Iranian Journal of Numerical Analysis and Optimization
Vol. 9, No. 1, (2019), pp 69–92
DOI:10.22067/ijnao.v9i1.68851

The network 1-median problem with discrete demand weights and traveling times

M. Abareshi and M. Zaferanieh*

Abstract

In this paper, the 1-median location problem on an undirected network with discrete random demand weights and traveling times is investigated. The objective function is to maximize the probability that the expected sum of weighted distances from the existing nodes to the selected median does not exceed a prespecified given threshold. An analytical algorithm is proposed to get the optimal solution in small-sized networks. Then, by using the central limit theorem, the problem is studied in large-sized networks and reduced to a nonlinear problem. The numerical examples are given to illustrate the efficiency of the proposed methods.

Keywords: Facility location; 1-median problem; probabilistic weights; probabilistic traveling times.

1 Introduction

In the classical deterministic p-median problem, the purpose is to locate p new facilities on the links or nodes of a given network G, so that the total weighted distances from all nodes that are considered as demand points to the nearest facility, is minimized. The p-median problem on general networks is NP-hard; see [8]. In a tree network, Kariv and Hakimi [12] proved that the problem can be solved in $O(p^2n^2)$ time. Tamir [18] improved the

*Corresponding author
Received 21 November 2017; revised 25 July 2018; accepted 29 August 2018
M. Abareshi
Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran. e-mail: abareshi66@gmail.com

M. Zaferanieh
Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran. e-mail: m.zaferanieh@hsu.ac.ir

time complexity on tree networks to $O(pn^2)$. Hakimi [10] demonstrated that the classical 1-median problem has nodal optimal solution, and Goldman [9] presented a linear time algorithm to find the optimal solution. For the case $p = 2$, an $O(n \log n)$ time algorithm was provided by Gavish and Sridhar in [7].

Determining precise demand weight of each node may be impossible. For example, the number of patients arising from different regions going into a hospital cannot be predicted precisely. Therefore, it is desirable to treat demand weights as random variables. Frank considered a probabilistic case, in which the demand weights are random variables with arbitrary known or estimated probability distributions, [5]. He discussed the median and center problems on a network with independent random demand weights and extended the classical 1-median problem to the maximum probability absolute median ($MPAM$) problem. The $MPAM$ problem seeks to find a new facility maximizing the probability that the total weighted distances from all nodes to the new facility is less than or equal to a given threshold. He also defined a maximum probability center to maximize the probability that the maximum weighted distance is less than a prespecified threshold value. Later, the results for the maximum probability median were extended by Frank to the case where demand weights are correlated normal random variables, [6].

Berman and Wang [2] extended the problem to some single facility location problems as maximum probability 1-median, 1-anti-median, 1-center, and 1-anti-center problems with independent discrete random weights and presented efficient algorithms to solve these problems. In [3], They also studied the p-facility location problem with maximum probability and formulated it as a linear integer programming problem, then determined the solutions in large-scaled networks by using the normal approximation.

Puga and Tancrez [16] studied a location-inventory problem for the design of large supply chain networks with uncertain demand and proposed a heuristic algorithm to solve the problem. Bieniek [4] considered the single source capacitated facility location problem with independent and identically distributed random demands. The unified of a priori solution for the locations of facilities and for the allocation of customers to the operating facilities was found.

The distances between nodes can also be considered as nondeterministic parameters. The uncertainty about the lengths of the links of a network often occurs when the lengths are measured in units of traveling times instead of geographical distances. Mirchandani and Odoni studied the p-median problem when traveling times on the links are discrete random variables, [13]. The p-median problem with continuous random lengths of links, was studied by Handler and Mirchandani, [11]. They formulated the problem for locating p medians and provided an algorithm for the 1-median problem. The uncertainty in link traveling times can also be considered in some other location problems. Berman et al [1] studied the maximum covering location problem on a network with traveling time uncertainty represented by different traveling time scenarios. The proposed models were applied to the analysis of the location of fire stations in Toronto, Canada.

In this paper, a new case is proposed in which the weights of nodes and the traveling times on links are both discrete independent random variables. The problem is to find the location of the new facility in such a way the expected sum of weighted distances is less than a prespecified value with maximum probability. The problem is studied in both small and large-sized networks. In Section 2, the problem formulation is stated and an expected 1-median with the maximum probability is defined. The solution approach for small-sized networks is stated in Section 3, and a precise algorithm to get the optimal solution is presented; then a numerical example is provided. In Section 4, by using the normal approximation, the solution is determined in large-sized networks and a numerical example is presented. Summary and conclusions are stated in the last section.

2 Problem formulation

Let $G = (N, E)$ be a network, where N is the set of nodes and E is the set of links with $|N| = n$ and $|E| = m$. The weight of each node $v_h \in N$, $h = 1, \ldots, n$ is denoted by w_h. Let the link connecting two nodes v_i and v_j and its length be denoted by e_{ij} and l_{ij}, respectively. In a deterministic network, the shortest distance between two points $y, x \in G$ is denoted by $d(y, x)$. Let x be a point on the link e_{ij} at distance x from node v_i; then the shortest distance between x and an arbitrary point y is given as follows:

$$d(y, x) = \min\{d(y, v_i) + x, d(y, v_j) + l_{ij} - x\}. \tag{1}$$

Hence the shortest distance from point y to $x \in e_{ij}$ is covered via node v_i or node v_j, whichever that is shorter. In a stochastic network, each link e_{ij} has a deterministic geographical length l_{ij}, that is a real positive fixed value. Moreover the traveling time along the link e_{ij} is a random variable dependent on the traveling speed. In this case, the distances are computed in terms of traveling times. It is clear that the traveling speed along each link may be changed under such various conditions as traffic congestion.

To introduce the traveling time of a link, we use the inverse of traveling speed. Let the traveling time and inverse of traveling speed along the link e_{ij} be denoted by T_{ij} and SP_{ij}, respectively. If SP_{ij} is a discrete random variable, T_{ij} is also a discrete random variable equal to $SP_{ij} l_{ij}$. In this paper, the random values for the inverse of traveling speed of the link e_{ij} are considered to be independent from the other links and chosen from the set $\{sp_{ij}^1, sp_{ij}^2, \ldots, sp_{ij}^{U_{ij}}\}$, where U_{ij} is the number of all states of traveling speed in the link e_{ij}. The relevant probability to sp_{ij}^u on the link e_{ij} is P_{ij}^u for $u = 1, \ldots, U_{ij}$.

Because of random values for traveling times, the network G gets into different state-based networks G_r. Let $G_{state} = \{G_r,\ r = 1, \ldots, R\}$, where $R = \prod_{e_{ij} \in E} U_{ij}$. We define

$$WD_r(x) = \sum_{h=1}^{n} w_h d_r(v_h, x), \quad r = 1, \ldots, R \tag{2}$$

as the total weighted traveling time from all nodes to the point x in the rth state $G_r \in G_{state}$, in which $d_r(v_h, x)$ is the distance between v_h and $x \in G_r$, that is determined by equation (1) noting that the traveling time of each link $e_{ij} \in E$ is computed by $sp_{ij}^r l_{ij}$. Then the expected 1-median problem is defined as below.

Definition 1. *The expected* 1*-median problem in network* G *with probabilistic distances seeks to find a point* x *which minimizes*

$$E(WD(x)) = \sum_{r=1}^{R} P_r WD_r(x), \tag{3}$$

where $WD(x)$ *is the random variable denotes the weighted traveling time from all nodes to the point* x *having the values* $WD_r(x),\ r = 1, \ldots, R$ *computed by expression* (2)*, and* P_r *is the probability of the* r*th state in set* G_{state}*. The function* $E(WD(x))$ *is known as the expected 1-median function.*

In continue, an independent discrete random weight W_h is allocated for each node $v_h \in N$. The discrete values of W_h are denoted by $w_h[k],\ k = 1, \ldots, K_h$, where K_h is the number of different situations for the weight of node $v_h \in N$ and it is assumed that $w_h[k] \leq w_h[k+1]$ for $k = 1, \ldots, K_h - 1$. The probability corresponding to $w_h[k]$ is denoted by $P'_h[k],\ k = 1, \ldots, K_h$. Let T be a preselected admissible threshold. The proposed objective function is maximizing the probability that the expected 1-median function (3) does not exceed the given threshold T. Here T is an upper level, which is given by the decision maker.

Definition 2. *Let the weights of nodes and the lengths of links be independent discrete random variables with known probabilities. A point* $x \in G$ *is an expected* 1*-median with the maximum probability if it maximizes the following probabilistic objective function:*

$$P'(\sum_{r=1}^{R} P_r WD_r(x) \leq T) \tag{4}$$

and consequently

$$P'(\sum_{r=1}^{R} P_r \sum_{h=1}^{n} W_h d_r(v_h, x) \leq T),$$

where $P'(.)$ *is the probability corresponding to the random weight vector of nodes.*

The expected 1-median with maximum probability is NP-hard on general networks; see [2].

2.1 Calculating a lower bound

In this section, we provide a lower bound approximation for the objective function (4) at each point x of G. The lower bound can be calculated as an expected function, which would be readily calculated in comparison with the probabilistic objective function (4); see Lemma 1.

Lemma 1. *If* $\mu_h = \sum_{k=1}^{K_h} w_h[k] P'_h[k]$, $h = 1, \ldots, n$ *is the expected value of* W_h*, then the probability* $P'(\sum_{r=1}^{R} P_r WD_r(x) < T)$ *is greater than or equal to* $1 - \sum_{r=1}^{R} P_r \sum_{h=1}^{n} \frac{\mu_h}{T} d_r(v_h, x)$.

Proof. Based on Markov's inequality, if X is a nonnegative random variable and $a \geq 0$, then $P(X \geq a) \leq \dfrac{E(X)}{a}$ or in other words $P(X < a) \geq 1 - \dfrac{E(X)}{a}$, where $E(X)$ is the expected value of X. Therefore, Since $W_1, \ldots, W_n$ are independent random variables, the following inequalities hold:

$$\begin{aligned} P'(\sum_{r=1}^{R} P_r \sum_{h=1}^{n} W_h d_r(v_h, x) < T) &\geq 1 - E(\sum_{r=1}^{R} P_r \sum_{h=1}^{n} \frac{W_h}{T} d_r(v_h, x)) \\ &= 1 - \sum_{r=1}^{R} P_r \sum_{h=1}^{n} \frac{E(W_h)}{T} d_r(v_h, x) \\ &= 1 - \sum_{r=1}^{R} P_r \sum_{h=1}^{n} \frac{\mu_h}{T} d_r(v_h, x), \end{aligned}$$

which completes the proof. □

Lemma 1 determines a lower bound approximation for the function (4), when a given point is available as the solution for the expected 1-median with maximum probability problem. This helps to correct the available value of T and estimates a more reasonable amount to the preselected threshold. To determine the solution of the expected 1-median with maximum probability, we define the *antipode points* which are originally introduced by Frank, [6]. Using the antipode points, we bring forward the *state-based primary regions* in links, which are useful to estimate the distances; see [2].

Definition 3. *For each node $v_h \in N$ and each state $r, r = 1, \ldots, R$, the point $z^{ij}_{r,h}$ on the link e_{ij} is called an antipode associated with the node v_h in state r, if the following equation holds:*

$$z^{ij}_{r,h} sp^r_{ij} + d_r(v_h, v_i) = (l_{ij} - z^{ij}_{r,h}) sp^r_{ij} + d_r(v_h, v_j).$$

An antipode $z^{ij}_{r,h}$ on the link e_{ij} provides two regions $[v_i, z^{ij}_{r,h}]$ and $[z^{ij}_{r,h}, v_j]$, where the distances between the points on the link e_{ij} in each of these regions and the node v_h in state r can be distinctly computed. Considering all antipode points and gathering them up together yield some basic intervals on e_{ij}, which we call them the state-based primary regions; see Definition 4.

Definition 4. *The set of all points between two consecutive antipodes $z^{ij}_{r',l}$ and $z^{ij}_{r'',k}$ in the link e_{ij} associated with the node $v_l \in N$ in the state r' and node $v_k \in N$ in the state r'', $(r', r'' \in \{1, \ldots, R\})$ is called a state-based primary region $[z^{ij}_{r',l}, z^{ij}_{r'',k}]$.*

Recall that the state-based primary regions are special with a remarkable property that the distances from any node v_h in each state r to all points $x \in [z^{ij}_{r',l}, z^{ij}_{r'',k}]$ are computed either by $x sp^r_{ij} + d_r(v_h, v_i)$ or $(l_{ij} - x) sp^r_{ij} + d_r(v_h, v_j)$ and it remains unchanged throughout this region. By using this property, the 1-median objective function in the state r, that is, $WD_r(x) = \sum_{h=1}^{n} w_h d_r(v_h, x)$, in each state-based primary region can be computed.

Next, a reformulation of the expected 1-median problem with maximum probability is given. Since the weights of nodes are discrete random variables, there are different situations in network. Let the set of all situations for the vector of weights be denoted by $W_{state} = \{W^s | W^s = (w^s_1, \ldots, w^s_n), s = 1, \ldots, S\}$, where $S = \prod_{h=1}^{n} K_h$ and W^s denotes the vector of weights in situation s. Therefore, the weight of node v_h, $h = 1, \ldots, n$, in situation s, $s = 1, \ldots, S$ and its probability are denoted by w^s_h and $P^{'s}_h$, respectively. For all $W^s \in W_{state}$ and $x \in G$, the following characterization function is provided to calculate the objective function (4):

$$Y_{W^s}(x) = \begin{cases} 1, & \sum_{r=1}^{R} P_r \sum_{h=1}^{n} w^s_h d_r(v_h, x) \le T, \\ 0 & \textit{otherwise}. \end{cases} \tag{5}$$

By applying this characterization function, the objective function (4) would be rewritten as follows:

$$\max_{x \in G} f(x) = \sum_{s=1}^{S} Y_{W^s}(x) P'_{W^s}, \tag{6}$$

where $P'_{W^s} = \prod_{v_h \in N} P^{'s}_h$; see [2].

3 The solution approach

In this section, two lemmas are provided to determine the optimal solution of the expected 1-median problem with maximum probability when the threshold T is selected from some regular intervals. Furthermore, when T is selected out of these regular intervals, an algorithm is represented to find the optimal solution; see [2]. The 1-anti-median and 1-median problems are used in Lemmas 2 and 3 to indicate some regular intervals for threshold T, where all points in the network G are optimal with probabilistic objective function equal to 1 and zero, respectively. In 1-anti-median problem, the purpose is to find the location of a facility in the network, where the total weighted distances from all nodes to the facility is maximized. The point x is 1-anti-median, if it maximizes $\sum_{h=1}^{n} w_h d(v_h, x)$, where the weights and distances are fixed values. For more details and the solution approaches, see [14].

Let x_r be the optimal solution to the following 1-deterministic-anti-median problem:

$$\max_{x \in G_r} \sum_{h=1}^{n} w_h[K_h] d_r(v_h, x),$$

for $r = 1, \ldots, R$, where $w_h[K_h]$ is the largest probabilistic weight corresponding to node v_h. Also let x'_r be the optimal solution to the following 1-deterministic-median problem:

$$\min_{x \in G_r} \sum_{h=1}^{n} w_h[1] d_r(v_h, x),$$

for $r = 1, \ldots, R$, where $w_h[1]$ is the smallest probabilistic weight corresponding to node v_h. Lemma 2 provides an upper level value to the threshold T, while Lemma 3 provides a lower level value to it. If T is selected greater than the estimated upper level or smaller than the estimated lower level, then the maximum probability expected 1-median solution would be readily determined.

Lemma 2. *If* $A = arg\max_{r=1,\ldots,R} \sum_{h=1}^{n} w_h[K_h] d_r(v_h, x_r)$ *while* x_A *is the corresponding* 1*-deterministic-anti-median solution to the following problem:*

$$\max_{x \in G_A} \sum_{h=1}^{n} w_h[K_h] d_A(v_h, x)$$

and T *is given in such a way to satisfy the constraint*

$$\sum_{h=1}^{n} w_h[K_h] d_A(v_h, x_A) \leq T,$$

then each point $x \in G$ *is the expected* 1*-median solution with the maximum probability equal to* 1.

Proof. The weights of the nodes $v_h \in N$ are assumed to be arranged in ascending order; therefore, the following inequalities hold:

$$w_h[1] < w_h[2] < \cdots < w_h[K_h].$$

Considering the above inequalities together with the definition of x_A yields the following inequalities for an arbitrary x:

$$\begin{aligned}
&\sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h[k] d_r(v_h, x) \\
&\quad \le \sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h[K_h] d_r(v_h, x) \le \sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h[K_h] d_r(v_h, x_r) \\
&\quad \le \sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h[K_h] d_A(v_h, x_A) = \sum_{h=1}^{n} w_h[K_h] d_A(v_h, x_A) \le T.
\end{aligned}$$

Hence $Y_{W^s}(x) = 1$ for all $W^s \in W_{state}$ and $x \in G$; therefore, the objective function value at every point $x \in G$ is equal to 1, that is, all $x \in G$ are optimal. □

Lemma 3. *Let $B = arg\min_{r=1,\ldots,R} \sum_{h=1}^{n} w_h[1] d_r(v_h, x'_r)$, where x'_B is the 1-deterministic-median solution to the following problem:*

$$\min_{x \in G_B} \sum_{h=1}^{n} w_h[1] d_B(v_h, x).$$

If $T < \sum_{h=1}^{n} w_h[1] d_B(v_h, x'_B)$, then all points $x \in G$ are optimal to the expected 1-median problem with maximum probability equal to zero.

Proof. Considering the definitions of x'_B, the following inequalities hold:

$$\begin{aligned}
T &< \sum_{h=1}^{n} w_h[1] d_B(v_h, x'_B) = \sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h[1] d_B(v_h, x'_B) \\
&\le \sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h[1] d_r(v_h, x'_r) \le \sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h[1] d_r(v_h, x) \\
&\le \sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h[k] d_r(v_h, x).
\end{aligned}$$

The foregoing inequalities yield $Y_{W^s}(x) = 0$ for all $x \in G$ and $W^s \in W_{state}$; therefore, all points $x \in G$ are optimal with probability equal to zero. □

Lemma 4 provides a sufficient condition to ignore some links, which along them, the characterization function (5) is equal to zero.

Lemma 4. *Let v_i and v_j be two adjacent nodes, where $Y_{W^s}(v_i) = Y_{W^s}(v_j) = 0$, for some $W^s \in W_{state}$; then $Y_{W^s}(x)$ is also equal to zero for all $x \in e_{ij}$.*

Proof. By the assumption, $Y_{W^s}(v_i) = Y_{W^s}(v_j) = 0$; therefore, the following inequalities hold:

$$\sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h^s d_r(v_h, v_i) > T,$$

$$\sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h^s d_r(v_h, v_j) > T.$$

Furthermore, each function $d_r(v_h, x)$ is concave in each link e_{ij}; see Figure 1. Therefore, the function:

$$\sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h^s d_r(v_h, x)$$

is concave on the link e_{ij}; see [14]. This yields the inequality

$$\sum_{r=1}^{R} P_r \sum_{h=1}^{n} w_h^s d_r(v_h, x) > T$$

for all $x \in e_{ij}$ and consequently $Y_{W^s}(x) = 0$, which completes the proof. □

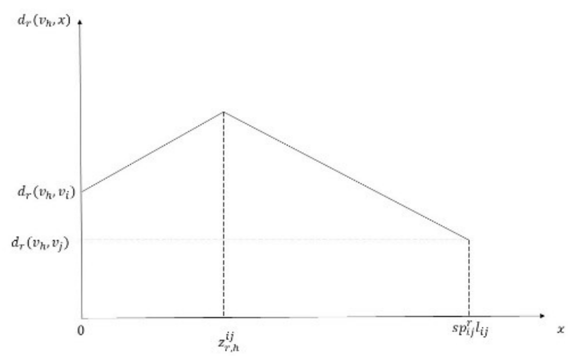

Figure 1: The figure of function $d_r(v_h, x)$ on the link e_{ij}.

Let the threshold T be selected between the upper and lower bound values provided by Lemmas 2 and 3,

$$\sum_{h=1}^{n} w_h[1] d_B(v_h, x'_B) \leq T < \sum_{h=1}^{n} w_h[K_h] d_A(v_h, x_A).$$

In this case, the solution of the expected 1-median problem with maximum probability would be determined by defining some new points x in network links, where the characterization function $Y_{W^s}(x)$ for the weight vectors $W^s \in W_{state}$ would change. First some special points in stated-based primary regions, called jump-points, are specified as introduced in [2]. Then it is shown the objective function (6) would be changed only at the jump-points. The shortest path from the points in a state-based primary region on the link e_{ij} to each node v_h would be obtained through the node v_i or v_j without switching to the other node. Hence, the value of objective function (6) can be readily computed in all jump-points in each state-based primary region. Therefore, the optimal solution in each primary region would be determined. By comparing all solutions in the state-based primary regions together, the local optimal solution in each link would be obtained.

Let $[z^{ij}_{r',l}, z^{ij}_{r'',k}]$ be a state-based primary region associated with the nodes l and k in the link e_{ij} in states r' and r'', respectively. Then the set of nodes N and the set of states corresponding to the traveling times of links, G_{state}, can be partitioned into two sets $L^-_{r'}$ and $K^+_{r''}$ as follows:

$$L^-_{r'} = \{(h,r) | z^{ij}_{r,h} \leq z^{ij}_{r',l}\}, \qquad K^+_{r''} = \{(h,r) | z^{ij}_{r,h} \geq z^{ij}_{r'',k}\}.$$

Indeed, the set $L^-_{r'}$ includes the pair indices (h,r), where h corresponds to the node $v_h \in N$ and r corresponds to $G_r \in G_{state}$, so that the position of antipode $z^{ij}_{r,h}$ in the link e_{ij} is before the position of the antipode $z^{ij}_{r',l}$ associated with the node v_l in the state r'. The set $K^+_{r''}$ is defined similarly. Therefore, for $(h,r) \in L^-_{r'}$, the distance between any point $x \in [z^{ij}_{r',l}, z^{ij}_{r'',k}]$ and the node v_h in the state r is determined through the node v_j, that is, $d_r(v_h, x) = sp^r_{ij}(l_{ij} - x) + d_r(v_h, v_j)$, where sp^r_{ij} is the inverse of the traveling speed on the link e_{ij} in the state r. The corresponding value for $d_r(v_h, x)$, where $(h,r) \in K^+_{r''}$, would be determined through node v_i in a similar manner.

As an example, consider the small network shown in Figure 2 with three state-based networks G_1, G_2, and G_3 with $sp^r_{12} = 0.5$, $sp^r_{14} = 0.3$, $sp^r_{23} = 0.3$, $sp^r_{24} = 0.5$ for $r = 1,2,3$ and $sp^1_{13} = sp^1_{34} = 0.2$, $sp^2_{13} = sp^2_{34} = 0.3$ and $sp^3_{13} = sp^3_{34} = 0.4$. Consider the link e_{12}, then the set of antipodes associated with the node 3 is $\{z^{12}_{1,3}, z^{12}_{2,3}, z^{12}_{3,3}\} = \{3.7, 2.7, 2.7\}$ and with the node 4 is $\{z^{12}_{1,4}, z^{12}_{2,4}, z^{12}_{3,4}\} = \{3.4, 4.1, 4.8\}$; see Figure 2. For the primary region $[z^{12}_{2,3}, z^{12}_{1,4}] = [2.7, 3.4]$, we have $L^-_2 = \{(2,1),(2,2),(2,3),(3,2),(3,3)\}$ and $K^+_1 = \{(1,1),(1,2),(1,3),(3,1)\ (4,1),(4,2),(4,3)\}$. Note that for all states r, the antipodes corresponding to the nodes 1 and 2, are the points $l_{12} = 6$ and 0, respectively. Now consider one of the weight vectors $W^s \in W_{state}$ with components $w^s_h, h = 1, \ldots, n$. So for any point $x \in [z^{ij}_{r',l}, z^{ij}_{r'',k}]$, the following relationships hold [2]:

$$\sum_r P_r \sum_h w^s_h d_r(v_h, x) \leq T$$

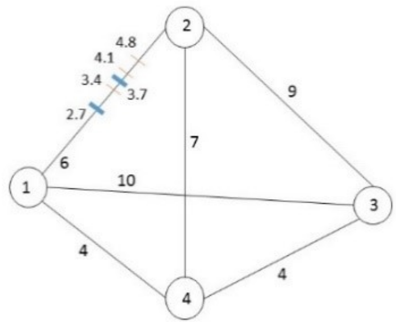

Figure 2: The state-based primary regions in the link e_{12} in a small-sized network.

$$\Rightarrow \sum_{(r,h)\in L^-_{r'}} P_r w^s_h d_r(v_h, x) + \sum_{(r,h)\in K^+_{r''}} P_r w^s_h d_r(v_h, x) \le T$$

or

$$\sum_{(r,h)\in L^-_{r'}} P_r w^s_h (d_r(v_h, v_j) + sp^r_{ij}(l_{ij} - x)) + \sum_{(r,h)\in K^+_{r''}} P_r w^s_h (d_r(v_h, v_i) + sp^r_{ij} x) \le T$$

$$\Rightarrow \sum_{(r,h)\in L^-_{r'}} P_r w^s_h (d_r(v_h, v_j) + sp^r_{ij} l_{ij}) + \sum_{(r,h)\in K^+_{r''}} P_r w^s_h d_r(v_h, v_i)$$
$$+ \Big(\sum_{(h,r)\in K^+_{r''}} P_r w^s_h sp^r_{ij} - \sum_{(h,r)\in L^-_{r'}} P_r w^s_h sp^r_{ij}\Big) x \le T. \tag{7}$$

Definition 5. *If inequality (7) is binding at a point x^{W^s}, then x^{W^s} is called a jump-point with respect to weight vector $W^s \in W_{state}$.*

If we set

$$\sum_{(r,h)\in L^-_{r'}} P_r w^s_h (d_r(v_h, v_j) + sp^r_{ij} l_{ij}) = A^{W^s}_{r',L}, \qquad \sum_{(r,h)\in K^+_{r''}} P_r w^s_h d_r(v_h, v_i) = B^{W^s}_{r'',K}$$

and

$$\sum_{(h,r)\in K^+_{r''}} P_r w^s_h sp^r_{ij} - \sum_{(h,r)\in L^-_{r'}} P_r w^s_h sp^r_{ij} = C^{W^s}_{(r',L),(r'',K)},$$

then

$$x^{W^s} = \frac{T - A^{W^s}_{r',L} - B^{W^s}_{r'',K}}{C^{W^s}_{(r',L),(r'',K)}}.$$

Let J be the set of all jump-points in state-based primary region $[z^{ij}_{r',l}, z^{ij}_{r'',k}]$ unified with two boundary points $\{z^{ij}_{r',l}, z^{ij}_{r'',k}\}$.

Lemma 5. *The expected 1-median objective function would change only in the points belonging to the set J.*

Proof. The objective function value would change in some point x, if the value of $Y_{W^s}(x)$ is changes from 0 to 1 or vice versa for some vectors W^s. This guarantees that these points belong to the set J. □

If a local optimal solution in a state-based primary region in some links is given, Lemma 6 provides some optimal intervals in this region. The coefficient of variable x in inequality (7) is a criterion, which determines the optimal intervals; for more details see [2].

Lemma 6. *Let $x^{W^s} \in [z^{ij}_{r',l}, z^{ij}_{r'',k}]$ be the local optimal jump-point while $y_1^{W^1}$ and $y_2^{W^2}$ are the left and right consecutive jump-points to x^{W^s}, respectively. Then the optimal interval in $[z^{ij}_{r',l}, z^{ij}_{r'',k}]$ is determined as follows:*

1. *If $C^{W^s}_{(r',L),(r'',K)} \geq 0$ and $C^{W^1}_{(r',L),(r'',K)} \leq 0$, then the interval $[y_1^{W^1}, x^{W^s}]$ is optimal.*
2. *If $C^{W^s}_{(r',L),(r'',K)} \leq 0$ and $C^{W^2}_{(r',L),(r'',K)} \geq 0$, then the interval $[x^{W^s}, y_2^{W^2}]$ is optimal.*
3. *If $C^{W^s}_{(r',L),(r'',K)} \geq 0$ and x^{W^s} is the first jump-point after $z^{ij}_{r',L}$, then the interval $(z^{ij}_{r',L}, x^{W^s}]$ is optimal, where the starting point of the interval can be distinctly verified.*
4. *If $C^{W^s}_{(r',L),(r'',K)} \leq 0$ and x^{W^s} is the last jump-point before $z^{ij}_{r'',K}$, then the interval $[x^{W^s}, z^{ij}_{r'',K})$ is optimal, where the end point of the interval can be distinctly verified.*

Proof. The coefficient of variable x in inequality (7) is $C^{W^s}_{(r',L),(r'',K)}$. If $C^{W^s}_{(r',L),(r'',K)} \geq 0$ and $C^{W^1}_{(r',L),(r'',K)} \leq 0$, then $Y_{W^s}(x) = 1$ for all $x \leq x^{W^s}$ and $Y_{W^1}(x) = 1$ for all $x \geq y_1^{W^1}$. For other vectors W', which their respective jump-points x lie outside the interval $[y_1^{W^1}, x^{W^s}]$, the values of $Y_{W'}(x)$ remain unchanged in this interval. Note that if $C^{W^1}_{(r',L),(r'',K)} > 0$ and $C^{W^s}_{(r',L),(r'',K)} > 0$, then $Y_{W^1}(x^{W^s}) = 0$. Therefore, $f(x^{W^s}) < f(y_1^{W^1})$, which contradicts the optimality of point x^{W^s}. The other cases can be proved similarly. Note that the rest of states for $C^{W^s}_{(r',L),(r'',K)}$, $C^{W^1}_{(r',L),(r'',K)}$, and $C^{W^2}_{(r',L),(r'',K)}$ are impossible, since they contradict the optimality of point x^{W^s}. □

3.1 The Algorithm: finding a global optimal solution

In this section, Algorithm 1 is represented to describe that how a local optimal solution can be obtained on an arbitrary link and consequently a global optimal solution is determined. Recall that if two arbitrary adjacent nodes, say v_i and v_j, have the same zero objective function values, then all points on the link e_{ij} have also objective values equal to zero. Therefore, these edges should be ignored in the proposed algorithm. In other side, as described in Lemmas 2 and 3, the solution can be determined directly for spacial values of T. The idea of Algorithm 1 is to specify the state-based primary regions on all links and obtain the local optimal solution in all regions. Hence, G_{state} the set of all state-based situations for the network G, , and the set of all weight situations, W_{state}, are determined. Then the primary regions are located by computing the antipodes while the set of jump-points are determined in each primary region, as well. The corresponding value of the objective function (6) is estimated for all jump-points x by evaluating $Y_{W^s}(x)$ for vectors $W^s \in W_{state}$. By comparing these solutions, the optimal solution in each primary region and consequently on each link would be determined. Finally, the global optimal solution in the network would be determined by comparing the local optimal solutions on all links, while the optimal regions are fully specified by Lemma 6.

In the represented algorithm, the maximum number of state-based primary regions in each link is $(n-2)R+1$ and the maximum number of jump-points in each link is $S((n-2)R+1)$, where S and R are the number of states for the weights of nodes and the traveling times of edges, respectively. To specify the solution, the values of $Y_{W^s},\ s=1,\dots,S$ for each jump-point should be estimated. Therefore, if any link is not ignored by Lemma 4, in the worst case, the time of algorithm in network would be $mS^2((n-2)R+1)$. This time would be so long, particularly for large-sized networks. In section 5, we propose a method to solve the problem in large-sized networks without using the Algorithm 1.

3.2 Numerical example

In the following, a numerical example is provided. The optimal solutions are estimated for different values of threshold T by performing the steps of Algorithm 1 in MATLAB R2014a, and the sensitivity analysis for different values of weights is proposed. Using Lemmas 2 and 3, some regular intervals for threshold T are introduced, where the optimal solutions along them would be readily obtained.

Example 1. In this example, we study an squared network given in Figure 3 with 25 nodes and 40 links. The lengths of the links have been written next to the links while the weights of the nodes with their relevant probabilities are shown in Table 1. The traveling speeds in different states for each link e_{ij} with their relevant probabilities are given in Table 2.

Algorithm 1 Determining the expected 1-median with maximum probability in the link e_{ij}

1: Determine the solution of the deterministic problems $\min_{r=1,\ldots,R}\min_{x'\in G_r}\sum_{h=1}^{n} w_h[1]d_r(v_h,x')$ and $\max_{r=1,\ldots,R}\max_{x\in G_r}\sum_{h=1}^{n} w_h[K_h]d_r(v_h,x)$, set their optimal values as L and U, respectively. If $T<L$ or $T\geq U$, then the optimal solutions are determined by Lemmas 3 and 2, respectively. Otherwise go to step (2).

2: Consider the set of all state-based situations for the network G as $G_{state}=\{G_r,\ r=1,\ldots,R\}$ in which $R=\prod_{e_{ij}\in E}U_{ij}$ with relevant probabilities $P_r=\prod_{e_{ij}\in E}P^r_{ij}$, where P^r_{ij} is the probability that the link e_{ij} has the traveling time $d_r(v_i,v_j)=sp^r_{ij}l_{ij}$.

3: Arrange the demand weight situations in the set $W_{state}=\{W^s|\ W^s=(w^s_1,\ldots,w^s_n), s=1,\ldots,S\}$ with relevant probabilities $P'_{W^s}=\prod_{v_h\in N}P'^s_h$ for $s=1,\ldots,S$.

4: Determine and sort the set of $Z^{ij}=\{z^{ij}_{r,h}|h=1,\ldots,n,\ h\neq i,j,\quad r=1,\ldots,R\}\cup\{0,l_{ij}\}$ on the link $e_{ij}\in E$ in ascending order. Rename the members of Z^{ij} as $z_k, k=1,\ldots,|Z^{ij}|$.

5: Set $W'_{state}=W_{state}\backslash\{W^s|Y_{W^s}(v_i)=Y_{W^s}(v_j)=0\}$ and $k=1$.

6: Determine the state-based primary regions $[z_k,z_{k+1}]$, where $z_k,z_{k+1}\in Z^{ij}$, and the jump-point set $J_k=\{x^k_d|x^k_d=x^{W^s}\in[z_k,z_{k+1}]_{for\ some\ vector}\ W^s\in W'_{state},\ d=1,\ldots,D\}$, where $D=|J_k|$. Set $q=1,\ d=1$ and $f(x^k_d)=0$.

7: If $Y_{W^q}(x^k_d)=1$, where W^q is the qth member of W'_{state}, set $f(x^k_d)=f(x^k_d)+P'_{W^q}$, else go to step (8).

8: Set $q=q+1$. If $q\leq|W'_{state}|$, then go to step (7), else set $d=d+1$ and go to step (9).

9: If $d\leq D$, set $f(x^k_d)=0$, $q=1$, and go to step (7), else go to step (10).

10: Set $x^k_{opt}=\arg\max_d f(x^k_d)$.

11: Set $k=k+1$, if $k\leq|Z^{ij}|-1$, go to step (6), else go to step (12).

12: Set $x_{e_{ij}}=argmax_{k=1,\ldots,|Z^{ij}|}f(x^k_{opt})$.

The upper and lower bound values provided by Lemmas 2 and 3 are 5716.435 and 310.2833, respectively. Therefore, if $T\geq 5716.435$ or $T<310.2833$, then all points of network G are optimal with probabilities equal to one and zero, respectively. In addition, using the proposed algorithm, it can be numerically verified that, for values of $T\leq 1050$ and $T\geq 1900$, the optimal objective function value is equal to zero and one, respectively.

To verify Algorithm 1 in details, we consider the threshold $T=1100$. By applying Lemma 4, only the links $e_{7,12}, e_{8,13}, e_{11,12}, e_{12,13}, e_{12,17}, e_{13,14}$ and

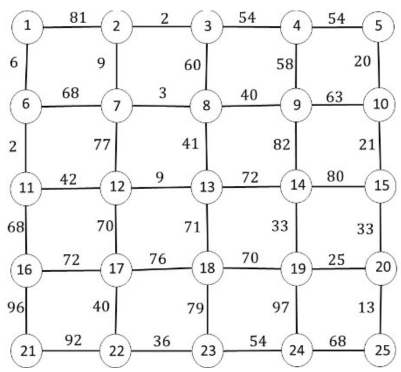

Figure 3: A squared network with 25 nodes.

Table 1: The weights of nodes in different states

nodes	1	2	3	4	5	6	7	8	9	10	11	12	13
state 1	75	35	33	59	47	57	56	72	23	86	78	62	53
state 2	69	68	18	88	74	54	28	5	90	90	40	50	96
state 3	44	36	4	67	9	50	57	25	72	41	32	55	4
state 4	47	94	57	28	80	43	82	27	7	38	50	78	39
state 5	75	28	33	76	43	33	53	72	60	44	41	15	68
nodes	14	15	16	17	18	19	20	21	22	23	24	25	Probability
state 1	82	18	51	75	48	27	72	8	32	46	74	4	0.2
state 2	77	18	10	54	6	9	25	88	51	1	66	73	0.25
state 3	57	6	19	16	69	23	41	27	56	51	63	62	0.15
state 4	101	32	38	97	107	46	90	68	7	28	24	87	0.1
state 5	47	17	36	9	57	29	58	44	48	66	72	48	0.3

Table 2: The different traveling speeds

State	1	2	3	4	5	6
link traveling speed	$100+i+j$	$70+i+j$	$50+i+j$	$40+i+j$	$80+i+j$	$90+i+j$
probability	0.15	0.2	0.1	0.25	0.23	0.07

$e_{13,18}$ have to be searched for state $s = 3$ and the other states $s = 1, 2, 4, 5$ as well as the other links would be ignored. We follow the steps of algorithm for links $e_{12,13}$ and $e_{13,14}$. The antipodes of the link $e_{12,13}$ are sorted ascending according to their distance from vertex 12, as follows:

$$Z^{12,13} = \{0,\ 5.5061,\ 5.6330,\ 5.8108,\ 5.8753,\ 5.9289,\ 5.9742,\ 9\}.$$

The jump-points obtained by equality (7) are all outside the primary regions and the only jump-points of the link $e_{12,13}$ would be its antipodes. Computing the values of $\sum_{r=1}^{6} P_r \sum_{h=1}^{25} w_h^3 d_r(v_h, z_k)$ for $z_k \in Z^{12,13},\ k = 1, \ldots, 8$ results in $Y_{W^3}(z_k) = 1$. So all points in the link $e_{12,13}$ are optimal with the objective function value equals to 0.15. The other evaluated link is $e_{13,14}$, where its antipode set is

$$\begin{aligned} Z^{13,14} = \{&0,\ 1.5085,\ 1.6971,\ 1.9164,\ 2.1749,\ 2.8602,\ 3.3285,\ 3.8673,\ 5.3977,\\ &7.4310,\ 8.1409, 8.7211,\ 9.2043,\ 33.5780,\ 34.0291,\ 34.6129,\\ &34.8126,\ 34.9741,\ 35.1074, 44.9147,\ 45.7261,\ 45.7886,\ 46.9351,\\ &46.9441,\ 47.4125,\ 47.8139,\ 48.0984,\ 48.1616,\ 48.2600,\ 48.5373,\\ &48.6747,\ 48.9968,\ 49.1737,\ 49.2538,\ 49.4117,\\ &49.6128,\ 49.7849,\ 69.6898,\ 71.4483,\ 72\} \end{aligned}$$

Again, the jump-points obtained by equality (7) are outside the primary regions and the set of jump-points includes just the antipodes. For antipodes

$$\{0,\ 1.5085,\ 1.6971,\ 1.9164,\ 2.1749,\ 2.8602,\ 3.3285,\ 3.8673,\ 5.3977\} \subset Z^{13,14}$$

the value of Y_{W^3} is equal to 1. Therefore, the optimal interval obtained by Lemma 6 is $[0\ ,\ 5.3977]$ with optimal probability 0.15. Using Algorithm 1, the solutions for different values of T with their objective (Obj) function values are determined and inserted in Table 3. The obtained optimal intervals (Int) in corresponding links with running times (t) are also reported. Moreover, the variations of the objective function (4) with respect to different values of T are released in Figure 4.

Next, we develop a sensitivity analysis to the solutions by certain perturbations in the weights of nodes and traveling speeds of links. The weights of all nodes are assumed to be equal to 75 in all 5 states with relevant probabilities $\{0.55,\ 0.1,\ 0.1,\ 0.1,\ 0.15\}$ while the lengths of all links are considered equal to 50. In addition, 6 states are considered for links' traveling speeds that are assumed to be the same for all links equal to $\{100,\ 70,\ 50,\ 40,\ 80,\ 90\}$ in states $r = 1, \ldots, 6$ with relevant probabilities given in Table 2. The weight of node 1 is multiplied by 100 in the first state, that is, $w_1^1 = 7500$. This alteration results in the solution of maximum probability expected 1-median problem for $T = 3700$ being attained in node 13 with probability 0.45. Moreover, $Y_{W^s}(x^*) = 1$ for the states $s = 2, 3, 4, 5$ and $Y_{W^1}(x^*) = 0$, where x^*

denotes the optimal solution, also the interval $[0,\ 6.8090]$ in the link $e_{13,18}$ is optimal.

By increasing the value of T from $T = 3700$ to $T = 6000$, the optimal value of the objective function remains unchanged while the optimal solutions would be spread in the network. For $T = 6000$, all nodes except 1, 5, 21 and 25 lie in the optimal solution set. If $T = 6100$, only the node 1 is optimal with objective function value 1. In this case, the value of T is larger than the expected sum of weighted distances from all nodes to node 1 in all states $s = 1, 2, 3, 4, 5$. Notice that if the median is in node 1, the expected sum of weighted distances from all nodes to node 1 is the same in all states $s = 1, \ldots, 5$.

Next, we multiply the weight of node 25 in state 5 by 100 while the other information remain unchanged. For $T = 3700, \ldots, 6000$, the optimal value of the objective function is 0.3. The solution for $T = 3700$ is node 13, by increasing the value of T to $T = 6000$ the value of optimal probability does not change but the number of optimal points is increased. For $T = 6000$ all nodes expect 1, 5, 21 and 25 lie in the optimal points' set. For $T = 6100$ the optimal probability is changed to 0.85 whereas the optimal solution set is reduced to node 1.

If the probability of the first and fifth states are changed together, that is, $P'_{W^1} = 0.15$ and $P'_{W^5} = 0.55$, then the solution would be in the last node with optimal probability 0.85. Also, if the first and the fifth states of nodes' weights have just the same probability equal to 0.35, then for $T = 6100$, both nodes 1 and 25 would be optimal with optimal probability 0.65.

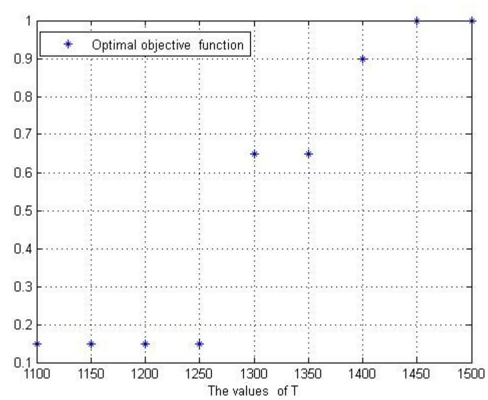

Figure 4: Variations of objective function (4) for different values of T

Table 3: The solutions for different values of T

$T = 1100$ $Obj = 0.15$	*link* *Int*	$e_{12,13}$ [0 , 9]	$e_{13,14}$ [0 , 5.3977]							$t = 0.51$
$T = 1150$ $Obj = 0.15$	*link* *Int*	$e_{8,13}$ [28.8133 , 41]	$e_{11,12}$ [34.9193 , 42]	$e_{12,13}$ [0 , 9]	$e_{12,17}$ [0 , 11.0125]	$e_{13,14}$ [0 , 9.2043]				$t = 0.49$
$T = 1200$ $Obj = 0.15$	*link* *Int*	$e_{8,13}$ [8.5634 , 41]	$e_{11,12}$ [32.5787 , 42]	$e_{12,13}$ [0 , 9]	$e_{12,17}$ [0 , 13.3575]	$e_{13,14}$ [0 , 9.2043]	$e_{13,18}$ [0 , 71]			$t = 0.66$
$T = 1250$ $Obj = 0.15$	*link* *Int*	$e_{2,7}$ [6.8588 , 9]	$e_{6,7}$ [65.7371 , 68]	$e_{7,8}$ [0 , 3]	$e_{7,12}$ [64.0216 77]	$e_{8,9}$ [0 , 4.1260]	$e_{8,13}$ [0 , 41]	$e_{11,12}$ [0 , 42]	$e_{12,13}$ [0 , 9]	$t = 1.51$
$T = 1250$ $Obj = 0.15$	*link* *Int*	$e_{12,17}$ [0 , 13.3575]	$e_{13,14}$ [0 , 9.2043] [71.4483 , 72]	$e_{13,18}$ [0 , 30.9657]	$e_{14,19}$ [0 , 33]	$e_{17,18}$ [71.4691 , 76]	$e_{18,19}$ [0 , 11.1206] [66, 8237 , 70]	$e_{19,20}$ [0 , 3.3058]	$e_{20,25}$ 0	$t = 1.51$
$T = 1300$ $Obj = 0.65$	*link* *Int*	$e_{12,13}$ [7.0710 , 9]	$e_{13,14}$ [0 , 2.1749]							$t = 2.29$
$T = 1350$ $Obj = 0.65$	*link* *Int*	$e_{7,12}$ [74.6709 , 77]	$e_{11,12}$ [38.7520 , 42]	$e_{12,13}$ [0 , 9]	$e_{13,14}$ [0 , 7.4310]					$t = 3.25$
$T = 1400$ $Obj = 0.9$	*link* *Int*	$e_{13,14}$ 0								$t = 3.66$
$T = 1450$ $Obj = 1$	*link* *Int*	$e_{12,13}$ [5.5061 , 9]	$e_{13,14}$ [0 , 2.8602]							$t = 4.58$
$T = 1500$ $Obj = 1$	*link* *Int*	$e_{12,13}$ [0 , 9]	$e_{13,14}$ [0 , 9.2043]							$t = 5.66$

4 Central limit theorem

In this section, the expected 1-median problem with maximum probability in the large-sized networks is considered. Considering all state-based primary regions, determining the jump-points, and finding the value of objective function in all of them make a large number of computations to be done that practically is useless. Therefore, we state an approximation method, which provides a near optimal solution to the expected 1-median with maximum probability, the approach proposed in [3].

Let the mean and variance of W_h, which is the independent discrete random weight of node $v_h \in N$, be denoted by μ_h and σ_h^2, respectively. Using Lemma 7, the mean and variance of term $\sum_{r=1,\dots,R} P_r \sum_{h=1,\dots,n} W_h d_r(v_h, x)$ are determined. Then by applying the central limit theorem, the problem is reduced to a nonlinear fractional programming problem.

Lemma 7. *Based on the central limit theorem, the term $\sum_{r=1,\dots,R} P_r \sum_{h=1,\dots,n} W_h d_r(v_h, x)$ in large-sized networks has the normal distribution with the expected mean value*

$$\mu_{EWD} = \sum_{r=1,\dots,R} P_r \sum_{h=1,\dots,n} \mu_h d_r(v_h, x)$$

and variance $\sigma_{EWD}^2 = \sum_{r=1,\dots,R} P_r^2 \sum_{h=1,\dots,n} \sigma_h^2 d_r^2(v_h, x)$.

Therefore, $\dfrac{\sum_{r=1,\dots,R} P_r \sum_{h=1,\dots,n} W_h d_r(v_h, x) - \mu_{EWD}}{\sigma_{EWD}} \sim N(0, 1)$, which yields the following:

$$\begin{aligned} &P'\Big(\sum_{r=1,\dots,R} P_r \sum_{h=1,\dots,n} W_h d_r(v_h, x) \le T\Big) \\ &= P'\Big(\frac{\sum_{r=1,\dots,R} P_r \sum_{h=1,\dots,N} W_h d_r(v_h, x) - \mu_{EWD}}{\sigma_{EWD}} \le \frac{T - \mu_{EWD}}{\sigma_{EWD}}\Big) \\ &= \phi\Big(\frac{T - \mu_{EWD}}{\sigma_{EWD}}\Big), \end{aligned}$$

where ϕ is the cumulative distribution function of the standard normal distribution. By considering the ascending property of ϕ, the main problem can be transformed into

$$\max_x \frac{T - \sum_{r=1,\dots,R} P_r \sum_{h=1,\dots,n} \mu_h d_r(v_h, x)}{\sqrt{\sum_{r=1,\dots,R} P_r^2 \sum_{h=1,\dots,n} \sigma_h^2 d_r^2(v_h, x)}}. \tag{8}$$

The above nonlinear fractional programming problem can be solved by using the existed approaches such as parametric or direct methods; see [17]. But here we use a method that considers the problem in three cases. Let x^*

be the optimal solution to the problem (8); then three cases can be occurred associated with $T-\mu_{EWD}(x^*)$ that are discussed in details below; see [3].

1. **First case**: $T-\mu_{EWD}(x^*) > 0$. Setting $T-\mu_{EWD}(x) = \frac{1}{\gamma}$, the problem is transformed into the following problem:

$$\min_x \sum_{r=1,\dots,R} P_r^2 \sum_{h=1,\dots,n} \sigma_h^2 \gamma^2 d_r^2(v_h, x) \tag{9}$$
$$s.t. \quad T\gamma - \sum_{r=1,\dots,R} P_r \sum_{h=1,\dots,n} \mu_h \gamma d_r(v_h, x) = 1.$$
$$\gamma \geq 0$$

   To solve the above problem, the optimal point x in each state-based primary region should be determined. Let $[z^{ij}_{r',l}, z^{ij}_{r'',k}]$ be a state-based primary region associated with the nodes l and k in states r' and r'' in the link e_{ij}, respectively. Then for all $x \in [z^{ij}_{r',l}, z^{ij}_{r'',k}]$ and an arbitrary state r and node v_h, either $d_r(v_h, x) = d_r(v_h, v_i) + sp^r_{ij}x$ or $d_r(v_h, x) = d_r(v_h, v_j) + sp^r_{ij}(l_{ij} - x)$. Substituting $d_r(v_h, x)$ in problem (9) and setting $\gamma x = x^{'}$, the problem can be rewritten as a second order problem with one linear constraint in each primary region, which can be solved by using such nonlinear convex programming algorithms as Lagrangian dual method or linearization approaches; see [15]. Comparing the objective function values of the optimal solutions in all intervals yields the global optimal solution in each link and consequently in the network.

2. **Second case**: $T-\mu_{EWD}(x^*) = 0$. The second case is considered when the first one is infeasible. In this case, it is sufficient to solve the following problem:

$$\min_x \sum_{r=1,\dots,R} P_r \sum_{h=1,\dots,n} \mu_h d_r(v_h, x). \tag{10}$$

   Using the concavity of the objective function as mentioned in the proof of Lemma 4, this problem has at least a nodal optimal solution. So it is sufficient to search among the nodes of the network. If the optimal value of the objective function (10) equals to T, then the obtained solution is optimal to the fractional programming problem (8) with the optimal value equal to zero. Otherwise, the third case should be considered.

3. **Third case**: $T-\mu_{EWD}(x^*) < 0$. If the first and second cases are infeasible, this case is considered similar to the first one, but the constraint is changed as follows:

$$\sum_{r=1,\ldots,R} P_r \sum_{h=1,\ldots,n} \mu_h \gamma d_r(v_h, x) - T\gamma = 1,$$

and the objective function (9) should be maximized. The local optimal solutions of the resulted problem can also be obtained in each primary region by using such nonlinear programming algorithms as the Lagrangian dual method or linearization approaches; see [15].

Example 2. To illustrate the efficiency of the proposed method based on central limit theorem, a test problem from the address ($http://people.brunel.ac.uk/mastjjb/jeb/info.html$) with 100 nodes and 396 links is considered. Four scenarios with probabilities $\{0.1, 0.25, 0.45, 0.2\}$ for the network links' traveling speeds have been considered. For each link e_{ij} the traveling speed in state $r = 1, 2, 3, 4$ is assumed as $sp_{ij}^r = (r + i + j)^{0.25}$.

We have considered 1000 different situations for vector of weights with discrete uniform distribution and each of components for random weights of nodes are taken from the interval $[0, 850]$. Using the calculated mean and variance for $W_h, h = 1, \ldots, 100$, the proposed method based on central limit theorem is applied for different values of threshold $T = 10^4(1300 + 50i); i = 1, \ldots, 10$. In each case, the resulted nonlinear problems are solved by using the interior-point solver in Matlab R2014a, and the best optimal solutions are determined. Then, the optimality of the solutions is verified by comparing the values of estimated objective function $\phi(\dfrac{T - \mu_{EWD}(x^*)}{\sigma_{EWD}(x^*)})$ by the central limit theorem with the values obtained from the objective function (6) for the obtained solutions x^*; see Figure 5.

In addition, the variations of the estimated relative error $\dfrac{|\phi(\dfrac{T - \mu_{EWD}(x^*)}{\sigma_{EWD}(x^*)}) - f(x^*)|}{f(x^*)}$, where $f(x^*)$ is computed by (6) for the optimal solution x^* and T is selected from the set $\{10^4(1300 + 50i); i = 1, \ldots, 10\}$ is depicted in Figure 6. As it is seen, increasing the value of T results in increasing the objective function optimal value and therefore decreasing the relative error.

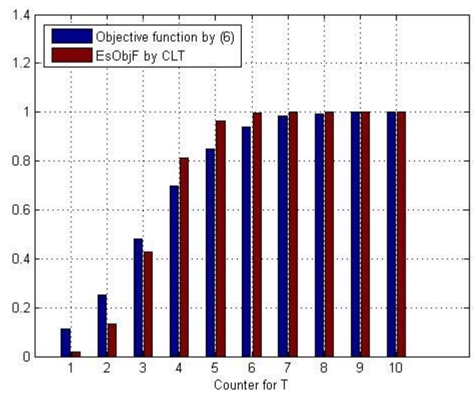

Figure 5: Comparison between $\phi(\dfrac{T - \mu_{EWD}(x^*)}{\sigma_{EWD}(x^*)})$ and objective function (6).

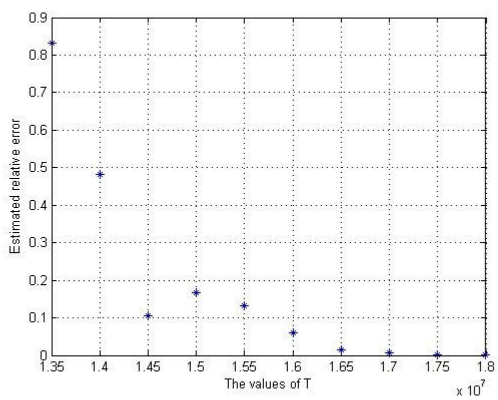

Figure 6: Variations of relative error for different values of T.

5 Summary and conclusions

The median problem is one of the most common problems in location theory specially with nondeterministic parameters. In this paper, the 1-median problem on a network with independent discrete demand weights and traveling times is investigated. The objective function is devoted to maximizing the probability that the expected 1-median function does not exceed a given threshold. First, a precise algorithm to obtain the optimal solution in small-sized networks is presented. Next, by using the central limit theorem, an approach to find the optimal solution in large-sized networks is proposed, where the original problem is reduced to some linear and nonlinear problems with a linear constraint. The numerical examples are given to illustrate the efficiency of the proposed methods.

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بررسی مساله ۱-میانه بر روی شبکه ای با طول یال و زمان حرکت گسسته

مریم ابارشی و مهدی زعفرانیه

دانشگاه حکیم سبزواری، دانشکده ریاضی و علوم کامپیوتر، گروه ریاضی کاربردی

دریافت مقاله ۳۰ آبان ۱۳۹۶، دریافت مقاله اصلاح شده ۳ مرداد ۱۳۹۷، پذیرش مقاله ۷ شهریور ۱۳۹۷

چکیده : در این مقاله، مساله ۱-میانه بر روی شبکه های درختی بدون جهت با طول یال و زمان حرکت تصادفی گسسته بررسی می شود. تابع هدف مساله به دست آوردن بیشینه احتمالی است که مجموع وزنی مورد انتظار فاصله رئوس شبکه تا مکان بهترین سرویس دهنده از یک مقدار کران بالای از پیش تعیین شده کمتر باشد. یک الگوریتم تحلیلی برای حل مساله در شبکه های کوچک و متوسط طراحی شده است. اما برای حل مساله در شبکه های بزرگ قضیه حد مرکزی به کار گرفته شده است که در نتیجه آن مساله اصلی به یک مساله غیرخطی تبدیل می شود. مثالهای ارائه شده در بخش پایانی نشان دهنده تضمین دقت و کارایی روشهای پیشنهادی هستند.

کلمات کلیدی : مساله مکانیابی؛ مساله ۱-میانه؛ وزنهای احتمالی؛ زمان های سفر احتمالی.