{\frac{1}{2}} - (1.3530e - 12)x^{\frac{3}{2}} \\ + (2.8673e - 14)x + (1.0324e - 12)x^2 + 2.1429e - 01) + 0.33335.$$

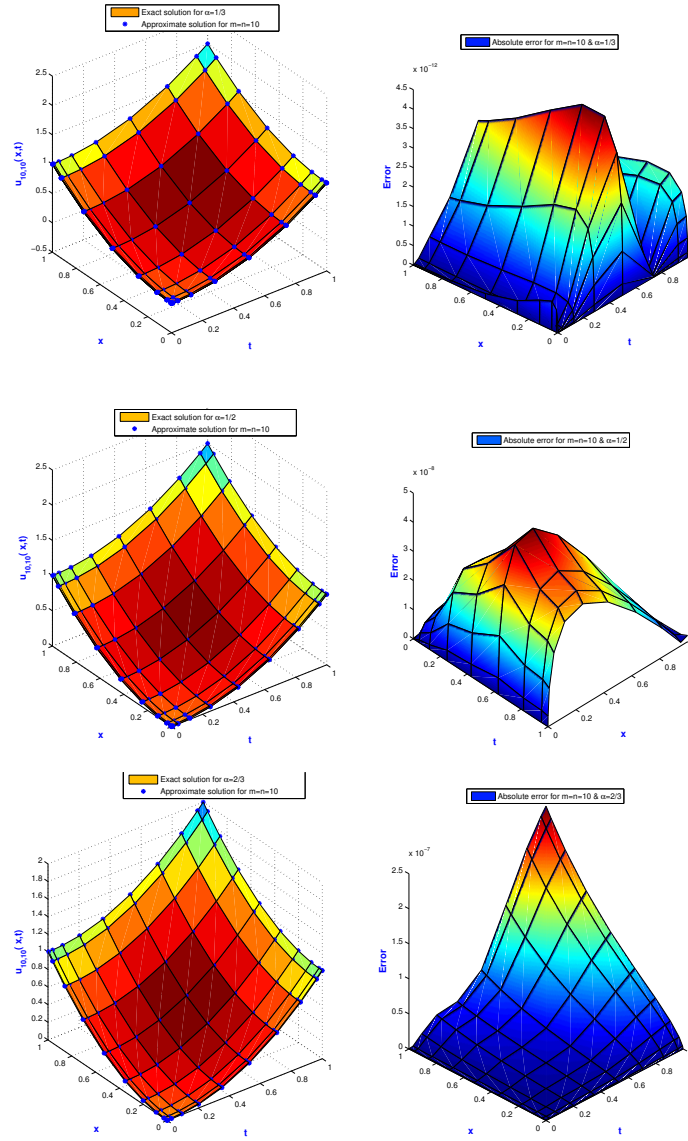


Figure 5: Comparison between the exact solution and the approximate solution with absolute error function $E_{m,n}$ for $m = n = 10$ and $\alpha = \frac{1}{3}, \alpha = \frac{1}{2}, \alpha = \frac{2}{3}$ for Example 3

Figure 4 shows the comparison between the exact solution and the approximate solution with absolute error function $E_{m,n}(x, t) = |u(x, t) - u_{m,n}(x, t)|$ for $m = 4, n = 4$, and $\alpha = 1/2$. Moreover, Figure 5 shows the comparison between the exact solution and the approximate solution with absolute error function $E_{m,n}$ for $m = 10, n = 10$ and $\alpha = 1/3, \alpha = 1/2, \alpha = 2/3$. Also, It can be seen that there is a high level of accuracy.

Example 4. Consider the nonlinear FPDE

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} + u^2(x, t) + u(x, t) &= g(x, t), \quad 0 < \alpha \leq 1, \\ u(0, t) &= E_\alpha(-t^\alpha), \quad u(x, 0) = e^x, \quad (x, t) \in (0, 1] \times (0, 1], \end{aligned}$$

where

$$g(x, t) = e^{2x}(E_\alpha(-t^\alpha))^2.$$

The exact solution is $u(x, t) = e^x E_\alpha(-t^\alpha)$, where $E_\alpha(-t^\alpha)$ is the Mittag-Leffler function defined in [12].

The numerical results are shown in Figures 6 and 7. Figure 6 shows the comparison between the exact solution and the approximate solution with absolute error function $E_{m,n}$ for various values of m, n , and α . Also, Figure 7 shows the logarithmic graphs of $\varepsilon_{m,n}$ ($\log_{10} \varepsilon_{m,n}$) for various values of α . It is seen that the absolute error converges to zero by increasing m and n .

Table 4: Maximum absolute error for various values of m, n and α for Example 5

| $\varepsilon_{m,n}$ | $\alpha = 0.2$ | $\alpha = 0.7$ | $\alpha = 0.8$ |
|-----------------------|----------------|----------------|----------------|
| $\varepsilon_{3,3}$ | $1.3809e + 00$ | $2.5202e - 02$ | $1.5926e - 02$ |
| $\varepsilon_{4,4}$ | $6.7523e - 01$ | $2.2084e - 03$ | $1.1443e - 03$ |
| $\varepsilon_{5,5}$ | $1.5944e - 01$ | $1.8012e - 04$ | $7.4059e - 05$ |
| $\varepsilon_{6,6}$ | $5.5797e - 02$ | $1.3632e - 05$ | $4.3405e - 06$ |
| $\varepsilon_{7,7}$ | $1.6813e - 02$ | $9.6897e - 07$ | $2.3238e - 07$ |
| $\varepsilon_{8,8}$ | $4.9588e - 03$ | $6.9251e - 08$ | $1.1452e - 08$ |
| $\varepsilon_{9,9}$ | $1.5654e - 03$ | $4.4995e - 09$ | $5.2297e - 10$ |
| $\varepsilon_{10,10}$ | $4.5928e - 04$ | $2.7060e - 10$ | $2.2264e - 11$ |
| $\varepsilon_{11,11}$ | $1.3696e - 04$ | $1.5228e - 11$ | $8.9617e - 13$ |
| $\varepsilon_{12,12}$ | $4.0219e - 05$ | $8.0180e - 13$ | $1.5720e - 13$ |
| $\varepsilon_{13,13}$ | $1.1339e - 05$ | $1.5720e - 13$ | $1.7497e - 13$ |
| $\varepsilon_{14,14}$ | $3.2635e - 06$ | $1.1945e - 13$ | $4.4941e - 13$ |
| $\varepsilon_{15,15}$ | $9.0875e - 07$ | $2.3048e - 12$ | $8.7752e - 13$ |

Example 5. Consider the nonlinear time-fractional gas dynamics equation [17]

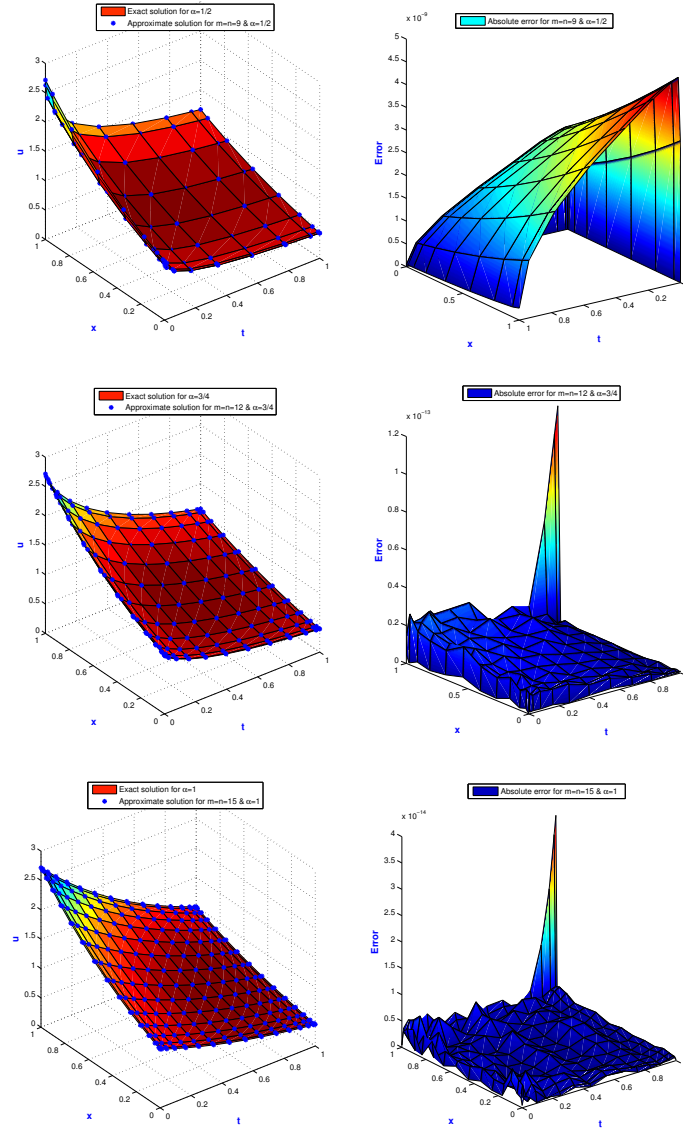
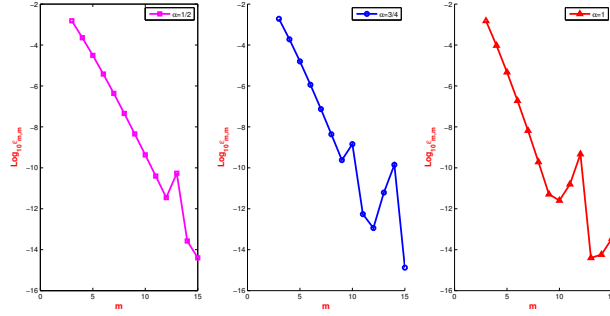


Figure 6: Comparison between the exact solution and the approximate solution with absolute error function $E_{m,n}$ for various values of m, n and α for Example 4

Figure 7: The logarithmic graphs of $\epsilon_{m,m}$ for Example 4

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t) \frac{\partial u(x,t)}{\partial x} - u(1-u) = 0, \quad 0 < \alpha \leq 1,$$

$$u(0,t) = E_\alpha(t^\alpha), \quad u(x,0) = e^{-x}, \quad (x,t) \in (0,1] \times (0,1].$$

The exact solution is $u(x,t) = e^{-x} E_\alpha(t^\alpha)$.

Figure 8 shows the approximate solutions obtained by the proposed method for $t = 1$ and various values of α . Similar results were obtained by Tamsir and Srivastava [17], by employing the fractional reduced differential transform method. For $\alpha = 1$, the exact solution is $u(x,t) = e^{t-x}$; see [7]. Figure 9 shows the comparison between the exact solution and the approximate solution with the absolute error function for $\alpha = 1$. We obtain a high level of accuracy, similar results in [17, 7], and we see that these results confirm the theoretical results. Also Figures 10, 11, and 12 show the comparison between the exact solution and the approximate solution with the absolute error function for various values of m, n , and $\alpha = 0.2, 0.7, 0.8$. In addition, Table 4 shows the maximum absolute error $\epsilon_{m,n}$ for various values of m, n , and $\alpha = 0.2, 0.7, 0.8$. Figure 13 shows the logarithmic graphs of $\epsilon_{m,n}$ ($\text{Log}_{10}\epsilon_{m,n}$). It is seen that the absolute error converges to zero by increasing m and n .

Example 6. Consider the inhomogeneous fractional convection-diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} + u^2(x,t) + u(x,t) = g(x,t), \quad 0 < \alpha \leq 1,$$

$$u(0,t) = t^\alpha, \quad u(x,0) = x^2, \quad u(1,t) = 1 + t^\alpha, \quad (x,t) \in (0,1) \times (0,1],$$

where

$$g(x,t) = \Gamma(1+\alpha) + (x^2 + t^\alpha)^2 + t^\alpha + x^2 + 2x - 2.$$

The exact solution is $u(x,t) = x^2 + t^\alpha$.
The numerical results are shown in Figure 14.

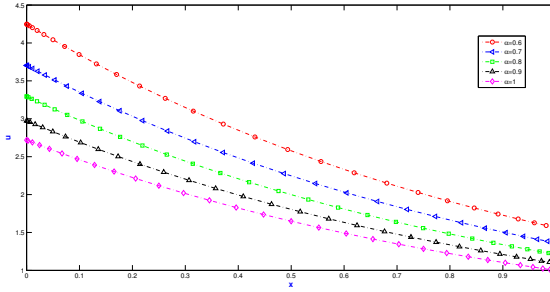


Figure 8: The approximate solutions obtained by proposed method for $t = 1$ and various values of α for Example 5

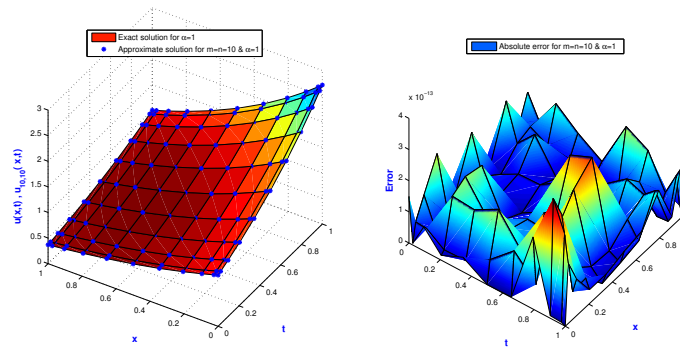


Figure 9: Comparison between the exact solution and the approximate solution with absolute error function $E_{10,10}$ for $\alpha = 1$ for Example 5

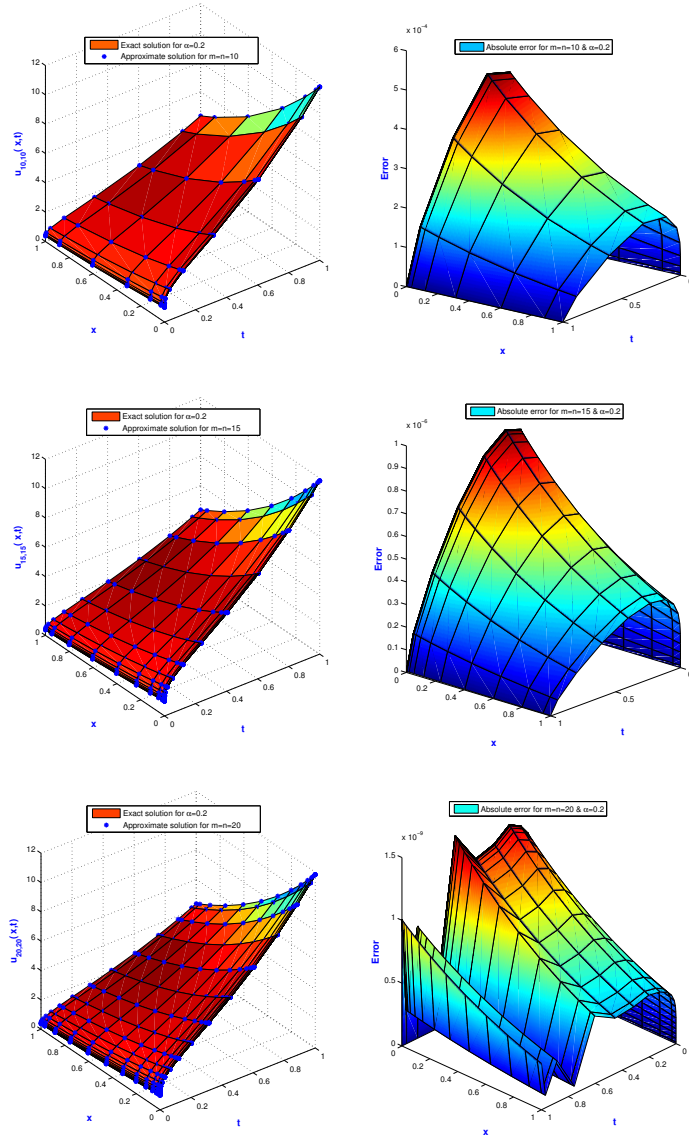


Figure 10: Comparison between the exact solution and the approximate solution with absolute error function $E_{m, n}$ for $m = n = 10, 15, 20$, and $\alpha = 0.2$ for Example 5

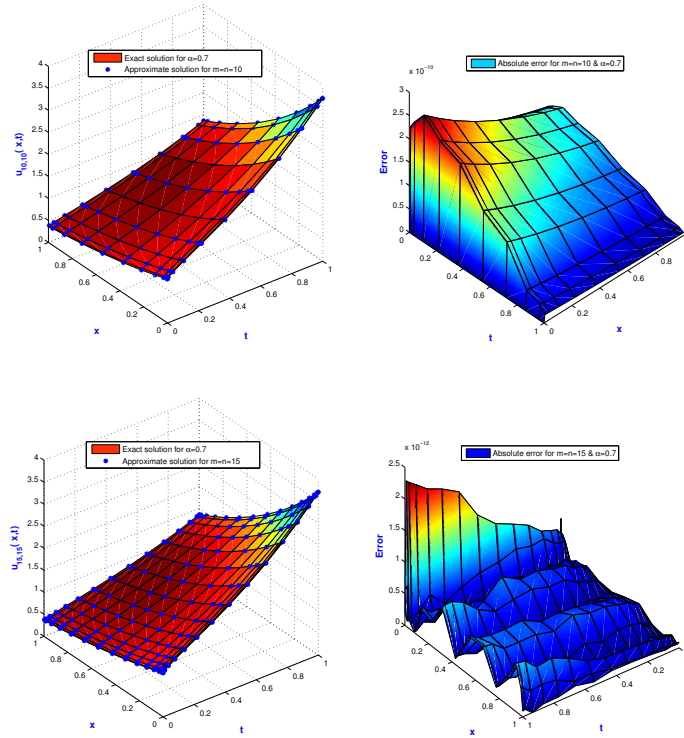


Figure 11: Comparison between the exact solution and the approximate solution with absolute error function $E_{m,n}$ for $m = n = 10, 15$, and $\alpha = 0.7$ for Example 5

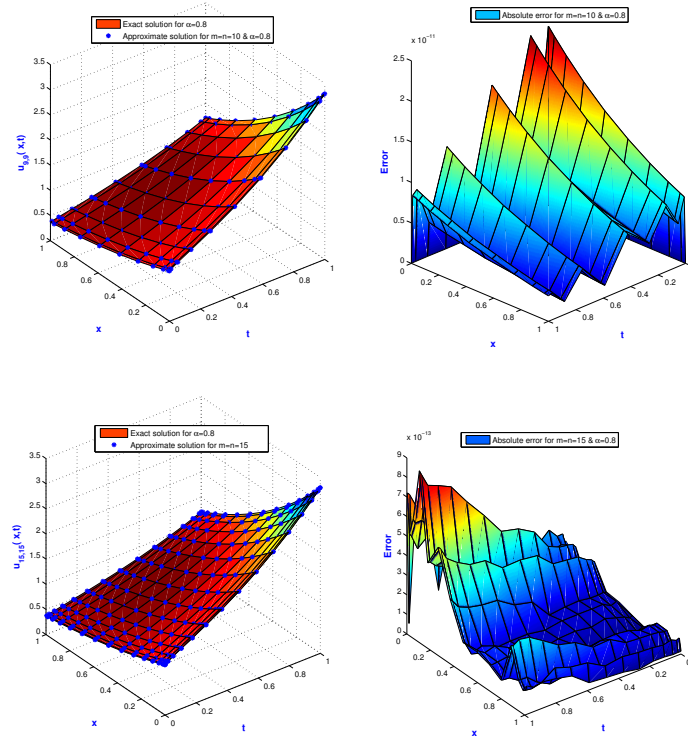


Figure 12: Comparison between the exact solution and the approximate solution with absolute error function $E_{m,n}$ for $m = n = 10, 15$, and $\alpha = 0.8$ for Example 5

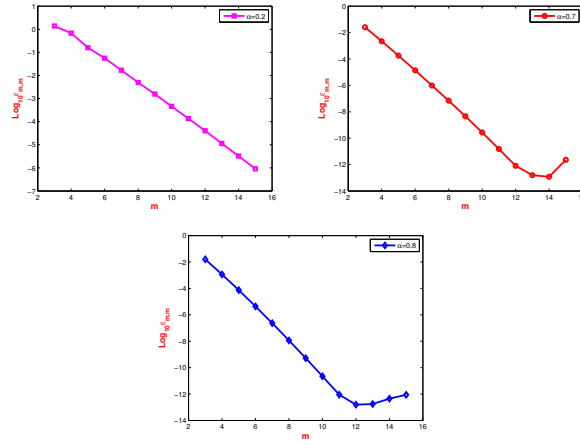


Figure 13: The logarithmic graphs of $\epsilon_{m,n}$ for $\alpha = 0.2, 0.7$ and $\alpha = 0.8$ for Example 5

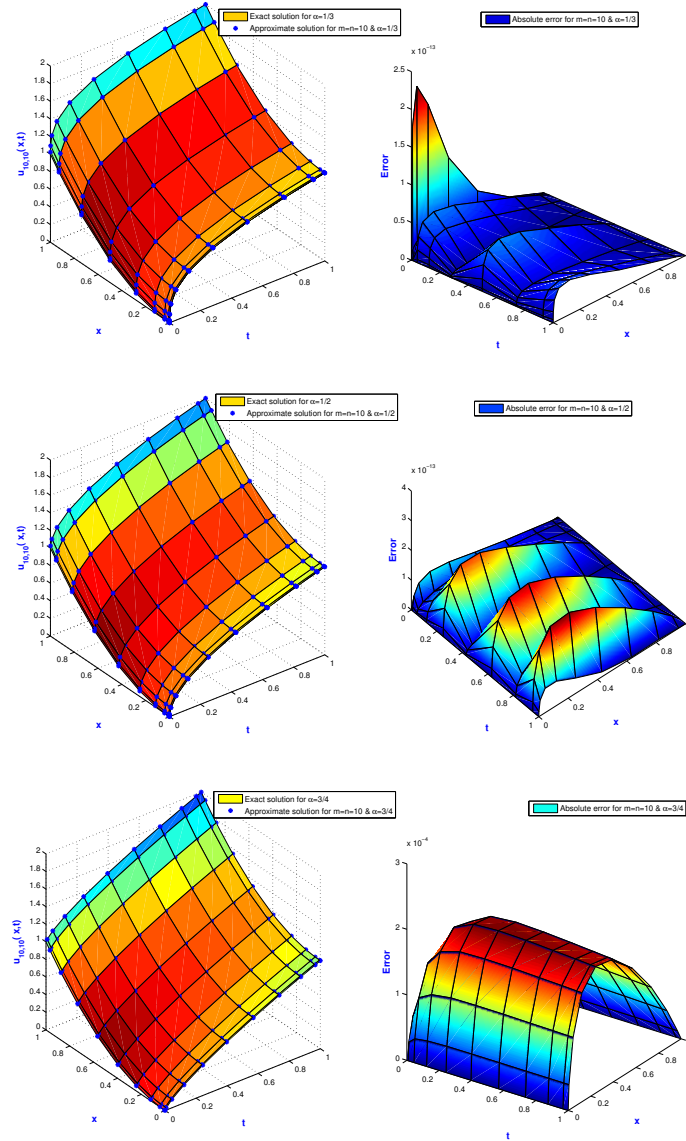


Figure 14: Comparison between the exact solution and the approximate solution with absolute error function $E_{10,10}$ at $\alpha = \frac{1}{3}, \frac{1}{2}$ and $\frac{3}{4}$ for Example 6

7 Conclusion

In this paper, we applied a basis of 2D-FMLPs to obtain the numerical solution of linear and nonlinear FPDEs with variable coefficients. To get the unknown coefficients 2D-FMLPs, we used the operational matrix of fractional derivatives of FMLPs together with the Tau method. The results of the numerical examples and the comparison with other methods showed the efficiency and accuracy of the proposed method.

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