A noniterative domain decomposition method for the forward-backward heat equation

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Abstract

A nonoverlapping domain decomposition technique applied to a finite difference method is presented for the numerical solution of the forward-backward heat equation in the case of one-dimension. While the previous attempts in dealing with this problem have been based on an iterative domain decomposition scheme, the current work avoids iterations. Also a physical matching condition is suggested to avoid difficulties caused by the interface boundary nodes. Furthermore, we obtain a square system of equations. In addition, the convergence and stability of the proposed method are investigated. Some numerical experiments are given to show the effectiveness of the proposed method.

1 Introduction

The analysis and numerical solution of the forward-backward heat equation (FBHE) has been under consideration for more than three decades; see [1, 8, 12, 20]. This problem appears in many applications such as boundary layer problems in fluid dynamics and steady state computation, plasma

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physics, and studies of the propagation of an electron beam through the solar corona (see [7] and the references therein). It also arises in computational fluid dynamics and randomly accelerated particle problem, and some more applications are mentioned in [1]. The forward-backward problem has been solved by various methods such as finite difference method (FDM) [7, 6, 13, 23], transformation to a system of first order differential equations [17], least square approach [3], Galerkin finite element [2, 10, 9, 16], and radial basis function meshless method [4].

The application of the domain decomposition method (DDM) together with FDM for this problem in the case of one-dimension is widely seen in the literature (see, for example, [7, 22]). This method was also applied to the two-dimensional FBHE, and convergence of the iterative method was considered in [23].

In addition, the theoretical aspects of the underlying problem have been studied by some authors [5, 14, 19]. Daoud [7] analyzed the convergence of the overlapping Schwarz waveform relaxation method for solving the FBHE.

Kuznetsov [15] proved the uniqueness of the entropy solution for a nonlinear forward-backward parabolic equation with Dirichlet boundary condition and showed that initial and final conditions must be formulated in the form of inequalities. Paronetto [20] considered the existence and uniqueness of the solution for elliptic problems with a small parameter and proved that the solution converges to the solution of a mixed type equation, elliptic-parabolic, parabolic both forward and backward. In the above-mentioned methods, the domain is usually divided into two subdomains, each of which is associated with either a standard forward or backward problem. A numerical method is then applied to each subproblem. A primary approximate solution is assumed on the interior boundary followed by solving each subproblem and updating the interface boundary solution iteratively. While the previous DDM-FDM has been applied based on an iterative scheme, the current work is based on applying a nonoverlapping DDM and employing an FDM without using iterations. Some advantages of the proposed method read as follows. Firstly, an initial approximate solution on the interface boundary, which may affect the convergence, is not required. Secondly, the iteration, which may considerably increase the computational costs, is avoided. In the suggested method, the domain is divided into two nonoverlapping subdomains, and a forward or a backward finite difference formulation is applied to each subproblem according to the sort of the initial conditions. The local algebraic equations produced by the subproblems are then assembled in a global system via employing a physical matching condition on the interface and obtaining a square system of equations by removing the virtual boundary nodes. This paper is organized as follows: The forward-backward equation is introduced in Section 2. The DDM will be applied to the underlying problem in Section 3. The convergence and stability of the proposed method will be discussed in Section 4. Finally some numerical results will be presented in Section 5.

2 The forward-backward equation

A boundary value problem of a forward-backward parabolic equation in one-dimension is introduced as

$$
\begin{aligned}
xu_t - u_{xx} = f(x,t), & \quad (x,t) \in \Omega = (-1,1) \times (0,1), \\
u(-1,t) = g_{-1}(t), & \quad \text{for all } t \in [0,1], \\
u(1,t) = g_1(t), & \quad \text{for all } t \in [0,1], \\
u(x,0) = u_0(x), & \quad \text{for all } x \in [0,1], \\
u(x,1) = u_1(x), & \quad \text{for all } x \in [-1,0],
\end{aligned} \tag{1}
$$

where $u_0(x)$, $u_1(x)$, $g_{-1}(t)$, and $g_1(t)$ are known functions with $u_0(1) = g_1(0)$ and $u_1(-1) = g_{-1}(1)$.

The existence, uniqueness, and stability of the above forward-backward problem have been considered by some authors [1, 2, 3, 18]. Authors of [1] used the energy method to prove the existence and uniqueness of a weak solution to FBHE on some special spaces equipped with some suitable norms. Lu and Wen [18] studied the existence and uniqueness of a weak solution to (1) on a certain Hilbert space. Aziz and Liu [3] reduced (1) to a first-order symmetric-positive system of differential equations and proved existence and uniqueness to the solution of the equivalent problem using the theory of symmetric-positive systems previously presented by Friedrichs [11] for a strong solution, under certain smoothness assumptions.

In order to apply the FDM, we produce a set of grids as follows: Let $h = \dfrac{1}{M}$, let $x_i = ih$ for $i = 0, \pm 1, \ldots, \pm(M-1)$, let $\tau = \dfrac{1}{N}$, and let $t_j = j\tau$ for $j = 0, \ldots, N$.

3 Domain decomposition method

In this section, the DDM is applied to the forward backward problem in two steps. First, the domain is divided into two subdomains and the algebraic equations are formed for each subregion. Then, the local formulations are assembled to obtain the global system of equations.
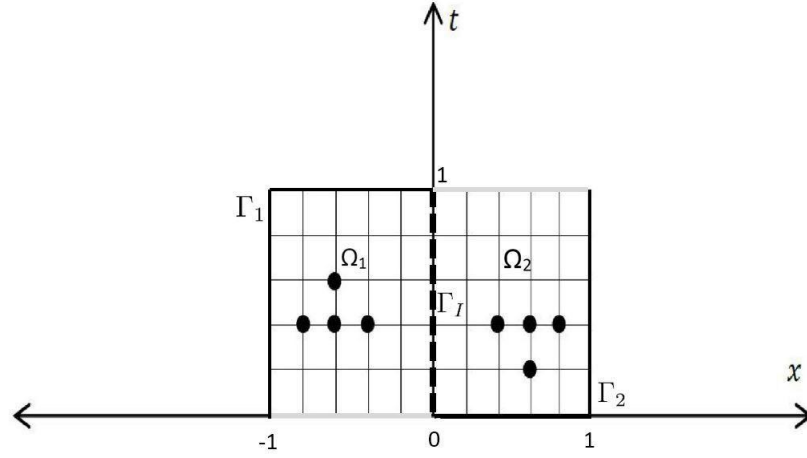
Figure 1: The domain partitioning and the pattern of the finite difference schemes.

3.1 Domain partitioning

Let the domain Ω be divided into two nonoverlapping subregions, $\Omega_1 = (-1, 0) \times (0, 1)$ and $\Omega_2 = (0, 1) \times (0, 1)$ with the real boundaries Γ_1 and Γ_2 and an interface boundary Γ_I in between (see Figure 1). In order to deal with this problem, we apply a forward and a backward difference scheme to the domains Ω_1 and Ω_2, respectively. Employing the first order forward formulation for time derivative and the second order central difference scheme for the spatial derivative at the mesh points in Ω_1, we obtain

$$ih\frac{u_{i,j+1} - u_{i,j}}{\tau} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + f_{i,j},$$
$$0 \le j \le N - 1, \quad -M + 1 \le i \le -1,$$
$$u_{i,N} = u_1(ih), \quad -M + 1 \le i \le 0,$$
$$u_{-M,j} = g_{-1}(j\tau), \quad 0 \le j \le N. \tag{2}$$

where $u_{i,j}$ denotes the approximation of $u(ih, j\tau)$ and $f_{i,j} = f(ih, j\tau)$.

Using the first order backward formulation and the second order central difference scheme for the mesh points in Ω_2, the following equations are achieved:

$$ih\frac{u_{i,j+1} - u_{i,j}}{\tau} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + f_{i,j+1},$$
$$0 \le j \le N - 1, \quad 1 \le i \le M - 1,$$
$$u_{i,0} = u_0(ih), \quad 0 \le i \le M - 1,$$
$$u_{M,j} = g_1(j\tau), \quad 0 \le j \le N, \tag{3}$$

The pattern of the finite difference schemes used in (2) and (3) is shown in Figure 1.

Note that the above equations produce two separate underdetermined systems of equations, as the boundary conditions on the interface Γ_I are not prescribed. In the next subsection, we propose a method in which the local equations are assembled in a global system with a square coefficient matrix.

3.2 Global formulation

We now globalize the problem by assembling the local equations constructed by the subproblems. As mentioned above, the systems obtained by equations (2) and (3) are underdetermined as the number of unknowns is equal to $2[N(M-2) + N - 1]$, whereas there are only $2N(M-2)$ equations, taking into account the boundary conditions imposed on the interface boundary have different values for two subproblems.

Let $u_{0,j}^{(k)}$, $k = 1, 2$, be the solutions of the subproblems associated with Ω_k at $(0, j\tau)$, $j = 1, \ldots, N$. In order to achieve a system with a square coefficient matrix, we use a sort of physical matching condition on the interface, that is, $u_{0,j} = u_{0,j}^{(1)} = u_{0,j}^{(2)}$, which guarantees a continuous solution on the virtual boundary. We now make a square system by eliminating the nodal values located on the interface. To do so, we utilize the central difference scheme for spatial derivative at the mesh points on Γ_I in order to make interaction between two subproblems, that is,

$$-\left(\frac{u_{1,j} - 2u_{0,j} + u_{-1,j}}{h^2}\right) = f_{0,j}, \quad 1 \le j \le N - 1. \tag{4}$$

Note that the first term of the governing equation in (1) vanishes since on Γ_I, $x = 0$ holds. By combining this equation with (2) and (3) in the case of $i = 1$ and $i = -1$, we can eliminate the nodal values on the line $x = 0$ from the eventual system of equations.

To this end, we consider the first equation of (3) in the case of $i = 1$, that is,

$$h\frac{u_{1,j+1} - u_{1,j}}{\tau} = \frac{u_{2,j+1} - 2u_{1,j+1} + u_{0,j+1}^{(2)}}{h^2} + f_{1,j+1}, \quad 0 \le j \le N - 1, \tag{5}$$

where the superscript indicates the subdomain considered.

Replacing j by $j + 1$ in (4) and solving it for $u_{0,j+1} = u_{0,j+1}^{(1)}$, we obtain

$$u_{0,j+1}^{(1)} = \frac{h^2}{2}f_{0,j+1} + \frac{1}{2}u_{1,j+1} + \frac{1}{2}u_{-1,j+1}, \quad 0 \le j \le N - 1. \tag{6}$$

Substituting (6) into (5), we find that

$$h\frac{u_{1,j+1} - u_{1,j}}{\tau} = \frac{u_{2,j+1} - \frac{3}{2}u_{1,j+1} + \frac{1}{2}u_{-1,j+1}}{h^2} + \frac{3}{2}f_{0,j+1}, \qquad (7)$$
$$j = 0, \ldots, N-1.$$

Repeating a similar process for the line $i = -1$ in the first equation of (2), we achieve a similar equation without presence of the term $u_{0,j+1}$ as follows:

$$-h\frac{u_{-1,j+1} - u_{-1,j}}{\tau} = \frac{u_{-2,j} - \frac{3}{2}u_{-1,j} + \frac{1}{2}u_{1,j}}{h^2} + \frac{3}{2}f_{0,j}, \quad j = 1, \ldots, N. \qquad (8)$$

Assembling equations (2) and (3) with the revised equations (7) and (8), a linear system of size $2N(M-2)$ is obtained as $PU = F$, where $P = \begin{bmatrix} P_{LL} & P_{LR} \\ P_{RL} & P_{RR} \end{bmatrix}$, $U = \begin{bmatrix} U_L \\ U_R \end{bmatrix}$, $F = \begin{bmatrix} F_L \\ F_R \end{bmatrix}$,

$$P_{LL} = \begin{bmatrix} A_L & & & \\ B_L & \ddots & & \\ & \ddots & A_L & \\ & & B_L & A_{L'} \end{bmatrix}, \qquad P_{RR} = \begin{bmatrix} A_R & & & \\ B_R & \ddots & & \\ & \ddots & A_R & \\ & & B_R & A_{R'} \end{bmatrix},$$

$$P_{LR} = \begin{bmatrix} & & C_L & 0 \\ & \ddots & \ddots & \\ C_L & 0 & & \\ 0 & & & \end{bmatrix}, \qquad P_{RL} = \begin{bmatrix} & & C_R & 0 \\ & \ddots & \ddots & \\ C_R & 0 & & \\ 0 & & & \end{bmatrix},$$

$$A_L = \begin{bmatrix} 2r - (-M+1)h & -r & & & \\ -r & 2r - (-M+2)h & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & 2r - (-2)h & -r \\ & & & -r & \frac{3}{2}r - (-h) \end{bmatrix},$$

$$A_{L'} = \begin{bmatrix} 2r - (-M+1)h & -r & & \\ -r & 2r - (-M+2)h & -r & \\ & \ddots & \ddots & \ddots \\ & & -r & 2r - (-h) \end{bmatrix},$$

$$C_L = \begin{bmatrix} & & 0 \\ & \ddots & \\ -\frac{r}{2} & & \end{bmatrix} \quad ; \quad B_L = \begin{bmatrix} (-M+1)h & & \\ & \ddots & \\ & & -h \end{bmatrix},$$

$$A_R = \begin{bmatrix} \frac{3}{2}r + h & -r & & & & \\ -r & 2r + 2h & -r & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -r & 2r + (M-2)h & -r \\ & & & & -r & 2r + (M-1)h \end{bmatrix},$$

$$A_{R'} = \begin{bmatrix} 2r + h & -r & & & & \\ -r & 2r + 2h & -r & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -r & 2r + (M-2)h & -r \\ & & & & -r & 2r + (M-1)h \end{bmatrix},$$

$$C_R = \begin{bmatrix} & & -\frac{r}{2} \\ & \ddots & \\ 0 & & \end{bmatrix} \quad ; \quad B_R = \begin{bmatrix} -h & & \\ & \ddots & \\ & & (-M+1)h, \end{bmatrix},$$

where $r = \dfrac{\tau}{h^2}$.

The indices L and R denote the portions of the matrices associated with the left and right subregions, respectively. For instance, P_{LR} represents the portions associated with coupling the nodal values of subproblem 1 (left side) to the nodal values of subproblem 2 (right side).

Having solved the above linear system and obtained the interior solution of the subproblems, the interface boundary solution can be found by using (6).

4 Convergence

In order to show the convergence of the proposed method, we use Lax's equivalence theorem [21]. To do so, we need to seek the stability condition and consistency of this method. We first check the stability.

Theorem 1. The finite difference equations (2)–(4) are unconditionally stable.

Proof. Substituting $u_{p,q} = e^{\beta ph}\xi^q$ into the difference equation (2), we find that

$$ph(\xi - 1) = r\xi(2\cos(\beta h) - 2) = r\xi(-4r\sin^2(\frac{\beta h}{2})).$$

Since $ph < 1$, we can write

$$\xi(4r\sin^2(\frac{\beta h}{2}) + 1) > 1.$$

Using the stability condition, $|\xi| \leq 1$, we have

$$\frac{1}{4r \sin^2(\frac{\beta h}{2}) + 1} < |\xi| \leq 1.$$

Clearly $0 \leq \xi \leq 1$ for all $r > 0$ and all β. Therefore, the equations are unconditionally stable. For equation (3), the same result can be obtained. $\quad\square$

The consistency of the method is considered in the following theorem.

Theorem 2. The finite difference equations (2)–(4) are consistent and they have the local truncation error of $O(hk) + O(h^2)$.

Proof. Let $F_{ij}(u) = 0$ represent the difference equation approximating the partial differential equation (1) at the (i, j)th mesh point, with exact solution u, and let U be the exact solution of equation (1). Then we have the local truncation error T_{ij} at the mesh point $(ih, j\tau)$ as follows:

$$\begin{aligned}
T_{ij} &= F_{ij}(U) \\
&= ih\frac{U_{i,j+1} - U_{i,j}}{k} - \frac{U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1}}{h^2} - f(i, j+1).
\end{aligned}$$

By Taylor's expansion, we have

$$\begin{aligned}
T_{ij} &= ih(\frac{\partial U}{\partial t})_{i,j+1} - (\frac{\partial^2 U}{\partial x^2})_{i,j+1} - f(i, j+1) \\
&\quad + i\frac{hk}{2}(\frac{\partial^2 U}{\partial t^2} + \frac{k}{3}\frac{\partial^3 U}{\partial t^3} + \cdots) - \frac{h^2}{12}(\frac{\partial^4 U}{\partial x^4} - \frac{h^2}{20}\frac{\partial^6 U}{\partial x^6} + \cdots).
\end{aligned}$$

Hence

$$T_{ij} = O(hk) + O(h^2).$$

$\square$

4.1 Higher order finite difference schemes

The accuracy of the solution can be improved by employing higher order finite difference approximations. To do so, we first apply the previous finite difference formulation to the nodes located on the lines $x = ih$ for $i = M - 1$ and $i = -M + 1$ as follows:

$$ih\frac{u_{i,j+1} - u_{i,j}}{\tau} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + f_{i,j}.$$

Now there is sufficient number of nodal values to use the following scheme for the other nodes:

$$-ih\frac{u_{i,j+2} - 4u_{i,j+1} + 3u_{i,j}}{2\tau}$$
$$= \frac{-u_{i+2,j} + 16u_{i-1,j} - 30u_{i,j} + 16u_{i,j-1} - u_{i,j-2}}{12h^2} + f_{i,j}. \quad (9)$$

Applying this formula improves the accuracy of the solution, taking into account that the order of the scheme (9) is $O(hk^2) + O(h^4)$. As will be observed, the numerical errors are remarkably declined in the case of using this formula.

5 Numerical results

We now present some numerical results in order to show the effectiveness of the proposed method.

Consider equation (1) with $u_0(x) = 0$, $u_1(x) = 0$, $g_{-1}(t) = 0$, $g_1(t) = 0$, and

$$f(x,t) = \begin{cases} 2x(x^2 - 1)t[(t-1)^2 - 4x^2 + t(t-1)] - 2t^2[(t-1)^2 - 24x^2 + 4], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad x \geq 0, \quad t \in [0,1], \\ 2x(x^2 - 1)(t-1)(2t^2 - t - 4x^2) - 2(t-1)^2(t^2 - 24x^2 + 4), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad x \leq 0, \quad t \in [0,1], \end{cases}$$

The exact solution of the above problem is given by

$$u(x,t) = \begin{cases} (x^2 - 1)t^2[(t-1)^2 - 4x^2], & x \geq 0, \, t \in [0,1], \\ (x^2 - 1)(t^2 - t - 4x^2)(t-1)^2, & x \leq 0, \, t \in [0,1]. \end{cases}$$

To measure the accuracy of the results, the maximum error (Max error) and the root-mean-square error (RMSE) are employed, that is,

$$Max\ error = \max_{j=1}^{N}|\hat{u}_j - u_j|, \quad RMSE = \sqrt{\frac{1}{N}\sum_{j=1}^{N}(\hat{u}_j - u_j)^2},$$

where N is the number of nodes and u_j and $\hat{u}_j$ denote the exact and approximate solutions at the jth node. The numerical solution of the above equation was obtained in two cases: Scheme (I), the FDM proposed in (2) and (3), and scheme (II), the FDM suggested in (9). The numerical errors are listed in Tables 1 and 2, respectively, for the cases (I) and (II). The agreement of the approximate and the exact solutions together with the configuration of the error functions is shown in Figures 2 and 3, respectively, for the cases (I) and (II). The numerical errors demonstrate reasonable accuracy in both cases, especially in the case of using the scheme (II).

Table 1: Numerical errors produced by the new method using scheme (I).

M	N	Max error	RMSE	M	N	Max error	RMSE
10	10	3.37E-2	8.40E-3	30	15	9.81E-3	4.80E-3
20	10	1.43E-2	6.89E-3	40	20	7.40E-3	3.70E-3
30	10	1.51E-2	7.68E-3	50	25	6.01E-3	2.95E-3
40	10	1.54E-2	8.02E-3	60	30	4.91E-3	2.50E-3
50	10	1.56E-2	8.20E-3	70	35	4.27E-3	2.22E-3
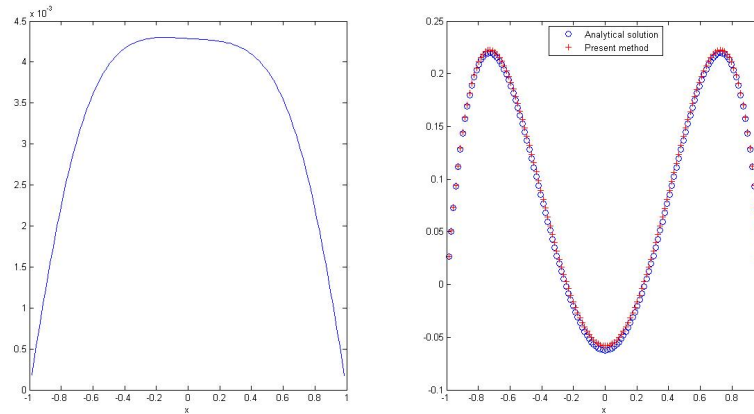
Figure 2: Comparing the exact solution and the numerical solution by scheme (I) with $M = 70$ and $N = 35$ and configuration of the error function.

In order to show the efficiency of the new method, we compare the numerical results with those presented in [13]. The errors obtained by scheme (II) and those presented in [13] are listed in Table 3 using the same mesh size and time steps. It is observed that although in some cases our method produces more accurate solutions, overall, the difference between the errors is not significant. However, the new method has some advantages over the previous ones, that is, iterative methods. In the previous works, an initial solution should be carefully selected, otherwise the convergence of the iterative method is not guaranteed. In addition, as seen in Table 3, in the previous work, the solution is obtained at the expense of performing some iterations, which considerably increase the computational costs. Moreover, a parameter is needed in the iterative method (see [13]). Choosing this parameter is also important in order to achieve convergence.

It should be noted, in the suggested method, a global system of equations is solved rather than the local ones. Although, apparently, this may increase the computational costs, the resulting system has a sparse coefficient matrix,

Table 2: Numerical errors produced by the new method using scheme (II).

M	N	Max error	RMSE	M	N	Max error	RMSE
10	10	3.10E-2	8.53E-3	30	15	3.85E-3	9.12E-4
20	10	7.48E-3	2.01E-3	40	20	2.36E-3	5.18E-4
30	10	3.50E-3	1.18E-3	50	25	1.62E-3	3.33E-4
40	10	2.86E-3	1.10E-3	60	30	1.18E-3	2.32E-4
50	10	2.61E-3	1.00E-3	70	35	8.75E-4	1.71E-4

which can be treated by the efficient solvers. Moreover, the iterations required in the case of using local matrices, are avoided.
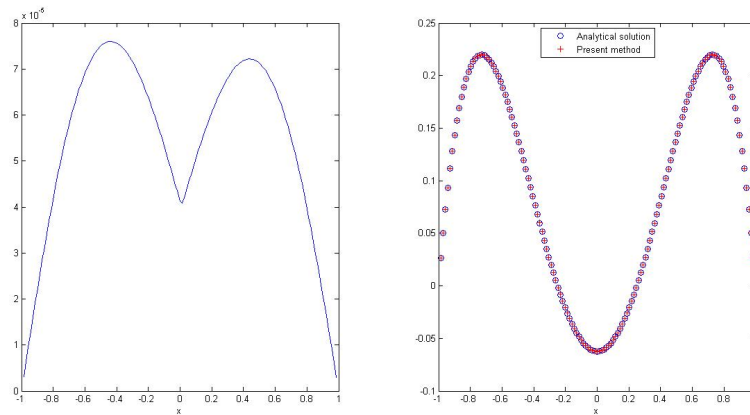

Figure 3: Comparing the exact solution and numerical solution by scheme (II) with $M = 70$, $N = 35$ and configuration of the error function.

6 Conclusions

A nonoverlapping DDM was applied to the FBHE in a one-dimensional case. The work was based on partitioning the spatial domain and considering each part as an independent forward or a backward subproblem. A forward and a backward finite difference schemes were employed for local problems followed by assembling the equations in a global one. The under determined system of equations, arising from the lack of the interface boundary conditions, was treated by a physical matching condition and a square system was obtained.

Table 3: Comparing the error values (Max error) of scheme (II) with the previous results produced by the iterative method with K denoting the number of iterations.

M	N	K	Iterative method	New method
4	4	24	1.36E-2	7.56E-2
8	16	38	3.68E-3	1.02E-2
16	64	63	9.22E-3	1.10E-3
32	256	100	2.42E-4	2.56E-4

Dealing with a single global system of equations, while avoiding iterations used in the previous attempts, led to increasing the computational efficiency and keeping a reasonable accuracy for the solution. The proposed method performed well in the one-dimensional case. The application of the present method to the case of two-dimensional is also under consideration.

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