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A shape-measure method for solving free-boundary elliptic systems with boundary control function

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Abstract

This article deals with a computational algorithmic approach for obtaining the optimal solution of a general free boundary problem governed by an elliptic equation with boundary control and functional criterion. After determining the weak solution of the system, the problem is converted into a variational format. Then, by using some aspects of measure theory, the method characterize the nearly optimal pair of domain and its related optimal control function at the same time. This method has many advantages such as strong linearity, automatic existence theorem and the ability of obtaining global solution. Two sets of numerical examples is also given.

Keywords: Elliptic controlled system; Weak solution; Atomic measure; Linear programming; Free boundary problem.

1 Introduction and background

The class of elliptic partial differential equations includes many important systems encountered in mechanics and geometry. Indeed an industrial setting time is spent setting up the allowable set of shapes (domains of equations) in order to get a feasible solution. The structural optimization of such systems has more commonly been applied in the automobile, marine and aerospace industries designing and even in a simple mechano-chemical model of a biomolecular processes (see [9], [1] and [2]). A large part of these problems deals with the free boundary problems when a part of domain's boundary is fixed and the rest is varied. The understanding of models of equations and free boundary problems (such as Monge-Amper equation) may have significant geometric even topological applications. For instance, for an inhomogeneous free boundary problem, Vogel obtained the convexity or starlikeness of variable part when the fixed part is convex or starlike [23]. Furthermore, Lancaster showed that the method of curves of constant direction could be used to investigate even the quasilinear and fully nonlinear

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elliptic free boundary problems [12]. Munch recently have done some works based on topological and numerical approaches (see for instance [14] and [15]). The main goal in structural optimization is to computerize the design process and therefore shorten the time it takes to create a new design or to improve an existing one. Therefore, the results on mechanical formulation of the problems, their functional analysis and on control theory have recently been combined. For a review of such results the reader is referenced to [9] and [10]. Indeed, the optimal control theory provides the basic techniques for computing the derivatives of criteria functions with respect to the boundary. Thus in essence, the optimal control and optimization may be applied when the control becomes associated with the shape of domain. In the former, most of the induced solution methods from control theory, were focused on applications of the principles of the calculus of variations (such as Hadamard works [8]). However in the latter, studies were made only for those problems with an explicit solution for PDE's. Eventually the methods were extended to problems of structural engineering; in particular, those were possible to be converted into an optimal control problem with control governed by the coefficients of PDE's [18]. In this manner the numerical methods (Finite Element, Finite Difference, Boundary Element,...) and Computer Aided Design technology within the optimization loop are used for fully computerizing the design loop. However, the problems are encountered with numerical discretization [18]. In 1986, Rubio in [19], introduced the embedding method for solving the optimal control problems governed by ordinary differential equations, by use of positive Radon measures; then it was employed to obtain the optimal control for a system governed by partial differential equations; like [11] for diffusion and [20] for elliptic systems. The method was based on the strong properties of measures and has many advantages.
Since 1999 till know, by helping of this method, we have solved different cases of the optimal shape design problems governed by elliptic systems (a brief report of this is given in [5]). After solving the problem in polar coordinates in [3], the based measure theoretical approached was improved to cartezian coordinates but for solving optimal shape problem in [4]. Then, we improved the method for optimal shape design problems with distributed control function in [5] To continue, in the present paper we introduce an approach for solving free boundary problems governed by elliptic equations with two special characteristics. Herein, the system is involved with a boundary control function as appears in industrial applications. Moreover, the geometry of domains is completely in the general case.

2 Problem in classical form

As a geometrical point of view, we consider $D \subset R^2$ as a bounded region with a piecewise-smooth, closed and simple boundary ∂D which consists of

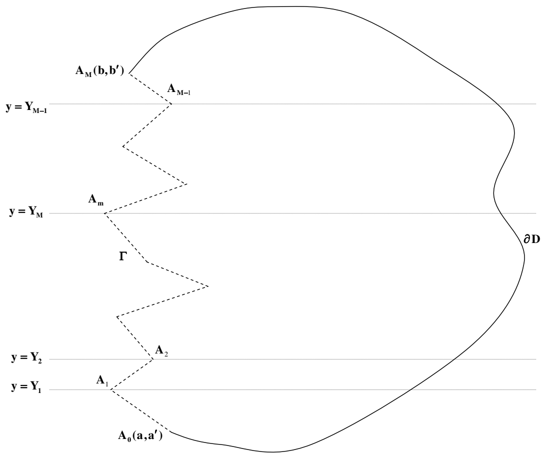

Figure 1: A general domain$D$ and its boundary ∂D.

a fixed and a variable part (cure), denoted by Γ. It is joining two known points $A = (a, a')$ and $B = (b, b')$ so that $a' \leq b'$ and it makes ∂D simple and closed (see Figure 1). By applying the idea of approximating a curve with broken lines, Γ can be approximated by a finite number of connected segments; we fix this number as $M + 1$. Therefore, each Γ can be considered as an $M + 1$ connected segments in which the initial and the final points of them belong to Γ (see Figure 1). If these points are denoted by $A_0 = A, A_1, \ldots, A_{M+1} = B$, then any appropriate region D (domain) can be represented by points $A_m = (x_m, y_m), m = 1, 2, \ldots, M$.
Indeed the variable part of ∂D (and hence D) is represented with $2M$ variables $x_1, x_2, \ldots, x_M$ and $y_1, y_2, \ldots, y_M$. We fix the y-component of each point A_m as $y_m = Y_m \in R$ for $m = 1, 2, \ldots, M$, so that $a' < Y_1 < Y_2 < \cdots < Y_M < b'$.[0] This would not decries the generalities; since even y_m is fixed, but x_m is varied and hence A_m could be any place on the half line $y(x) = Y_m$ with the vertex on the fixed part of D. Therefore the segments $A_m A_{m+1}, m = 0, 1, \ldots, M$, could be selected so that the set of them approximates Γ well enough.
Let $f \in C(D \times R)$ and $g \in C(D)$, be two given real valued functions. The above domain D is called *admissible* if the elliptic equation

$$\Delta u(X) + f(X, u) = g(X) \ , \ u_{|\partial D} = v, \tag{1}$$

has a bounded solution on the domain D; here it is also supposed that $X = (x, y) \in D$, $u : D \to R$ is a bounded trajectory function which takes values in

[0] For special case like $a' = b'$, one can fix the x-components of points A_m's instead of y-components.

the bounded set U, and $v : \partial D \to R$ is a bounded boundary control function, which is Lebesgue measurable and taks values in a bounded set V.

As mentioned, the variable part of ∂D can be approximated with M number of unknown corners. For a fixed positive integer M, the set of all admissible domains is denoted by $\mathcal{D}_M$. When $M \longrightarrow \infty$, if an appropriate optimal shape design problem in $\mathcal{D}_M$ has a minimizer, then this may tend in some topology to the minimizer over $\mathcal{D}$ (the set of all general admissible domains) if such exists. However things can go wrong; for instance: There may be no minimizer over $\mathcal{D}_M$; there may be no minimizer over $\mathcal{D}$; or both $\mathcal{D}$ and $\mathcal{D}_M$; the sequence of minimizer over $\mathcal{D}_M$ may not be convergent or may tend in some sense towards a curve that does not define a shape. Young in [24] has shown that their related subsequence of broken lines, tends to an infinitesimal zigzag (generalized curve). This is not (necessarily) an admissible curve. So the solution over $\mathcal{D}_M$ does not tend to the solution over $\mathcal{D}$, even in the weakly*-sense. Also, there is the important point that too oscillatory boundaries (like the infinitesimal zigzag) sometimes cause problem; Pironneau in [18] shows some of these problems. Hence, we prefer to fix the number M in this paper, and search for the optimal solution of the appropriate problems over $\mathcal{D}_M$.

For a given admissible domain $D \in \mathcal{D}_M$, let $f_1 : D \times U \longrightarrow R$ and , $f_2 : \partial D \times V \longrightarrow R$ be two continuous, non- negative, real-valued functions; further, we assume that there is a constant $L > 0$ so that $\mid f_1(X, u(X)) \mid \leq L \mid u \mid$. We define the functional performance criteria, as

$$\mathbf{I}(D, v) = \int_D f_1(X, u(X))\, dX + \int_{\partial D} f_2(s, v(s))\, ds, \tag{2}$$

where u is the bounded solution of (1). We also define $\mathbf{F}$ as the set of all pairs of (D, v) where $D \in \mathcal{D}_M$ and v is the boundary control function. With the above assumption, we are going to solve the following optimal shape design problem on $\mathbf{F}$:

$$\begin{aligned} &\textbf{Minimize}: && \mathbf{I}(D, v) = \int_D f_1(X, u(X))\, dX + \int_{\partial D} f_2(s, v(s))\, ds \\ &\textbf{Subject to}: && \Delta u(X) + f(X, u) = g(X)\ ,\ u_{|\partial D} = v, \end{aligned} \tag{3}$$

(For some industrial applications of this problem, the reader can have a look on [18]).

To identify the optimal domain in $\mathcal{D}_M$, D^*, and its associated optimal control function, $v^*_{D^*}$, we apply the method which we call shape-measure [5]. This approach characterizes the optimal pair of domain and its related optimal control function in two stages; first for a given domain $D \in \mathcal{D}_M$, by applying the embedding method and use of the Radon measures power, the related optimal control problem will be solved. Then in the next stage, a standard minimization algorithm will be applied to determine the nearly optimal pair of domain and its related optimal control function at the same time.

3 Problem in new formulation

In general, even for a fixed domain, it is difficult to characterize a classical bounded solution for the elliptic equation (1). Therefore, one can change the problem into the other form in which a bounded *weak* (generalized) solution of (1) is involved.

Proposition 1. : *Let u be the classical solution of (1), then we have the following integral equality:*

$$\int_D (u\Delta\psi + \psi f)\, dX - \int_{\partial D} v(\bigtriangledown\psi.\mathbf{n})\, ds = \int_D \psi g\, dX,\ \forall\psi \in H_0^1(D). \qquad (4)$$

that here $\mathbf{n}$ is the outward unit vector on ∂D.

Proof. By multiplying (1) with the function $\psi \in H_0^1(D)$ (the set of functions in the Sobolev space of order 1 in which they are zero on ∂D), integrating over D, and then using the Green's formula (see [13]), one can obtain the equality (4). □

Now, by regarding [5], let D be a fixed domain; then the mentioned optimal free boundary problem changes into an optimal control one in which the same functional as $\mathbf{I}$ must be minimized over the set of all admissible pairs of trajectory and control functions on D. We define $\Omega = D\times U$ and $\omega = \partial D\times V$; then, a bounded weak solution and its corresponded control function define a pair of positive and linear functional $u(\cdot) : F \longrightarrow \int_D F(X, u(X))dX$ and $v(\cdot) : G \longrightarrow \int_{\partial D} G(s, V(s))ds$ on $C(\Omega)$ and $C(\omega)$ respectively. As shown in [19] and [20], the Riesz Representation Theorem [21] shows that there are measures μ_u and ν_v so that:

$$\mu_u(F) = u(F),\ \forall F \in C(\Omega)\ ;\ \nu_v(G) = v(G),\ \forall F \in C(\omega).$$

So far, we have just changed the appearance of the problem. Indeed the transformation between the pair of trajectory and controls, (u, v), and the pair of measures (μ_u, ν_v), is injection (see [19]). Now we extend the underlying space and consider the minimization of the problem over the set of all pairs of measures (μ, ν) in $\mathcal{M}^+(\Omega)\times\mathcal{M}^+(\omega)$ satisfying the mentioned conditions plus the extra properties $\mu(\xi) = \int_D \xi(X)\, dX = a_\xi$ and $\nu(\tau) = \int_{\partial D} \tau(s)\, ds = b_\tau$; these are deduced from the definition of an admissible pair (u, v) and they indicate that the measures μ and ν project on the (x, y)-plan and real line respectively, as Lebesgue measures. We remind the reader that here it is supposed $\xi : \Omega \longrightarrow R$ in $C(\Omega)$ depends only on variable $X = (x, y)$ (i.e. $\xi \in C_1(\Omega)$), and $\tau : \omega \longrightarrow R$ in $C(\omega)$ depends only on variable s (i.e. $\tau \in C_1(\omega)$).

Therefore, we are going to solve the following problem:

$$
\begin{array}{lll}
\textbf{Minimize}: & \mathbf{i}(\mu,\nu) := \mu(f_1) + \nu(f_2) & \\
\textbf{Subject to}: & \mu(F_\psi) + \nu(G_\psi) = c_\psi, & \forall \psi \in H_0^1(D); \\
& \mu(\xi) = a_\xi, & \forall \xi \in C_1(\Omega); \\
& \nu(\tau) = b_\tau, & \forall \tau \in C_1(\omega), \qquad (5)
\end{array}
$$

where $F_\psi = u\Delta\psi + \psi f$, $G_\psi = -v(\bigtriangledown\psi.\mathbf{n} \mid_{\partial D})$ and $c_\psi = \int_D \psi g \, dX$. This new formulation has some advantages; for instance, it is linear in respect to the unknown measure, and if we denote $Q \subset \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ as the set of all pairs of measures (μ, ν) satisfying the conditions mentioned in (5), then Q is compact in the sense of the weak* topology (see for instance [5]). Moreover the function $(\mu, \nu) \in Q \longrightarrow \mu(f_1) + \nu(f_2) \in R$ is continuous. Thus by Proposition $II.1$ of [19], the problem (5) definitely has a minimizer in Q. The theoretical measure problem (5) is an infinite-dimensional linear program problem; even there is no any identified method for obtaining the solution directly, but its solution can be achieved by choosing the countable sets of functions that are uniformly dense (total), in the appropriate spaces. Let $\{\psi_i : i = 1, 2, 3, \dots\}$, $\{\xi_j : j = 1, 2, 3, \dots\}$, and $\{\tau_l : l = 1, 2, 3, \dots\}$, be total sets in the spaces $H_0^1(D)$, $C_1(\Omega)$ and $C_1(\omega)$ respectively. By choosing just a finite number of these functions, the problem (5) is changed into the following one:

$$
\begin{array}{lll}
\textbf{Minimize}: & \mathbf{i}(\mu,\nu) = \mu(f_1) + \nu(f_2) & \\
\textbf{Subject to}: & \mu(F_i) + \nu(G_i) = c_i, & i = 1, 2, \dots, M_1; \\
& \mu(\xi_j) = a_j, & j = 1, 2, \dots, M_2; \\
& \nu(\tau_l) = b_l, & l = 1, 2, \dots, M_3, \qquad (6)
\end{array}
$$

where $F_i := F_{\psi_i}, G_i := G_{\psi_i}, c_i := c_{\psi_i}$, $a_j := a_{\xi_j}$ and $b_l := b_{\tau_l}$. As proved in [5] Theorem 2, the solution of (6) tends to the solution of (5) whenever $M_1, M_2, M_3 \longrightarrow \infty$; hence the solution of (5) can be approximated by one from (6) when the positive integers M_1, M_2 and M_3 are chosen large enough. Now one can construct a suboptimal pair of trajectory and control functions for the functional $\mathbf{i}$ via the optimal solution, (μ^*, ν^*), of (6).

4 Atomic measures and discretization

The problem (6) is a semi-infinite linear programming problem; the number of equations is finite but the underlying space is not a finite-dimensional space. Despite of some possibility for solving such problems (for instance see [7]), it is much more convenient if we could estimate its solution by a finite LP. The pair of optimal measures of (6) can be characterized by a result of Rosenbloom's work which is shown in [19]; by introducing appropriate dense subsets in Ω and ω, one can conclude that μ^* and ν^* have the form $\mu^* = \sum_{n=1}^{N} \alpha_n \delta(Z_n)$ and $\nu^* = \sum_{k=1}^{K} \beta_k \delta(z_k)$ where $Z_n, n = 1, 2, \ldots, N$, and $z_k, k = 1, 2, \ldots, K$, belong to dense subsets of Ω and ω respectively and $\delta(t)$ is the unitary atomic measure with support the singleton set $\{t\}$. Hence, by defining a discretization on Ω and ω with the nodes $Z_n = (x_n, y_n, u_n)$, $n = 1, 2, \ldots, N$, and $z_k, k = 1, 2, \ldots, K$, the solution of (6) can be obtained by solving the following problem in which its unknowns are the coefficients $\alpha_n, n = 1, 2, \ldots, N$, and $\beta_k, k = 1, 2, \ldots, K$.

$$
\begin{aligned}
\textbf{Minimize}: & \sum_{n=1}^{N} \alpha_n f_1(Z_n) + \sum_{k=1}^{K} \beta_k f_2(z_k) \\
\textbf{Subject to}: & \sum_{n=1}^{N} \alpha_n F_i(Z_n) + \sum_{k=1}^{K} \beta_k G_i(z_k) = c_i, \ i = 1, 2, \ldots, M_1; \\
& \sum_{n=1}^{N} \alpha_n \xi_j(Z_n) = a_j, \qquad j = 1, 2, \ldots, M_2; \\
& \sum_{k=1}^{K} \beta_k \tau_l(z_k) = b_l, \qquad l = 1, 2, \ldots, M_3; \\
& \alpha_n \geq 0, \qquad n = 1, 2, \ldots, N; \\
& \beta_k \geq 0, \qquad k = 1, 2, \ldots, K;.
\end{aligned}
\tag{7}
$$

The result of this problem introduces a pair of measures (μ^*, ν^*) that the value of $\mathbf{i}(\mu^*, \nu^*)$, will be minimum; this pair serves the suboptimal pair of trajectory and control functions $(u_{v_D^*}, v_D^*)$. Thus for the fixed domain D, the minimum value of the functional $\mathbf{I}$ in the problem (3) is approximated as $\mathbf{I}(D, v_D^*) \equiv \mathbf{i}(\mu^*, \nu^*)$.

5 Searching the optimal curve (domain)

For a given domain, we have explained that how one can find the optimal control v_D^* for the problem (3), so that the value of $\mathbf{I}(D, v_D^*)$ is minimum. To obtain the minimum value of the performance criterion $\mathbf{I}(D, v)$ on $\mathbf{F}$, for each domain $D \in \mathcal{D}_M$, as explained, the variable part of its boundary is defined by a set of points like $\{A_m = (x_m, Y_m), m = 1, 2, \ldots, M\}$. Thus, for a given $D \in \mathcal{D}_M$, by solving the appropriate finite linear programming problem in (7), the nearly optimal value for $\mathbf{I}(D, v)$ (i.e. $\mathbf{I}(D, v_D^*) \equiv \mathbf{i}(\mu^*, \nu^*)$) is calculated as a function of the variables $x_1, x_2, \ldots, x_M$. Consequently, one can define the following function, which is a vector function of variables $x_1, x_2, \ldots, x_M$:

$$
\mathbf{J} \, : \, D \in \mathcal{D}_M \longrightarrow \mathbf{I}(D, v_D^*) \in R. \tag{8}
$$

Now to find the optimal pair of domain (or variable curve) and its related control function in **F**, say $(D^*, v^*_{D^*})$, which solves the problem (3), it is enough to find the minimizer of **J**.
The global minimizer of the vector function **J**, say $(x_1^*, x_2^*, \ldots, x_M^*)$, can be identified by using one of the appropriate standard minimization search methods, like method introduced by Nelder and Mead in [16]; these algorithms usually need an initial set of components (initial domain) to start the process of minimization (we suppose that they give the global minimizer). Each time that the algorithm wants to calculate a value for **J**, a finite LP problem like (7) should be solved. Whenever it reaches to the minimum value for **J**, the minimizer $(x_1^*, x_2^*, \ldots, x_M^*)$ (the optimal curve) and therefore its associated optimal control function have been obtained. So, the optimal domain and its corresponding optimal control are determined at the same time; this is one of the main advantage of this method. Similar to the Proposition 5 of [5], one can easily prove that the method is convergence.

6 Numerical tests

For the following numerical works, we choose a countable total sets of functions in each spaces $H_0^1(D)$, $C_1(\Omega)$ and $C_1(\omega)$, that is, so that the linear combinations of these functions are uniformly dense (dense in the topology of uniform convergence) in the appropriate spaces. We know that the vector space of polynomials with the variable x and y, $P(x, y)$, is dense in $C^\infty(\overline{D})$; therefore the set

$$P_0(x, y) = \{p(x, y) \in P(x, y) \mid p(x, y) = 0, \forall (x, y) \in \partial D\},$$

is dense (uniformly) in $\left\{h \in C^\infty(\overline{D}) : h_{|_{\partial D}} = 0\right\} \equiv C_0^\infty(\overline{D})$. Since the set

$$Q(x, y) = \left\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots\right\}$$

is a countable base for the vector space $P(x, y)$, each elements of $P(x, y)$ and also $P_0(x, y)$, is a linear combination of the elements in $Q(x, y)$. By Theorem 3 of [13] page 131, the space $C^\infty(\overline{D})$ is dense in $H^1(D)$; thus the space $C_0^\infty(\overline{D})$ will be dense in $H_0^1(D)$. Consequently, the space $P_0(x, y)$ is uniformly dense in $H_0^1(D)$. Therefore, we define the function ψ_i for each i as

$$\psi_i(x, y) = q_D(x, y) q_i(x, y), \tag{9}$$

where q_i is an element of the countable set $Q(x, y)$ and $q_D(x, y)$ is a polynomial depended on D so that it is zero on ∂D (it will be defined separately for each example). Therefore $\psi_{i_{|_{\partial D}}} = 0$ and the set $\{\psi_i(x, y) : i = 1, 2, \ldots\}$ is total (uniformly dense in the topology of the uniform convergence) in $H_0^1(D)$.
Note: We remind the reader that Rubio and others (like Farahi and Kamyad

[6] and [11]) avoids to use the polynomials for such purposes; they usually prefer to apply the related functions defined by *sin* and *cos* or combinations of them. To be sure that these polynomials are suitable to determine the shape, we applied them first for determination the inside and the boundary of a circle (as an example of a shape) by applying the embedding method. For this purpose we used the Stock's theorem for these functions to show the relationship between the inside region and the boundary. The results was very good so that the most obtained inner points (28 from 30) was inside the circle and the rest (two other points) were close enough to the boundary.

For the second set (and also similarly the third one) of functions in (7), let L be a given positive integer number and divide D into L (not necessary equal) parts $D_1, D_2, \ldots, D_L$, so that by increasing L the area of each $D_s, s = 1, 2, \ldots, L$, will be decreased. Then, for each $s = 1, 2, \ldots, L$, we define:

$$\xi_s(x, y, u) = \begin{cases} 1 & (x, y) \in D_s \\ 0 & \text{otherwise} \end{cases}$$

These functions are not continuous, but each of them is the limit of an increasing sequence of positive continuous functions,$\{\xi_{s_k}\}$; then if μ is any positive Radon measure on Ω, $\mu(\xi_s) = \lim_{k\to\infty} \mu(\xi_{s_k})$. Now consider the set $\{\xi_j : j = 1, 2, \ldots, L\}$ of all such functions, for all positive integer L. The linear combination of these functions can approximate a function in $C_1(\Omega)$ arbitrary well (see [19] chapter 5).
By defining $U = V = [-1.0, 1.0]$ and $g(X) = 0$, in the following, two examples for the linear and nonlinear cases of the elliptic equations will be presented.

Example 6.1 Let the fixed part of the boundary of D consists of three sides of a unit square joining points $A = (1, 0), (0, 0), (0, 1)$ and $B = (1, 1)$; and a variable unknown curve joining points A and B, so that it makes ∂D simple and closed. Therefore $q_D(x, y)$ in (9) can be chosen as $xy(y-1)\prod_{l=1}^{M}(x - x_l + y - Y_l)$; in this manner, $\psi_i(x, y)$ is selected so that it is zero on each unknown corner $A_m = (x_m, Y_m)$ of Γ. Reminding that we are able to choose it in a way that it would be zero on each segment $A_{m-1}A_m$ (something which is done in the next example). Now we take $f_2(s, v) = 0$,

$$f_1(X, u) = \begin{cases} 400 & \text{-0.05} \le u \le 0.05 \\ \frac{1}{u^2} & \text{otherwise,} \end{cases}$$

and also $M = 8$, $Y_1 = 0.15$, $Y_2 = 0.25$, $Y_3 = 0.35$, $Y_4 = 0.45$, $Y_5 = 0.55$, $Y_6 = 0.65$, $Y_7 = 0.75$, $Y_8 = 0.85$. The control function is supposed to be zero on ∂D except the segment of line $y = 1$ which along this segment, $v(s)$ takes values in V, when $s \in [0, 1]$. Thus, in (7) we have $G_i = -(\frac{\partial \psi_i(s,y)}{\partial y})_{|y=1}$.
To set up the finite linear programming (7) for the next two cases, we choose $M_1 = 3$ and $M_2 = M_3 = 10$. Also the condition $0 \le x_m \le 2, m = 1, 2, \ldots, 8$, is applied by using the penalty method (see [22]). Moreover we put a dis-

cretization on Ω with $N = 1100$ nodes by points $Z_n = (x_n, y_n, u_n)$, $n = 1, 2, \ldots, N$. Because the control function is zero on ∂D except the segment of the line $y = 1$, we have put a discretization on ω with $K = 110$ nodes like $z_k = (s_k, v_k), k = 1, 2, \ldots, K$; these nodes have been chosen as $z_k = z_{11(i-1)+j}$ for $i = 1, 2, \ldots, 10$ and $j = 1, 2, \ldots, 11$, where $s_{11(i-1)+j} = \frac{(i-1)+0.5}{10}$ and $v_{11(i-1)+j} = \frac{2(j-1)}{10} - 1.0$. Hence the total number of variables in a similar problem to (7) is $1100 + 110 = 1210$. In the case of these concepts, we solved the following examples for the linear and nonlinear case of the elliptic equations; in each case we chose the subroutine $AMOEBA$ (see [17]) as the standard minimization algorithm with the initial valves $X_m = 1.0$, for $m = 1, 2, \ldots, 8$ (indeed, here the initial domain is selected as a unit square). Also, we applied the $E04MBF$ NAG-Library Routine for solving the appropriate finite linear program.

Linear Case: In this example for the linear case, we chose $f = 0$; then by applying the mentioned method, after 497 iteration we achieved the optimal value of $\mathbf{I} = 0.44432256772971$. The value of the variables in the final step was

$$X_1 = 0.044671, X_2 = 0.000003, X_3 = 0.000018, X_4 = 0.083868, X_5 = 0.004590,$$

$$X_6 = 1.181268, X_7 = 0.003360, X_8 = 1.291424.$$

According to the obtained results, the suboptimal control function, the initial and the final domain, and also the changes diagram of the objective function according to the number of iterations, have been plotted in the Figures 2, 3 and 4. We remind that one could do some smoothness and get better results (see Example 6.2 for instance).

Nonlinear case: By choosing $f = 5u^2$ and applying the other assumption as above, the example for the nonlinear case of the elliptic equations was solved. After 492 iterations, the optimal value was $\mathbf{I} = 0.44432182922939$ and the value of the variables in the final step was $X_1 = 0.044691, X_2 = 0.083889, X_3 = 0.004568, X_4 = 0.003356, X_5 = 0.000026, X_6 = 0.000001, X_7 = 1.181291, X_8 = 1.291379$. The results have introduced the suboptimal control function, the final domain and the changes diagram of the objective function which have been plotted in the Figures 5, 6 and 7.

Example 6.2 Let the fixed part of the boundary be the left half of the unite circle, joining the points $A = (0, -1)$ and $B = (0, 1)$. Hence for $M = 9$, $q_D(x, y)$ in (9) can be chosen as

$$(x+\sqrt{1-y^2})(x-5X_1(y+1))(x-5X_9(y-Y_9))\prod_{l=2}^{9}(x-X_{l-1}-5(X_l-X_{l-1})(y-Y_{l-1})),$$

where $Y_1 = -0.8$, $Y_2 = -0.6$, $Y_3 = -0.4$, $Y_4 = -0.2$, $Y_5 = 0$, $Y_6 = 0.2$, $Y_7 = 0.4$, $Y_8 = 0.6$, $Y_9 = 0.8$. This function is zero on all ∂D and thus in (7)

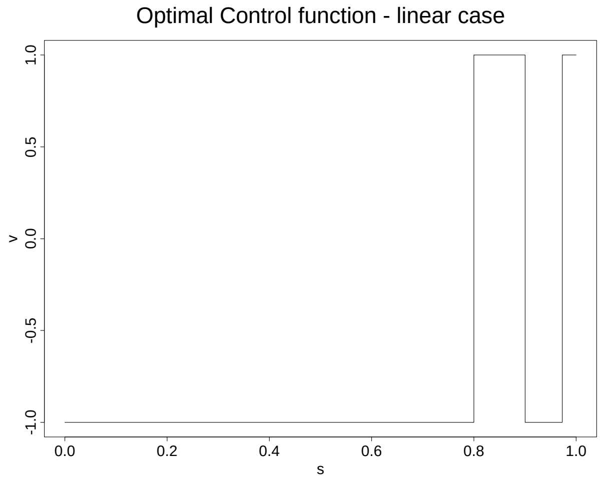

Figure 2: The optimal boundary control function for the linear case of Example 6.1.

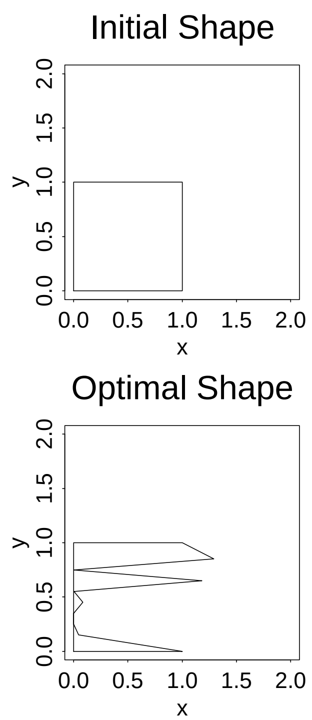

Figure 3: The initial and the optimal domain for the linear case of Example 6.1.

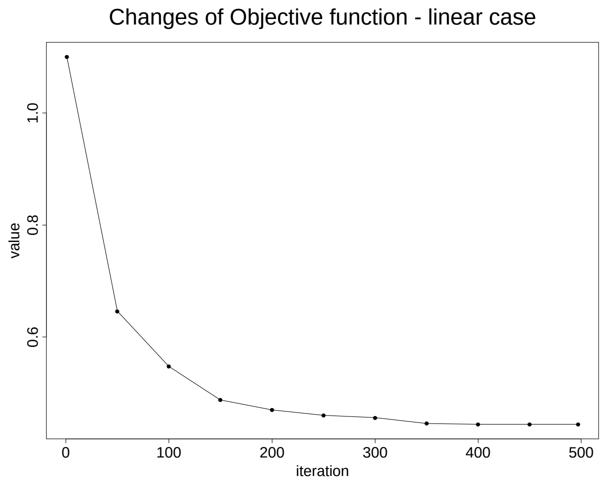

Figure 4: Change of the objective function according to iterations for the linear case of of Example 6.1.

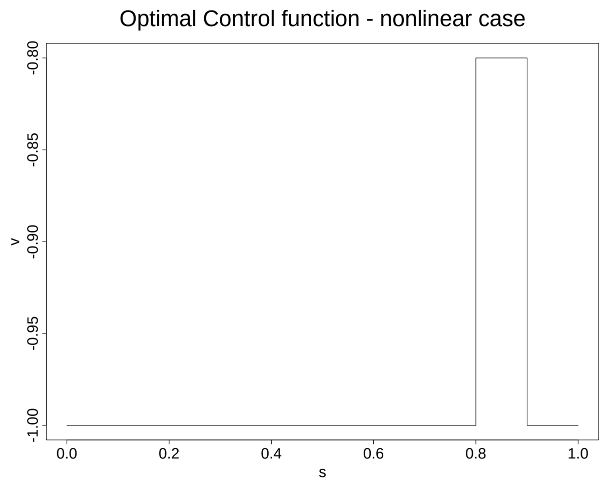

Figure 5: The optimal boundary control function for the nonlinear case of Example 6.1.

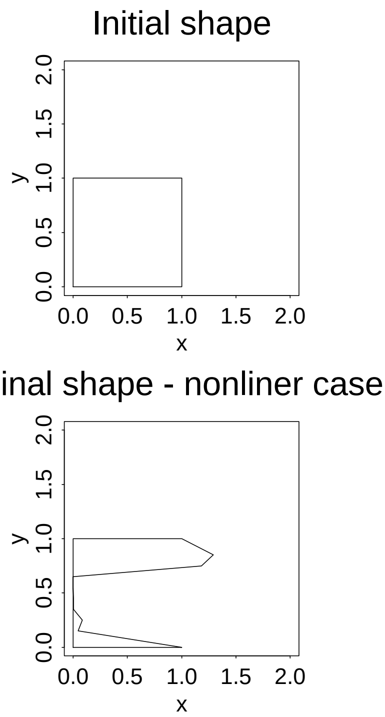

Figure 6: The initial and the optimal domain for the nonlinear case of Example 6.1.

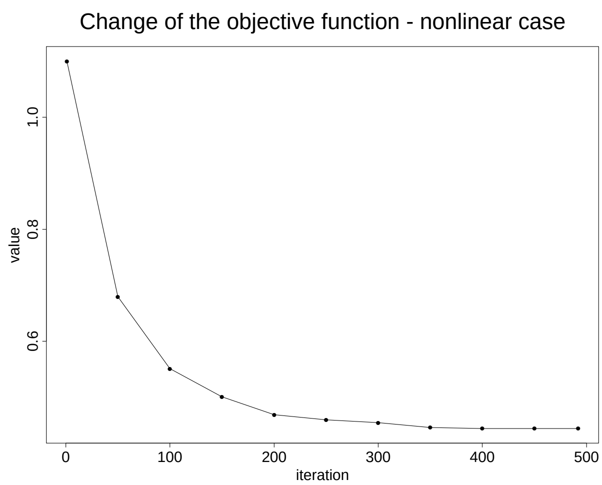

Figure 7: Change of the objective function according to iterations for the nonlinear case of Example 6.1.

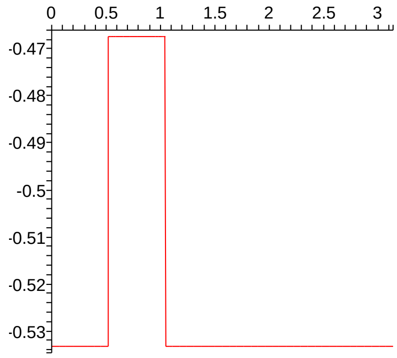

Figure 8: The optimal boundary control function for the linear case of Example 6.2.

we have $G_i = v\sqrt{1-y^2}\psi_{ix} - vy\psi_{iy}$. To show that the method is suitable enough even for the hard situations, we considered much more difficulties in conditions. Therefore, it is supposed that here the variables x_i's have an upper bounds $\sqrt{1-Y_i^2}$ and a lower bound which guaranteed that the variable points can not pass the left half of the unite circle. These conditions are applied by means of the penalty method (see [22]). By selecting

$$f_2(s,v) = \begin{cases} \sqrt{(X_i + 5(X_i - X_{i-1})(s - Y_{i-1})^2 + s^2}, & \mathrm{Y}_{i-1} \leq s \leq Y_i \\ s^2 - v^2 - 1, & 1 \leq s \leq 1+\pi, \end{cases}$$

$K = 3200$, $N = 11875$, $M_2 = 9$ and $M_3 = 15$ (9 equation for fixed boundary and the rest for Γ), an extra condition for the summation of α_i's is also considered to be sure that the domain is covered by characteristic functions perfectly. Thus Ω and ω are discretized by 15075 nodes in which 100 of them was chosen from the fixed part of boundary and also on each segment $A_{m-1}A_m$, 20 nodes was selected. Moreover the subroutines $AMOEBA$ and $DLLRRS$ from the Fortran library were used for solving the following examples.

Linear Case: Let $f_1 = 1 - u^2$ and $f = 0$ (hence the elliptic equation (1) is linear). The initial domain is selected as complete unite circle. After 785 iterations, the optimal value was converge to 19.9850134 and the optimal values of x_i's were: 0.941, 0.1868, 0.2767, 0.3863, 0.5033, 0.3845, 0.2770, 0.1870, 0.0945. The nearly optimal control and the optimal domain, before and after fitting a smooth curve by means of the natural cubic Spline, were plotted in Figures 8, 12 and 10 (by use of Maple9.5 software).

Nonlinear Case: By the above assumptions and choosing $f_1 = x+y+u-0.1$ and $f = 5u^2$, a nonlinear case of the problem is solved. The initial domain is

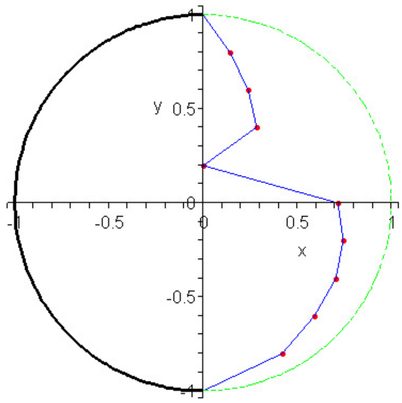

Figure 9: The optimal domain for the linear case of Example 6.2 before fitness.

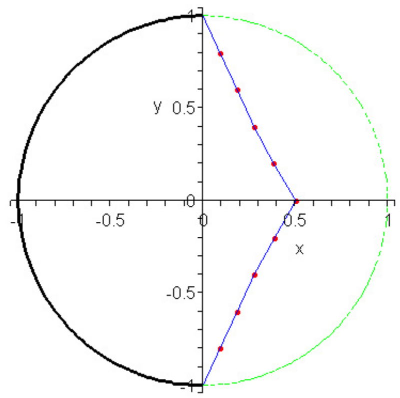

Figure 10: The optimal domain for the linear case of Example 6.2 after fitness.

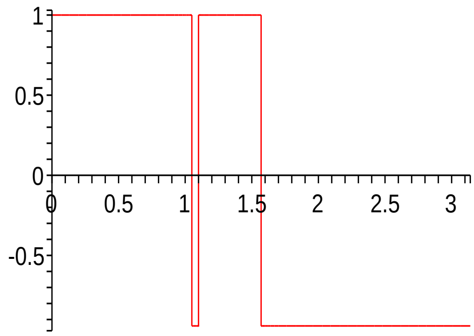

Figure 11: The optimal boundary control function for the nonlinear case of Example 6.2.

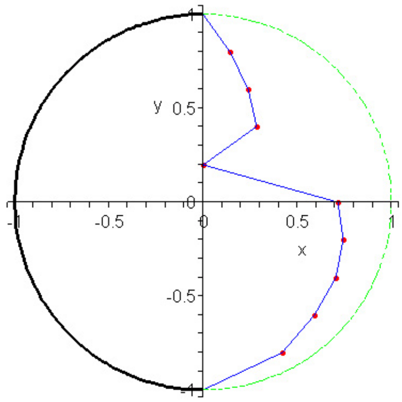

Figure 12: The optimal domain for the nonlinear case of Example 6.2 after fitness.

also selected as a unit circle and after 1303 iterations the optimal control and the optimal domain are obtained. The optimal control is plotted in Figure 11 and after fitting a smooth curve as above, the optimal domain for this case is shown in Figure 12. In this case, the optimal value was 10.699029.

7 Conclusions

Having continued our previous work; herein, we have shown that the mentioned Shape-measure method can be successfully applied for solving free boundary problems which involved with boundary control function. The method was able to characterize the optimal pair of domain and its related control function simultaneously; moreover, the optimal value for the general form of the objective function were determined in an easy way just by applying a standard search technique and also the simplex algorithm perfectly well. Presenting a linear treatment even for the extremely nonlinear problems was one of the main advantages of this method.

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List of symbols
D: a domain
∂D: boundary of D
u: solution of the elliptic system
v: boundary control
$\mathcal{D}_{\mathcal{M}}$: the set of admissible domains for fixed M
$\mathcal{D}$: the set of general admissible domains
$\mathbf{F}$: the set of all pairs of (D, v) where $D \in \mathcal{D}_M$
$\mathbf{n}$: is the outward unit vector on ∂D
$H^1(D)$: the Sobolev space of order 1
$H_0^1(D)$: set of functions in $H^1(D)$ in which they are zero on ∂D
$C(\Omega)$: the set of continuous and bounded functions on Ω
$C_1(\Omega)$: the set of functions in $C(\Omega)$ which depend only on the first variable
$\mathcal{M}^+(\Omega)$: the space of positive Radon measures on $C(\Omega)$
$P(x, y)$: the space of polynomials of x and y.

References

1. Alt, W. and Dembo, M. *Cytoplasm dynamics and cell motion: two-phase flow models*, Int. J. Mathematical Biosciences, 156 (1999), 207–228.

2. Berger, M. P. F. and Wong, W. K. *Apllied Optimal Designs*, John Wiley & Sons Ltd, 2005.

3. Fakharzadeh Jahromi, A. and Rubio, J. E. *Shapes and Measures*, Journal of Mathematical Control and Information, 16 (1999), 207-220.

4. Fakharzadeh Jahromi, A. and Rubio, J. E. *Shape-Measure Method for Solving Elliptic Optimal Shape Problems (Fixed Control Case)*, Bulletin of the Iranian Mathematical Society, 27 (2001), 41–63.

5. Fakharzadeh Jahromi, A. and Rubio, J. E. *Best Domain for an Elliptic Problem in Cartesian Coordinates by Means of Shape-Measure*, AJOP Asian J. of Control, 11, 5 (2009), 536–547.

6. Farahi, M. H. *The Boundary Control of the Wave Equation*, PhD thesis, Leeds University (1996).

7. Goberna, M. A. and Lopez, M. A. *Linear Semi-infinite Optimization*, Alicant Uiniversity, 1998.

8. Hadamard, J. *Lessons on the Calculus of Variation*. Gauthier-Villards, Paris, 1910. (in French).

9. Haslinger, J. and Neittaanamaki, P. *Finite Element Approximation for Optimal Shape Design: Theory and Applications*, Johan Wiley & Sons Ltd, 1988.

10. Haug, E. J. and Cea, J. *Optimization of Disstributed Parameter Structures*, Vols I and II, Sijthoff and Noordhoff, Alpen and Rijn, The Netherland, 1981.

11. Kamyad, A. V., Rubio, J. E. and Wilson, D. A. *The optimal control of multidimensional diffusion equation*, JOTA, 1, 70 (1991), 191–209.

12. Lancaster, K. E. *Qualitative behavior of solution of elliptic free boundary problems*, Pacific J. Maths., 154, 2 (1992), 297–317.

13. Mikhailov, V. P. *Partial Differential Equation*, MIR Publisher, Moscow, 1978.

14. Munch, A. *Optimal design of the support of the control for the 2-d wave equation: Numerical investigation*, Mathematical Modeling and Numerical Analysis, 5, 2 (2008), 331–351.

15. Munch, A. *Optimal internal dissipation of a damped wave equation using a topological approach*, Int. J. Appl. Math. Comput. Sci., 19, 1 (2009), 15–37.

16. Nelder, J. A. and Mead, R. *A simplex method for function minimization*, The Computer Journal, 7 (1964-65), 303-313.

17. Press, W. H.,Teukolsky, S. A., Vetterling, W. T. and Flannery, B. R. *Neumerical Recipes in Fortran: The art of scientific computing*, 2ed edition, Cambridge unversity press, 1992.

18. Pironneau, O. *Optimal Shape Design for Elliptic System*, Springer-Verlag, New York - Berlin - Heidelberg - Tokyo, 1983.

19. Rubio, J. E. *Control and Optimization: The Linear Treatment of Nonlinear Problems*, Manchester University Press, Manchester, 1986.

20. Rubio, J. E. *The global control of nonlinear elliptic equation. Journal of Franklin Institute*, 330, 1 (1993), 29–35.

21. Rudin, W. *Real and Complex Analysis*, Tata McGraw-Hill Publishing Co.Ltd, New Delhi, second edition, 1983.

22. Sun, W. and Ya-Xiang, X. *Optimization Theory and Methods: Nonlinear programming*, Springer, 2006.

23. Vogel, T. *A free boundary problem arising from galvanizing process*, SIAM J. Math. Mech. Anal., 16 (1985) 970–979.

24. Young, L. C. *Lectures on the Calculues of Variations and Optimal Control Theory*, W.B. Sunders Company, 1969.