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Research Article

A numerical approach for singular perturbation problems with an interior layer using an adaptive spline

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Abstract

An adaptive spline is used in this work to deal with singularly perturbed boundary value problems with layers in the interior region. To evaluate the layer behavior in the solution, a different technique on a uniform mesh is designed by replacing the first-order derivatives with nonstandard differences in the adaptive cubic spline. A tridiagonal solver is used to solve the tridiagonal system of the difference scheme. The fourth-order convergence of the approach is established. The validity of the suggested computational method is demonstrated through numerical experiments, which are compared to other methods in the literature. Layer profile is depicted in graphs.

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Keywords: Singular perturbation problem; Interior layer; Adaptive spline; Tridiagonal system.

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1 Introduction

A small parameter leads to singular perturbation problems (SPPs) in a variety of science and engineering problems in fluid mechanics, elasticity, aerodynamics, magneto-hydrodynamics, optimal control, and other domains of fluid motion [1]. WKB difficulties, the modeling of steady and unsteady viscous flow problems having big Reynolds numbers, magneto-hydrodynamics duct problems with high Hartman numbers, and so on are few notable examples. The numerical solutions to SPPs are often nontrivial because of the rapid development or decay (boundary/interior layer behavior) of solutions to dissipative problems. Here, we include a few references related to the interior layer problems. The authors in [2, 3, 5, 11, 16, 24, 25] provided a full theoretical, analytical, and numerical discussion of this topic. The authors of [8, 9] provided an informative overview of SPPs on layers in their survey publications. In [10], for the Emden–Fowler type equations, the author proposed a finite element collocation technique based on cubic B-splines. Farrell [4] provides a uniformly convergent scheme for the SPP with a turning point problem. Geng, Qian, and Li [6] suggested an approach based on the reproducing kernel method and the asymptotic expansion technique. Miller, O'Riordan, and Shishkin [12] explained how to solve convection-diffusion and reaction-diffusion problems using conventional techniques on Shishkin meshes. Natesan, Vigo-Aguiar and Ramanujam [13] split the region into two subdomains, the layer domain and the regular domain, and then resolved the layer problem with the fitted operator method and the regular problem with the finite difference technique. Navnit [14] proposed a fourth-order numerical approach for solving an SPP using an adaptive cubic spline. The same author in [15] proposed a general scheme for the solution of nonlinear SPP using an adaptive cubic spline. The author in [19] developed a nonuniform mesh optimal B-spline collocation method for the numerical solution of a singular two-point boundary value problem describing electro hydrodynamic flow of a fluid in a circular cylindrical conduit. In [20], two B-spline collocation methods were proposed to solve a class of nonlinear derivative dependent singular boundary value problems. Ramos [22] presented a locally analytical technique for solving SPPs with internal and boundary layers, as well as turning points.

With this motivation, in this paper, a higher order finite difference method on a uniform mesh for solving SPP turning point problems with an interior layer is proposed.

2 Statement of the problem

Consider a second-order SPP of the form

$$\varepsilon w''(t) + P(t)w'(t) + Q(t)w(t) = f(t), \qquad -1 \le t \le 1, \tag{1}$$

with boundary conditions

$$w(-1) = \alpha \qquad \text{and} \qquad w(1) = \beta, \tag{2}$$

where $0 < \varepsilon \ll 1$ is a positive perturbation parameter, α and β are finite constants, and $P(t)$, $Q(t)$, and $f(t)$ are considered to be suitably smooth functions. Based on the coefficient $P(t)$, the solution to (2) has a layer or turning point behavior. The presence of a turning point in the solution to the problem makes it substantially more difficult to handle a boundary or inner layer. In this paper, we examine at the situation where a problem's turning point results in a solution with an interior layer. Under assumptions

$$P(0) = 0, P'(0) \geq 0, \quad Q(t) \leq Q_0 < 0, \qquad \text{for all } t \in D = [-1,\ 1],$$

$$|P'(t)| \geq \frac{|P'(0)|}{2}, \qquad \text{for all } t \in D = [-1,\ 1],$$

the given turning point problem has a solution with interior layers at $t = 0$.

3 Adaptive spline

With grid points t_i in $[a, b]$, consider a mesh such that $\{\Omega : a = t_0 < t_1 < t_2 < \cdots < t_N = b$, where $h = t_i - t_{i-1}$ for $i = 1, 2, \ldots, N\}$. A function $\psi(t, \tau)$ interpolates $w(t)$ at the grid points t_i, depends on a variable, and leads to a cubic spline $\psi(t)$ in $[a, b]$ as $\tau \to 0$, named as an adaptive spline; see [7, 23]. The function $\psi(t, \tau)$ satisfies the equation

$$\varepsilon\psi''(t,\tau) - p\psi'(t,\tau) = \frac{t - t_{i-1}}{h}\left(\varepsilon\mathcal{M}_i - pm_i\right) + \frac{t_i - t}{h}\left(\varepsilon\mathcal{M}_{i-1} - pm_{i-1}\right), \tag{3}$$

where $t_{i-1} \leq t \leq t_i$, $\psi'(t,\tau) = m_i$, and $\psi''(t,\tau) = \mathcal{M}_i$. Solving (3) and using the interpolatory conditions $\psi(t_{i-1},\tau) = w_{i-1}$ and $\psi(t_i,\tau) = w_i$, we have
$\psi(s,\tau) = \mathcal{A}_i + \mathcal{B}_i e^{\tau z} - \frac{h^2}{\tau^3}\left[\frac{1}{2}\tau^2 z^2 + \tau z + 1\right]\left(\mathcal{M}_i - \frac{\tau}{h}m_i\right)$

$$+\frac{h^2}{\tau^3}\left[\frac{1}{2}\tau^2(1-z)^2 + \tau(1-z) + 1\right]\left(\mathcal{M}_{i-1} - \frac{\tau}{h}m_{i-1}\right), \tag{4}$$

where

$$\mathcal{A}_i\left(e^{\tau} - 1\right) = -x_i + x_{i-1}e^{\tau} - \frac{h^2}{\tau^3}\left[\left(\frac{\tau^2}{2} + \tau + 1\right) - \tau e^{\tau}\right]\left(\mathcal{M}_i - \frac{\tau}{h}m_i\right)$$

$$-\frac{h^2}{\tau^3}\left[\left(\frac{\tau^2}{2} - \tau + 1\right) - \tau\right]\left(\mathcal{M}_{i-1} - \frac{\tau}{h}m_{i-1}\right),$$

$$\mathcal{B}_i\left(e^{\tau}-1\right)=x_i-x_{i-1}e^{\tau}+\frac{h^2}{\tau^3}\left[(\frac{\tau}{2}+1)-\tau e^{\tau}\right](\mathcal{M}_i-\frac{\tau}{h}m_i)$$
$$+\left[(\frac{\tau}{2}-1)-\tau\right](\mathcal{M}_{i-1}-\frac{\tau}{h}m_{i-1}),$$

$\tau=\frac{Ph}{\varepsilon}$, and $z=\frac{s-s_{i-1}}{h}$. The spline function $\psi(t,\tau)$ on $[t_i,t_{i+1}]$ is acquired with replacing i by $(i+1)$ in (11). Utilizing the first or second derivative continuity conditions of $\psi(t,\tau)$ at $t=t_i$, we get the following relationship:

$$\begin{aligned}&\left(\mathcal{M}_{i+1}-\frac{\tau}{h}m_{i+1}\right)\left[e^{-\tau}(\frac{\tau^2}{2}+\tau+1)-1\right]\\&+(\mathcal{M}_i-\frac{\tau}{h}m_i)\left[e^{-\tau}\left(\frac{\tau^2}{2}-\tau-2\right)+\left(-\frac{\tau^2}{2}-\tau+2\right)\right]\\&+\left(\mathcal{M}_{i-1}-\frac{\tau}{h}m_{i-1}\right)\left[e^{-\tau}-1+\tau-\frac{\tau^2}{2}\right]\\&\quad=-\frac{\tau^2}{h^3}\left[e^{-\tau}w_{i+1}-\left(1+e^{-\tau}\right)w_i+w_{i-1}\right].\end{aligned}\tag{5}$$

Furthermore, the below relations are given for the adaptive splines:

$$(i)\qquad m_{i-1}=-h\left(\mathcal{A}_1\mathcal{M}_{i-1}+\mathcal{A}_2\mathcal{M}_i\right)+\frac{1}{h}\left(w_i-w_{i-1}\right),$$
$$(ii)\qquad m_i=h\left(\mathcal{A}_3\mathcal{M}_{i-1}+\mathcal{A}_4\mathcal{M}_i\right)+\frac{1}{h}\left(w_i-w_{i-1}\right),$$
$$(iii)\qquad \frac{\theta\ h}{2\tau}\mathcal{M}_{i-1}=-\left(\mathcal{A}_4m_{i-1}+\mathcal{A}_2m_i\right)+B_1\frac{(w_i-w_{i-1})}{h},$$
$$(iv)\qquad \frac{\theta\ h}{2\tau}\mathcal{M}_i=\left(\mathcal{A}_3m_{i-1}+\mathcal{A}_1m_i\right)+B_2\frac{(w_i-w_{i-1})}{h},$$

where

$$\mathcal{A}_1=\frac{1}{4}\left(1+\theta\right)+\frac{\theta}{20},\quad \mathcal{A}_2=\frac{1}{4}\left(1-\theta\right)-\frac{\theta}{20},$$
$$\mathcal{A}_3=\frac{1}{4}\left(1+\theta\right)-\frac{\theta}{20},\quad \mathcal{A}_4=\frac{1}{4}\left(1-\theta\right)+\frac{\theta}{20},$$
$$\mathcal{B}_1=\frac{1}{4}\left(1-\theta\right),\quad \mathcal{B}_2=-\frac{1}{2}\left(1+\theta\right),\quad \text{and}\quad \theta=\coth\left(\frac{\tau}{2}\right)-\frac{2}{\tau}.$$

We also obtain

$$\mathcal{A}_2\mathcal{M}_{i+1}+\left(\mathcal{A}_1+\mathcal{A}_4\right)\mathcal{M}_i+\mathcal{A}_3\mathcal{M}_{i-1}=\frac{1}{h^2}\left[w_{i+1}-2w_i+w_{i-1}\right].\tag{6}$$

Remark: In the limiting case when $\tau\to 0$, we have

$$\mathcal{A}_1=\mathcal{A}_4=\frac{1}{3},\quad \mathcal{A}_2=\mathcal{A}_3=\frac{1}{6},\quad \mathcal{B}_1=\frac{1}{2},\quad \mathcal{B}_2=-\frac{1}{2},\quad \theta=0,\quad \frac{\theta}{\tau}=\frac{1}{6},$$

and (8) reduces to the ordinary cubic spline scheme.

4 Description of the procedure

At the mesh point t_i, the suggested approach can be discretized by the convection-diffusion equation (2) as

$$\varepsilon\ \mathcal{M}_i = f(t_i) - P(t_i){w_i}'(t) - Q(t_i)w(t_i). \tag{7}$$

The above equations shall be replaced by (8), and by approximating the first order derivatives of t at the mesh points $t_1, t_2, \ldots, t_{N-1}$ as

$$w'_{i-1} \approx \frac{-w_{i+1} + 4w_i - 3w_{i-1}}{2h}, \qquad w'_{i+1} \approx \frac{3w_{i+1} - 4w_i + w_{i-1}}{2h},$$
$$w'_i \approx \left(\frac{1 + 2\eta h^2 Q_{i+1} + \eta h\left[3P_{i+1} + P_{i-1}\right]}{2h}\right) w_{i+1} - 2\eta\ \left[P_{i+1} + P_{i-1}\right] w_i$$
$$- \left(\frac{1 + 2\eta h^2 Q_{i-1} - \eta h\left[P_{i+1} + 3P_{i-1}\right]}{2h}\right) w_{i-1} + \eta\ h\left[f_{i+1} - f_{i-1}\right],$$

we get the tridiagonal system

$$L_i w_{i-1} + C_i\ w_i + U_i w_{i+1} = W_i \qquad \text{for}\ \ i = 1, 2, \ldots,\ N-1, \tag{8}$$

where

$$L_i = -\ \varepsilon - \frac{3}{2}\mathcal{A}_3\ P_{i-1}\ h$$
$$- \left(\mathcal{A}_1 + \mathcal{A}_4\right) P_i h\left[1 + 2\eta h^2 Q_{i-1} - \eta h\left(P_{i+1} + 3P_{i-1}\right)\right]$$
$$+ \frac{\mathcal{A}_2}{2} P_{i+1} + \mathcal{A}_3\ Q_{i-1} h^2$$

$$C_i = 2\varepsilon + 2\mathcal{A}_3\ P_{i-1}\ h - 4\left(\mathcal{A}_1 + \mathcal{A}_4\right)\ P_i h_i^2 \eta[P_{i+1} + P_{i-1}]$$
$$- 2\mathcal{A}_2\ P_{i+1} h + 2\left(\mathcal{A}_1 + \mathcal{A}_4\right)\ Q_i h^2$$

$$U_i = -\ \varepsilon - \frac{\mathcal{A}_3}{2}\ P_{i-1}\ h\ + \left(\mathcal{A}_1 + \mathcal{A}_4\right) P_i h\left[1 + 2\eta h^2 Q_{i-1} + \eta h\left(3P_{i+1} + P_{i-1}\right)\right]$$
$$+ \frac{3}{2}\mathcal{A}_2 \eta P_{i+1} h + \mathcal{A}_2 Q_{i+1} h^2$$

$$W_i = h^2\left[\left(\mathcal{A}_2 - 2\eta\left(\mathcal{A}_1 + \mathcal{A}_4\right)\ P_i h\right) f_{i+1} + 2\left(\mathcal{A}_1 + \mathcal{A}_4\right)\ f_i\right]$$
$$+ h^2\left[\left(\mathcal{A}_3 + 2\eta\left(\mathcal{A}_1 + \mathcal{A}_4\right)\ P_i h\right) f_{i-1}\right].$$

The tridiagonal system (10) is solved for $i = 1, 2, \ldots, N-1$, to obtain the approximations $w_1, w_2, \ldots, w_{N-1}$ of the solution $w(t)$ at $t_1, t_2, \ldots, t_{N-1}$.

5 Truncation error

Developed local truncation error associated with the scheme in (10) is

$$\begin{aligned} T_i(h) =& \varepsilon\left[1-\left(2\left(\mathcal{A}_1+\mathcal{A}_4\right)+\mathcal{A}_2+\mathcal{A}_3\right)\right] h^2 w''+\varepsilon\left(\mathcal{A}_3-\mathcal{A}_2\right) h^3 w''' \\ &+\left[\frac{\mathcal{A}_2+\mathcal{A}_3}{2}-4 \eta \varepsilon\left(\mathcal{A}_1+\mathcal{A}_4\right)-\frac{1}{6}\left[2\left(\mathcal{A}_1+\mathcal{A}_4\right)+\mathcal{A}_2+\mathcal{A}_3\right]\right] P_i h^4 w''' \\ &+\frac{\varepsilon}{12}\left[1-6\left(\mathcal{A}_2+\mathcal{A}_3\right)\right] h^4 w^{iv} \\ &-\frac{1}{12}\left(\mathcal{A}_3-\mathcal{A}_2\right)\left[P_i w^{iv}+2\left(P_i'+Q_i\right) w'''+6\left(P_i''+Q_i'\right) w''\right. \\ &\left.+2\left(P_i'''+3 Q_i'''\right) w'\right]+\left[2 Q_i'''\ w-2 f'''\right] h^5+O\left(h^6\right) . \end{aligned}$$

Thus for different values of A_2, A_3, A_1+A_4 in the scheme, (10) indicates different orders.
Remarks:
(i) If $\mathcal{A}_2=\mathcal{A}_3$, for any choice of arbitrary $\mathcal{A}_2, \mathcal{A}_1+\mathcal{A}_4$ with $(\mathcal{A}_1+\mathcal{A}_4)+\mathcal{A}_2=\frac{1}{2}$ and for any value of η , then the method is obtained for second order.
(ii) For $\mathcal{A}_2=\mathcal{A}_3=\frac{1}{12},(\mathcal{A}_1+\mathcal{A}_4)=\frac{5}{12}$, and $\eta=-\frac{1}{20\varepsilon}$, the fourth-order method is derived.

6 Convergence

The convergence analysis of the proposed method is discussed in this section. The system of equations (10) in the matrix form with the boundary conditions is

$$(D+F) W+G+T(h)=0, \tag{9}$$

where

$$D=[-\varepsilon, \quad 2\varepsilon, \quad -\varepsilon]=\begin{bmatrix} 2\varepsilon & -\varepsilon & 0 & 0 & \dots & 0 \\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & \dots & 0 \\ 0 & -\varepsilon & 2\varepsilon & -\varepsilon & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & -\varepsilon & 2\varepsilon \end{bmatrix}$$

and

$$F=[\widetilde{z}_i, \widetilde{v}_i, \widetilde{w}_i]=\begin{bmatrix} \widetilde{v}_1 & \widetilde{w}_1 & 0 & 0 & \dots & 0 \\ \widetilde{z}_2 & \widetilde{v}_2 & \widetilde{w}_2 & 0 & \dots & 0 \\ 0 & \widetilde{z}_3 & \widetilde{v}_3 & \widetilde{w}_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \widetilde{z}_{N-1} & \widetilde{v}_{N-1} \end{bmatrix}$$

in which

$$\begin{aligned}\widetilde{z}_i =& -\frac{3}{2}\mathcal{A}_3\ P_{i-1}\ h\ -\left(\mathcal{A}_1+\mathcal{A}_4\right)P_ih\left[1+2\eta h^2Q_{i-1}-\eta h\left(P_{i+1}+3P_{i-1}\right)\right]\\ &+\frac{\mathcal{A}_2}{2}P_{i+1}h+\mathcal{A}_3\ Q_{i-1}h^2\\ \widetilde{v}_i =& 2\mathcal{A}_3\ P_{i-1}\ h-4\left(\mathcal{A}_1+\mathcal{A}_4\right)\ P_ih^2\eta\left[P_{i+1}+P_{i-1}\right]\ -2\mathcal{A}_2\ P_{i+1}h\\ &+2\left(\mathcal{A}_1+\mathcal{A}_4\right)\ Q_ih^2\\ \widetilde{w}_i =& -\frac{\mathcal{A}_3}{2}\ P_{i-1}\ h\ +\left(\mathcal{A}_1+\mathcal{A}_4\right)P_ih\left[1+2\eta h^2Q_{i-1}+\eta h\left(3P_{i+1}+P_{i-1}\right)\right]P_i\\ &+\frac{3}{2}\mathcal{A}_2P_{i+1}h+\mathcal{A}_2Q_{i+1}h^2 \qquad \text{for all}\quad i=1,2,3,4,\ldots,N-1,\end{aligned}$$

and

$$G=\left[q_1-\widetilde{z}_1\alpha,q_2,q_3,\ldots,q_{N-1}-\widetilde{w}_{N-1}\beta\right],$$

in which

$$\begin{aligned}q_i =& h^2\left[\left(\mathcal{A}_2-2\eta\left(\mathcal{A}_1+\mathcal{A}_4\right)\ P_ih\right)f_{i+1}+2\left(\mathcal{A}_1+\mathcal{A}_4\right)\ f_i\right]\\ &+h^2\left[\left(\mathcal{A}_3+2\eta\left(\mathcal{A}_1+\mathcal{A}_4\right)\ P_ih\right)f_{i-1}\right] \qquad \text{for all } i=2,\ldots,N-1,\\ T(h) =& 0(h^6),\ \mathcal{A}_2=\mathcal{A}_3=\frac{1}{12},\quad \left(\mathcal{A}_1+\mathcal{A}_4\right)=\frac{5}{12},\quad \eta=-\ \frac{1}{20\varepsilon},\\ W =& \left[W_1,W_2,W_3,\ldots,W_{N-1}\right]^T,\quad T(h)=\left[T_1,T_2,\ldots,T_{N-1}\right]^T,\\ O =& \left[0,0,\ldots,0\right]^T.\end{aligned}$$

Let $w=\left[w_1,w_2,\ldots,w_{N-1}\right]^T\cong W$ satisfy the equation

$$\left(D+F\right)w+G=0. \tag{10}$$

Let $E=\left[e_1,e_2,\ldots,e_{N-1}\right]^T=w-W$, where $e_i=w_i-W_i,\ i=1,2,\ldots,N-1$, is the discretization error. Using (9) and (10), we obtain

$$\left(D+F\right)E=T(h). \tag{11}$$

Let $|Q(s)|\leq\xi_1$ and $|P(s)|\leq\xi_2$, where ξ_1 and ξ_2 are positive constants. If $\zeta_{i,j}$ is the (i,j)th element of F, then

$$\begin{aligned}|\zeta_{i,i+1}|=|\widetilde{w}_i|\leq\varepsilon&\left(h\left(\mathcal{A}_2+\left(\mathcal{A}_1+\mathcal{A}_4\right)\right)\xi_1+h^2\mathcal{A}_2\xi_2+4\left(\mathcal{A}_1+\mathcal{A}_4\right)\eta h^2\xi_1^2\right)\\ &+\left(2h^3\left(\mathcal{A}_1+\mathcal{A}_4\right)\eta\xi_1\xi_2\right) \qquad \text{for all } i=1,2,\ldots,N-2,\\ |\zeta_{i,i-1}|=|\widetilde{z}_i|\leq\varepsilon&\left(h(\mathcal{A}_2+\left(\mathcal{A}_1+\mathcal{A}_4\right))\eta_1+h^2\mathcal{A}_2\eta_2+4\left(\mathcal{A}_1+\mathcal{A}_4\right)\eta h^2\xi_1^2\right)\\ &+\left(2h^3\left(\mathcal{A}_1+\mathcal{A}_4\right)\eta\xi_1\xi_2\right) \qquad \text{for all } i=2,3,\ldots,N-1.\end{aligned}$$

Hence, for sufficiently small h, we have

$$|\zeta_{i,i+1}|\leq\varepsilon \qquad \text{for all } i=1,2,\ldots,N-2, \tag{12}$$

$$|\zeta_{i,i-1}| \le \varepsilon \qquad \text{for all } i = 2, 3, \dots, N-1. \tag{13}$$

Therefore, $(D+F)$ is irreducible (see [27]).
Let the sum of the values of the ith row of $(D+F)$ be S_i. Then

$$S_i = \varepsilon - \frac{\mathcal{A}_2 h}{2}\left(P_{i+1} - 3P_{i-1}\right) + h\left(\mathcal{A}_1 + \mathcal{A}_4\right) P_i + h^2\left(\mathcal{A}_2 Q_{i-1} + 2\left(\mathcal{A}_1 + \mathcal{A}_4\right) Q_i\right)$$
$$+ h^2\left(\mathcal{A}_1 + \mathcal{A}_4\right)\eta P_i\left(3P_{i-1} + P_{i+1}\right) - 2h^3\left(\mathcal{A}_1 + \mathcal{A}_4\right)\eta P_i Q_{i-1} \qquad \text{for } i = 1,$$
$$S_i = h^2\left(Q_{i-1} + 2\left(\mathcal{A}_1 + \mathcal{A}_4\right) Q_i + \mathcal{A}_2 Q_{i+1}\right) + 2h^3\left(\mathcal{A}_1 + \mathcal{A}_4\right) P_i \eta\left(Q_{i+1} - Q_{i-1}\right)$$
$$\text{for all } i = 2, 3, \dots, N-2,$$
$$S_i = \varepsilon + \frac{\mathcal{A}_2 h}{2}\left(P_{i-1} - 3P_{i+1}\right) - h\left(\mathcal{A}_1 + \mathcal{A}_4\right) P_i + h^2\left(\mathcal{A}_2 Q_{i-1} + 2\left(\mathcal{A}_1 + \mathcal{A}_4\right) Q_i\right)$$
$$- h^2\left(\mathcal{A}_1 + \mathcal{A}_4\right)\eta P_i\left(3P_{i+1} + P_{i-1}\right) - 2h^3\left(\mathcal{A}_1 + \mathcal{A}_4\right)\eta P_i Q_{i-1}$$
$$\text{for } i = N-1.$$

Let $\xi_{1^*} = \min_{1\le i\le N} |P(t_i)|$, let $\xi_1^* = \max_{1\le i\le N} |P(t_i)|$, let $\xi_{2^*} = \min_{1\le i\le N} |Q(t_i)|$, and let $\xi_2^* = \max_{1\le i\le N} |Q(t_i)|$.

Since ε is very small and $\varepsilon \propto o(h)$, for suitable small h, $(D+F)$ is monotone (see [26, 27]). Hence, $(D+F)^{-1}$ exists and $(D+F)^{-1} \ge 0$. Thus from (14), we get

$$||E|| \le ||(D+F)^{-1}||\,||T||. \tag{14}$$

Let (i,k)th element of $(D+F)^{-1}$ be $(D+F)^{-1}_{i,k}$, and define

$$||(D+F)^{-1}|| = \max_{1\le i\le N-1} \sum_{k=1}^{N-1} (D+F)^{-1}_{i,k}, \text{and } ||T(h)|| = \max_{1\le i\le N-1} |T(h)|. \tag{15}$$

Since $(D+F)^{-1}_{i,k} \ge 0$ and $\sum_{k=1}^{N-1} (D+F)^{-1}_{i,k} \bar{S}_k = 1,$ for all $i = 1, 2, \dots, N-1$, hence

$$(D+F)^{-1}_{i,1} \le \frac{1}{\bar{S}_1} < \frac{1}{h^2\left[\left(\mathcal{A}_2 + 2(\mathcal{A}_1 + \mathcal{A}_4)\right)\xi_{2^*} - 4(\mathcal{A}_1 + \mathcal{A}_4)\psi\xi_1^2\right]}, \tag{16}$$

$$(D+F)^{-1}_{i,N-1} \le \frac{1}{\bar{S}_N - 1} < \frac{1}{h^2\left[\left(\mathcal{A}_2 + 2(\mathcal{A}_1 + \mathcal{A}_4)\right)\xi_{2^*} - 4(\mathcal{A}_1 + \mathcal{A}_4)\psi\xi_1^2\right]}. \tag{17}$$

Furthermore, for all $i = 2, 3, \dots, N-2$,

$$\sum_{k=2}^{N-2} (D+F)^{-1}_{i,k} \le \frac{1}{\min_{2\le k\le N-2} \bar{S}_k} < \frac{1}{h^2\left[2\left(\mathcal{A}_2 + (\mathcal{A}_1 + \mathcal{A}_4)\right)\xi_{2^*}\right]}. \tag{18}$$

By utilizing (15)–(18) and using (14), we have

$$||E|| \leq O(h^4). \tag{19}$$

Hence, the method given in (10) is fourth-order convergent for

$$\mathcal{A}_2 = \mathcal{A}_3 = \frac{1}{12}, \qquad (\mathcal{A}_1 + \mathcal{A}_4) = \frac{5}{12}, \quad \text{and} \quad \eta = -\frac{1}{20\varepsilon}.$$

7 Numerical examples

Three examples with internal layer behavior are examined in this part to explain the concept computationally. These problems were chosen because they have received considerable attention in the literature. The numerical rate of convergence is computed using the formula given by Doolan, Miller, and Schilders [3] as $R^N = \frac{\log(E^N) - \log(E^{2N})}{\log(2)}$.

Example 1. [17] Consider

$$\varepsilon w''(t) + 2tw'(t) = 0, \qquad t \in (-1, 1),$$

with $w(-1) = -1$ and $w(1) = 1$.
Thus $w(t) = erf\left(\frac{t}{\sqrt{\varepsilon}}\right)$ is the exact solution to the problem.
Table 1 shows the computed solution's maximum absolute errors (MAEs). Figure 1 graphically depicts the numerical and exact solutions. The error plot in the solution of this example is shown in Figure 4.

Example 2. [21] Consider

$$\varepsilon w''(t) + 2(2t-1)w'(t) - 4w(t) = 0, \qquad t \in (0, 1),$$

with $w(0) = 1$ and $w(1) = 1$. Hence

$$w(t) = -\frac{e^{\frac{1}{2\varepsilon} - \frac{(1-2t)^2}{2\varepsilon}}\left(2e^{\frac{(1-2t)^2}{2\varepsilon}}\sqrt{2\pi}t\,erf\left(\frac{1-2t}{\sqrt{2\varepsilon}}\right) - e^{\frac{(1-2t)^2}{2\varepsilon}}\sqrt{2\pi}\,erf\left(\frac{1-2t}{\sqrt{2\varepsilon}}\right) - 2\sqrt{\varepsilon}\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\,erf\left(\frac{1}{\sqrt{2\varepsilon}}\right) + 2\sqrt{\varepsilon}}$$

is the exact solution to the problem.

The MAE of the solution is shown in Tables 2 and 3. Figure 2 shows the layer profile of the numerical and exact solutions. The error plot in the solution of this example is shown in Figure 5.

Example 3. [17] Consider

$$\varepsilon w'' + tw' - w = 0, \qquad -1 \leq t \leq 1,$$

with $w(-1) = 1$ and $w(1) = 2$.
The exact solution is

$$w(t) = \frac{2\sqrt{\varepsilon}\left(t + 3e^{-\frac{t^2-1}{2\varepsilon}}\right) + \left(e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}terf\left(\frac{1}{\sqrt{2\varepsilon}}\right) + 3e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}erf\left(\frac{t}{\sqrt{2\varepsilon}}\right)\right)}{2e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}erf\left(\frac{1}{\sqrt{2\varepsilon}}\right) + 4\sqrt{\varepsilon}}.$$

The MAE is shown in Table 4. Figure 3 shows the layer profile of the numerical and exact solutions. The error plot in the solution of this example is shown in Figure 6.

Table 1: MAE of the solution of Example 1

$\varepsilon\downarrow$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
present method						
2^{-5}	1.1881(-4)	7.3536(-6)	4.5818(-7)	2.8613(-8)	1.7889(-9)	1.1198(-10)
2^{-6}	4.8395(-4)	2.9414(-5)	1.8231(-6)	1.1455(-7)	7.1558(-9)	4.4732(-10)
2^{-7}	0.0019(-0)	1.1881(-4)	7.3536(-6)	4.5818(-7)	2.8613(-8)	1.7890(-9)
2^{-8}	0.0065(-0)	4.8395(-4)	2.9414(-5)	1.8231(-6)	1.1455(-7)	7.1558(-9)
2^{-9}	0.0203(-0)	0.0019(-0)	1.1881(-4)	7.3536(-6)	4.5818(-7)	2.8613(-8)
2^{-10}	0.0957(-0)	0.0065(-0)	4.8395(-4)	2.9414(-5)	1.8231(-6)	1.1455(-7)
R^N	3.8757	3.7514	4.0403	4.0120	3.9923	
Results in [17]						
2^{-5}	3.3962(-1)	4.9291(-1)	6.1199(-1)	7.0727(-1)	7.8166(-1)	8.3818(-1)
2^{-6}	2.6635(-1)	4.2332(-1)	5.5588(-1)	6.6268(-1)	7.4724(-1)	8.1197(-1)
2^{-7}	1.7078(-1)	3.3962(-1)	4.9291(-1)	6.1199(-1)	7.0727(-1)	7.8166(-1)
2^{-8}	9.5121(-2)	2.6635(-1)	4.2332(-1)	5.5588(-1)	6.6268(-1)	7.4724(-1)
2^{-9}	2.9280(-2)	1.7078(-1)	3.3962(-1)	4.9291(-1)	6.1199(-1)	7.0727(-1)
2^{-10}	1.1065(-1)	9.5121(-2)	2.6635(-1)	4.2332(-1)	5.5588(-1)	6.6268(-1)

Total elapsed time with maximum number of subintervals 2^{10} is 0.086497 sec.

Table 2: MAE of the solution of Example 2

$\varepsilon\downarrow$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
present method						
2^{-5}	8.6799(-6)	5.4123(-7)	3.3837(-8)	2.1151(-9)	1.3232(-10)	8.7138(-12)
2^{-6}	2.4578(-5)	1.5331(-6)	9.5748(-8)	5.9831(-9)	3.7396(-10)	2.3510(-11)
2^{-7}	6.7327(-5)	4.3399(-6)	2.7061(-7)	1.6919(-8)	1.0576(-9)	6.6184(-11)
2^{-8}	1.9559(-4)	1.2289(-5)	7.6654(-7)	4.7874(-8)	2.9916(-9)	1.8699(-10)
2^{-9}	5.6901(-4)	3.3663(-5)	2.1700(-6)	1.3531(-7)	8.4593(-9)	5.2878(-10)
2^{-10}	0.0015(-0)	9.7794(-5)	6.1444(-6)	3.8327(-7)	2.3937(-8)	1.4958(-9)
R^{-N}	3.9490	3.9924	4.0028	4.0010	4.0003	
Results in [18]						
2^{-5}	5.9701(-3)	3.3654(-3)	1.7391(-3)	8.7449(-4)	4.3697(-4)	2.1822(-4)
2^{-6}	5.3525(-3)	3.2322(-3)	1.7219(-3)	8.7336(-4)	4.3719(-4)	2.1834(-4)
2^{-7}	1.1177(-2)	2.9851(-3)	1.6827(-3)	8.6953(-4)	4.3725(-4)	2.1848(-4)
2^{-8}	2.5867(-2)	2.6763(-3)	1.6161(-3)	8.6093(-4)	4.3668(-4)	2.1860(-4)
2^{-9}	4.7842(-2)	5.5886(-3)	1.4925(-3)	8.4134(-4)	4.3477(-4)	2.1862(-4)
2^{-10}	7.5829(-2)	1.2934(-2)	1.3381(-3)	8.0805(-4)	4.3046(-4)	2.1834(-4)

Total elapsed time with maximum number of subintervals 2^{10} is 0.089720 sec.

Table 3: MAE of the solution of Example 2

$\varepsilon \downarrow$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
Present method						
2^{-6}	2.4578(-5)	1.5331(-6)	9.5748(-8)	5.9831(-9)	3.7396(-10)	2.3510(-11)
2^{-8}	1.9559(-4)	1.2289(-5)	7.6654(-7)	4.7874(-8)	2.9916(-9)	1.8699(-10)
2^{-10}	0.0015(-3)	9.7794(-5)	6.1444(-6)	3.8327(-7)	2.3937(-8)	1.4958(-9)
2^{-12}	4.4958(-3)	7.5528(-4)	4.8897(-5)	3.0722(-6)	1.9163(-7)	1.1969(-8)
2^{-14}	3.4263(-3)	1.7659(-3)	3.7764(-4)	2.4449(-5)	1.5361(-6)	9.5817(-8)
R^N	0.9562	2.2253	3.9492	3.9924	4.0028	
Results in [21]						
2^{-6}	3.630(- 3)	1.475(-3)	5.047(- 4)	1.508(- 4)	4.146(- 4)	1.089(- 4)
2^{-8}	3.987(- 3)	1.952(- 3)	9.168(- 4)	3.733(- 4)	1.272(- 4)	3.789(- 5)
2^{-10}	7.147(- 3)	1.979(- 3)	9.814(- 4)	4.866(- 4)	2.297(- 4)	9.359(- 4)
2^{-12}	5.562(- 3)	3.583(- 3)	9.837(- 4)	4.898(- 4)	2.444(- 4)	1.215(- 4)
2^{-14}	4.045(- 3)	2.778(- 3)	1.794(- 3)	4.908(- 4)	2.446(- 4)	1.222(- 4)

Table 4: MAE of the solution of Example 3

$\varepsilon \downarrow$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
present method						
2^{-5}	1.3020(-5)	8.1184(-7)	5.0756(-8)	3.1727(-9)	1.9854(-10)	1.3311(-11)
2^{-6}	3.6866(-5)	2.2996(-6)	1.4362(-7)	8.9747(-9)	5.6092(-10)	3.5138(-11)
2^{-7}	1.0099(-4)	6.5099(-6)	4.0592(-7)	2.5378(-8)	1.5863(-09)	9.9321(-11)
2^{-8}	2.9338(-4)	1.8433(-5)	1.1498(-6)	7.1811(-8)	4.4873(-09)	2.8046(-10)
2^{-9}	8.5352(-4)	5.0495(-5)	3.2550(-6)	2.0296(-7)	1.2689(-08)	7.9318(-10)
2^{-10}	2.2657(-3)	1.4669(-4)	9.2166(-6)	5.7490(-7)	3.5906(-08)	2.2437(-09)
R^N	3.9491	3.9924	4.0029	4.0010	4.0003	
Results in [18]						
2^{-5}	1.0203(-2)	5.5017(-3)	2.7453(-3)	1.3449(-3)	6.6344(-4)	3.2928(-4)
2^{-6}	9.1104(-3)	5.3503(-3)	2.7665(-3)	1.3581(-3)	6.6733(-4)	3.3030(-4)
2^{-7}	7.9110(-3)	5.1014(-3)	2.7508(-3)	1.3726(-3)	6.7246(-4)	3.3172(-4)
2^{-8}	2.2330(-2)	4.5552(-3)	2.6751(-3)	1.3832(-3)	6.7905(-4)	3.3366(-4)
2^{-9}	4.6794(-2)	3.9555(-3)	2.5507(-3)	1.3754(-3)	6.8632(-4)	3.3623(-4)
2^{-10}	7.7601(-2)	1.1165(-2)	2.2776(-3)	1.3376(-3)	6.9162(-4)	3.3953(-4)
Results in [17]						
2^{-5}	1.0111(-1)	1.3778(-1)	1.6169(-1)	1.7746(-1)	1.8802(-1)	1.9520(-1)
2^{-6}	5.3820(-2)	8.5843(-2)	1.0677(-1)	1.2048(-1)	1.2958(-1)	1.3573(-1)
2^{-7}	2.2863(-2)	5.0556(-2)	6.8889(-2)	8.0847(-2)	8.8730(-2)	9.4011(-2)
2^{-8}	5.6850(-3)	2.6910(-2)	4.2921(-2)	5.3386(-2)	6.0238(-2)	6.4792(-2)
2^{-9}	2.0288(-2)	1.1431(-2)	2.5278(-2)	3.4445(-2)	4.0424(-2)	4.4365(-2)
2^{-10}	2.9876(-2)	2.8425(-3)	1.3455(-2)	2.1461(-2)	2.6693(-2)	3.0119(-2)

Total elapsed time with maximum number of subintervals 2^{10} is 0.090109 sec.

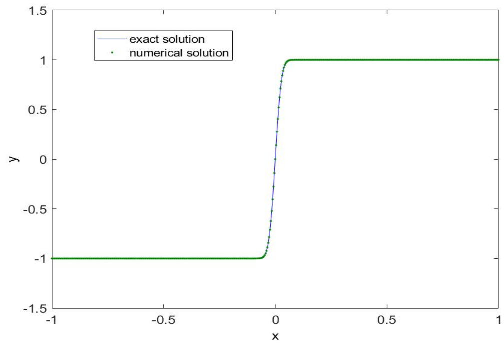

Figure 1: Numerical and exact solution of Example 1 for $\varepsilon = 2^{-10},\ h = 2^{-7}$.

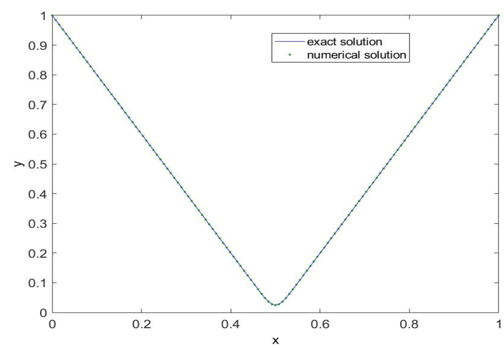

Figure 2: Numerical and exact solution of Example 2 for $\varepsilon = 2^{-10},\ h = 2^{-7}$.

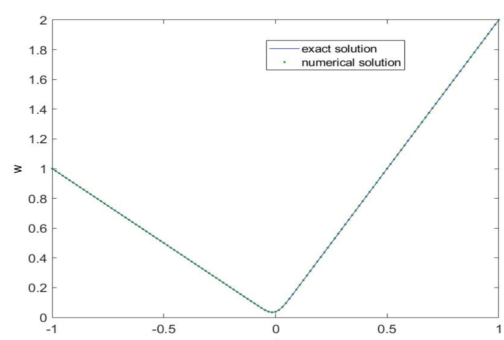

Figure 3: Numerical and exact solution of Example 3 for $\varepsilon = 2^{-10},\ h = 2^{-7}$.

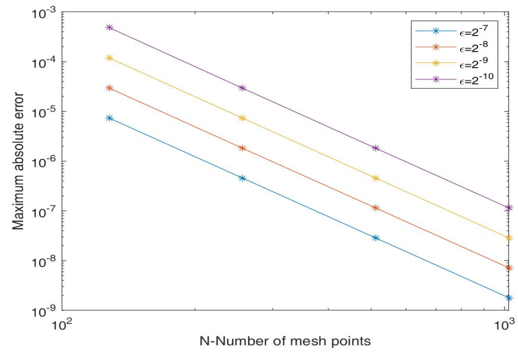

Figure 4: Log-Log scale for Example 1

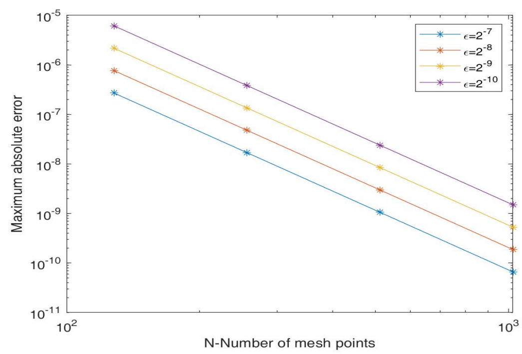

Figure 5: Log-Log scale for Example 2

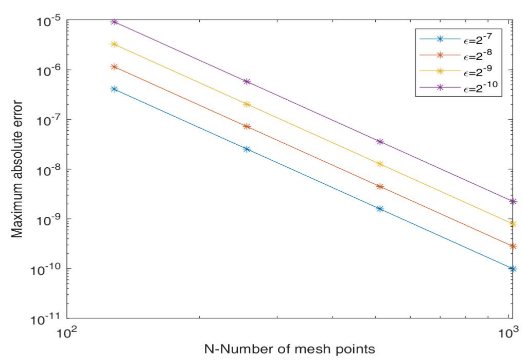

Figure 6: Log-Log scale for Example 3

8 Conclusion

In this paper, for solving singularly perturbed two-point boundary value problems with an interior layer, a higher-order finite difference approach was proposed. To derive the discretization equation, an adaptive cubic spline approach has been extended for a singularly perturbed boundary value problem with interior layers. It was produced by substituting nonstandard finite differences in the first-order derivatives of $w(t)$. The tridiagonal solver discrete consistent embedding was used to solve the discretization equation. Convergence was evaluated in the proposed method. To demonstrate the method, numerical problems have been solved, when ε is either small or large as compared to the mesh size h. To justify the method, numerical results were compared with the results taken by the methods given in [17, 18, 21]. For the layer behavior, the solutions were graphically displayed, and we discovered that the numerical solution closely matches the exact solution. This technique is simple to compute and requires little computational effort.

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