Iranian Journal of Numerical Analysis and Optimization
Vol. 12, No. 3 (Special Issue), 2022, pp 719–738
DOI:10.22067/ijnao.2022.78220.1168
https://ijnao.um.ac.ir/

How to cite this article
**Research Article**

# An efficient design for solving discrete optimal control problem with time-varying multi-delays

S.M. Abdolkhaleghzade, S. Effati* and S.A. Rakhshan

**Abstract**

The focus of this article is on the study of discrete optimal control problems (DOCPs) governed by time-varying systems, including time-varying delays in control and state variables. DOCPs arise naturally in many multistage control and inventory problems where time enters discretely in a natural fashion. Here, the Euler–Lagrange formulation (which are two-point boundary values with time-varying multi-delays) is employed as an efficient technique to solve DOCPs with time-varying multi-delays. The main feature of the procedure is converting the complex version of the discrete-time optimal control problem into a simple form of differential equations. Since the main problem is in discrete form, then the Euler–Lagrange equation changes to an algebraic system with initial and final conditions. The graphic representation of numerical simulation results shows that the proposed method can effectively and reliably solve DOCPs with time-varying multi-delays.

**AMS subject classifications (2020):** Primary 49M25; Secondary 93C55 , 37N35.

* Corresponding author

Received 15 August 2022; revised 3 October 2022; accepted 9 October 2022

Seyed Mostafa Abdolkhaleghzade
Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran. Email: mostafa.khaleghzade@gmail.com

Sohrab Effati
Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran.
Center of Excellence on Soft Computing and Intelligent Information Processing, Ferdowsi University of Mashhad, Mashhad, Iran Email: s-effati@um.ac.ir

Seyed Ali Rakhshan
Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran. Email: seyedalirakhshan@yahoo.com

**Keywords:** Discrete-time optimal control problem with time-varying delay, Euler–Lagrange equations, Pontryagin maximum principle.

## 1 Introduction

It is well known that discrete calculus is an important tool for describing natural phenomenas, which is expanded from classic calculus [18, 3, 7, 13]. By employing discrete calculus in optimal control problem (OCP), also well-known as discrete optimal control problem (DOCP), one can uniquely discover how to model natural phenomena. Discrete differential equations govern the dynamics of a dynamical system in a DOCP are one of the newest exciting mathematical challenges [16, 21, 22, 12].

The primary difference between continuous and discrete-time systems arises from the necessity to convert analog signals to digital values, as well as the time required for a computer system to calculate and execute the corrective action to the output.

A discrete time-control study on COVID-19 to address the quarantine and vital environmental loads has been explored in [2]. Mehraeen et al. [14] proposed an approach to obtain the optimal solutions based on the Hamilton–Jacobi–Isaacs equation for the discrete-time nonlinear system by using neural networks. In [11], the authors proposed an improved stability analysis method called a delay-mode-based functional method by weakening a condition in the Lyapunov–Krasovskii functional method. Adaptive dynamic programming as an effective intelligent control method has played an important role in seeking solutions for optimal control. Approximate dynamic programming techniques are used to solve the value function, and hence the optimal control policy, in discrete-time nonlinear OCPs having continuous state and action spaces; see([1, 5]). The adaptive dynamic programming algorithm was introduced in [20] for solving infinite-horizon undiscounted OCPs in discrete-time systems.

Discrete-time OCPs occur in many multi-stage control and scheduling problems, as may be expected. Originally, continuous-time OCPs can also be discretized suitably and subsequently formalized as discrete-time OCPs. Although due to the expansion of mathematical methods for solving continuous-time OCPs, this is not currently necessary. There are efficient methods for discrete-time OCPs in the literature.

To solve combined discrete-time OCPs and optimal parameter selection problems concerned with general constraints, a computational method was introduced in [4]. The DOCP for discrete-time linear system control constraint was investigated in [23], in which the control input is a one-dimensional variable whose range is contained in a bounded closed interval. Li, Teo, and Duan [10] considered a class of DDTOCP that contains nonlinear inequality constraints on both the state and control. In [19], authors discussed a delay optimal tracking control for discrete-time systems with quadratic performance indexes when they are affected by persisting disturbances.

This paper presents a novel approach to solving DOCP, including time-varying delays. A general formula of the structure for DOCP with time-varying delays can be considered as follows:

$$J(u(\cdot)) = \sum_{k=k_0}^{k_f-1} F(x_k, u_k, k), \tag{1}$$

subject to time-varying delay in a dynamic system

$$x_{k+1} = G(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k), \qquad k_0 \le k \le k_f, \tag{2}$$

with initial conditions:

$$\begin{aligned} x_k &= \phi_k, \qquad k_0 - \tau_{k_0} \le k \le k_0, \\ u_k &= \Theta_k, \qquad k_0 - \omega_{k_0} \le k \le k_0, \end{aligned} \tag{3}$$

where $x(\cdot)$ is the state variable vector, $u(\cdot)$ is the control variable vector, $k$ represents the time, $F$ and $G$ are given functionals, $k_0$ and $k_f$ are fixed, $\phi_k$ and $\Theta_k$ are specific functions, $\tau_k \ge 0$ is delay function for state variable, and $\omega_k \ge 0$ is delay function for control variable.

Whenever the associated dynamic system of DOCP depends on prior information at a particular time, it can be considered that it is the DOCP with time-varying delays. A realistic distributed assumption, instead of a traditional point-wise assumption, creates interesting cases of delays [17]. Discrete derivatives are essential for explaining physical phenomena with memories, as previous information about predators and even prey can have an impact on birth rates, rather than the current model of predator-prey relationships and hereditary traits; thus, DOCP with time-varying delays is applied to all physical processes with realistic distribution assumptions and experiences [6]. As it can be seen, the problem satisfying (1)–(3) includes the delay system. A delay system is a specific form of partial differential equation with infinite dimensions. Therefore, these types of mathematical problems are very important in engendering and physical sciences.

Generally, time-delays systems can be found in control systems, lasers, traffic models, metal cutting, transmission lines, epidemiology, cell cycle, protein, production population dynamics, and neuroscience. Therefore, it is important to propose a beneficial method for solving time-delays systems. Also, solving optimal control problems is complicated in normal mode, especially in non-linear modes. As a result, they become much more complicated in modes whose systems have time delays. So it is very valuable to work on such issues.

As a review of this paper, the framework of this paper is organized as follows:

Section 2 includes the proposed technique for solving DOCP with time-varying delays in state and control variables. Finally, Section 3 contains a number of numerical examples that demonstrate the model's effectiveness. We conclude in the last section.

## 2 Main results

There are several kinds of variational problems in calculus [9, 8]. Here, we propose the two-boundary value problem based on classical Euler–Lagrange equations to solve DOCP with time-varying delay. Therefore, we review some necessary definitions and theoretical concepts to derive our efficient technique.

**Definition 1.** Suppose that $x_k$ (respectively, $x_{k+1}$) takes on variations $\delta x_k$ (respectively, $\delta x_{k+1}$) from their optimal values $\bar{x}_k$ (respectively, $\bar{x}_{k+1}$) satisfying

$$x_k = \bar{x}_k + \delta x_k, \qquad x_{k+1} = \bar{x}_{k+1} + \delta x_{k+1}. \tag{4}$$

Now with these variations, the performance index (1) becomes

$$\begin{aligned} \hat{J} &= J(\bar{x}_{k_o}, k_o) = \sum_{k=k_0}^{k_f-1} F(\bar{x}_k, \bar{x}_{k+1}, k) \\ J &= J(x_{k_o}, k_o) = \sum_{k=k_0}^{k_f-1} F(\bar{x}_k + \delta x_k, \bar{x}_{k+1} + \delta x_{k+1}, k). \end{aligned} \tag{5}$$

**Definition 2.** The first variation $\delta J$ is the first order approximation of the increment $\Delta J = J - \hat{J}$. So, applying the Taylor series expansion of (5), we obtain

$$\delta J = \sum_{k=k_0}^{k_f-1} \frac{\partial F(\bar{x}_k, \bar{x}_{k+1}, k)}{\partial \bar{x}_k} \delta x_k + \frac{\partial F(\bar{x}_k, \bar{x}_{k+1}, k)}{\partial \bar{x}_{k+1}} \delta x_{k+1}. \tag{6}$$

**Theorem 1.** For $x_k$ to be a contender for an optimum, the first variation of $J$ must be zero on $x_k$, that is, $\delta J(x_k, \delta x_k) = 0$ for all admissible values of $\delta x_k$. This is a necessary condition. As a sufficient condition for minimum, we have the second variation $\delta^2 J > 0$, and for maximum, $\delta^2 J < 0$.

*Proof.* The researchers can consider the proof in detail in (see [8, p. 37]). □

**Lemma 1.** Suppose that $g_k$ is a function in which the domain and range are each a discrete set of values. If $g_k$ is a discrete function satisfying

$$\sum_{k=k_0}^{k_f} g_k \delta x_k = 0, \tag{7}$$

where the function $\delta x_k$ is discrete in the interval $[k_0, k_f]$, then $g_k = 0$ for every $k \in [k_0, k_f]$.

*Proof.* Let $g_{k_0} \neq 0$ for some $k_0$. Assume that $\delta x_s = 0$ if $s \neq k_0$ and $\delta x_{k_0} = 1$. Then $\delta$ is a discrete function. In addition, $\sum_k \delta x_k g_k = g_{k_0} = 0$, which is a contradiction. □

**Definition 3** (Gateaux derivative)**.** Suppose that $X$ and $Y$ are locally convex topological vector spaces, $U \subset X$ is open, and $f : X \to Y$. The Gateaux differential of $f$ at $u \in U$ in the direction $\psi \in X$, denoted by $df(u;\psi)$, is defined as

$$df(u;\psi) = \lim_{k\to 0} \frac{f(u+k\psi) - f(u)}{k} = \frac{d}{dk} f(u+k\psi)\big|_{k=0}. \tag{8}$$

If the limit (8) exists for every $\psi \in X$, then the function $f$ is called Gateaux differentiable at $u$ [15].

This paper investigates a structured strategy for finding the necessary optimality condition for the problem (1)–(3). It means that the DOCP with time-varying delays is analyzed in order to find the optimal control $u(\cdot)$ with the minimum performance index (1). Therefore, we investigate the necessary optimally condition of the DOCP with time-varying delays as follows.

**Theorem 2** (Necessary conditions for DOCP with time-varying delays)**.** Suppose that the DOCP defined by (1)–(3) with $k_0$, $x_{k_0}$, and $k_f$ is fixed. Also, suppose that $X$ is a locally convex topological vector spaces, and that $U \subset X$ is an open subset. In addition, assume that the following regularity conditions are satisfied:

- R1. $x_k, x_{k-\tau_k} \in X$;
- R2. $u_k, u_{k-\omega_k} \in U$;
- R3. $\tau_k : \mathbb{N} \longrightarrow \mathbb{N}$ and $\omega_k : \mathbb{N} \longrightarrow \mathbb{N}$ are natural-valued functions, and $\tau(\cdot), \omega(\cdot) \geq 0$ ;
- R4. $k_0 \in \mathbb{Z}$, $k_f \in \mathbb{Z}$, $\Theta : \mathbb{Z} \longrightarrow \mathbb{Z}$, and $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}$ are known;
- R5. $J$ is Gateaux differentiable at $u_k$;
- R6. $F$ and $G$ are locally convex topological vector spaces.

Then any solution $u(\cdot) \in U$ must satisfy the following conditions:

- N1. The state dynamics, for $k_0 \leq k \leq k_f$:

$$x_{k+1} = G\Big(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k\Big), \qquad k_0 \leq k \leq k_f. \tag{9}$$

N2. The adjoint dynamics:

$$\begin{cases} \dfrac{\partial F}{\partial x_k} - \lambda_k + \lambda_{k+1}^T \dfrac{\partial G}{\partial x_k} + \lambda_{k+1}^T \psi_k = 0,\ k > \tau_k, \\ \dfrac{\partial F}{\partial x_k} - \lambda_k + \lambda_{k+1}^T \dfrac{\partial G}{\partial x_k} = 0, \qquad\qquad O.W., \end{cases} \tag{10}$$

where

$$\begin{aligned} &\psi_k = \frac{\partial G}{\partial x_{k-\tau_k}}, \\ &F = F(x_k, u_k, k), \\ &G = G\Big(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k\Big). \end{aligned}$$

N3. The optimal control dynamics:

$$\begin{cases} \dfrac{\partial F}{\partial u_k} + \lambda_{k+1}^T \dfrac{\partial G}{\partial u_k} + \lambda_{k+1}^T \eta_k = 0 \qquad ,\ k > \omega_k, \\ \dfrac{\partial F}{\partial u_k} + \lambda_{k+1}^T \dfrac{\partial G}{\partial u_k} = 0, \qquad\qquad O.W., \end{cases} \tag{11}$$

where $\eta_k = \dfrac{\partial G}{\partial u_{k-\omega_k}}$.

N4. The Boundary conditions:

$$x_k = \phi_k, \qquad k \le k_0, \tag{12}$$

$$u_k = \Theta_k, \qquad k \le k_0, \tag{13}$$

$$\frac{\partial \mathcal{L}\big(x_{k-1}, x_{k-\tau_{k-1}-1}, x_k, u_{k-1}, u_{k-\omega_{k-1}-1}, \lambda_k\big)}{\partial x_k}\Big|_{k=k_f} = 0. \tag{14}$$

*Proof.* The required condition for the DOCP with time-varying delays is found by utilizing the variational method. Suppose that

$$\bar{J}(u(\cdot)) = \sum_{k=k_0}^{k_f-1} F(x_k, u_k, k) + \lambda_{k+1}^T \Big( G(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k) - x_{k+1} \Big), \tag{15}$$

where $\lambda(\cdot)$ is the Lagrange multiplier. Let $\delta x_k$, $\delta u_k$, $\delta x_{k-\tau_k}$, $\delta u_{k-\omega_k}$, and $\delta \lambda_k$ be the variation of $x_k$, $u_k$, $x_{k-\tau_k}$, $u_{k-\omega_k}$, and $\lambda_k$, respectively. We then define a family of curves as follows:

$$\begin{cases} x_k^\epsilon = x_k + \epsilon\delta x_k, \\ x_{k+1}^\epsilon = x_{k+1} + \epsilon\delta x_{k+1}, \\ x_{k-\tau_k}^\epsilon = x_{k-\tau_k} + \epsilon\delta x_{k-\tau_k}, \\ u_k^\epsilon = u_k + \epsilon\delta u_k, \\ u_{k-\omega_k}^\epsilon = u_{k-\omega_k} + \epsilon\delta u_{k-\omega_k}, \\ \lambda_{k+1}^\epsilon = \lambda_{k+1} + \epsilon\delta\lambda_{k+1}. \end{cases} \tag{16}$$

Let

$$\begin{aligned} L(k) &= L\big(x_k, x_{k+1}, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k\big) \\ &= F(x_k, u_k, k) + \lambda_{k+1}^T\Big(G(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k) - x_{k+1}\Big), \end{aligned} \tag{17}$$

and

$$\begin{aligned} L^\epsilon(k) &= L(x_k^\epsilon, x_{k+1}^\epsilon, x_{k-\tau_k}^\epsilon, u_k^\epsilon, u_{k-\omega_k}^\epsilon, k) \\ &= F(x_k^\epsilon, u_k^\epsilon, k) + (\lambda_{k+1}^\epsilon)^T\Big(G(x_k^\epsilon, x_{k-\tau_k}^\epsilon, u_k^\epsilon, u_{k-\omega_k}^\epsilon, k) - x_{k+1}^\epsilon\Big). \end{aligned} \tag{18}$$

Also note that according to Definition 3, we get

$$\begin{aligned} \delta\bar{J}(u_k; \delta u_k) &= \lim_{\epsilon\to 0}\frac{J(u_k + \epsilon\delta u_k) - J(u_k)}{\epsilon} \\ &= \sum_{k=k_0}^{k_f}\lim_{\epsilon\to 0}\frac{L^\epsilon(k) - L(k)}{\epsilon} = \sum_{k_0}^{k_f}\frac{d}{d\epsilon}L^\epsilon(k)\mid_{\epsilon=0}. \end{aligned} \tag{19}$$

The variational of functional $\bar{J}(u(\cdot))$ is given as

$$\begin{aligned} \delta\bar{J}(u(\cdot)) = \sum_{k=k_0}^{k_f-1}\frac{d}{d\epsilon}L^\epsilon(k)\mid_{\epsilon=0} = \sum_{k=k_0}^{k_f-1}\Big[&\frac{\partial L^\epsilon(k)}{\partial x_k^\epsilon}\frac{dx_k^\epsilon}{d\epsilon} + \frac{\partial L^\epsilon(k)}{\partial x_{k+1}^\epsilon}\frac{dx_{k+1}^\epsilon}{d\epsilon} \\ &+ \frac{\partial L^\epsilon(k)}{\partial x_{k-\tau_k}^\epsilon}\frac{dx_{k-\tau_k}^\epsilon}{d\epsilon} + \frac{\partial L^\epsilon(k)}{\partial u_k^\epsilon}\frac{du_k^\epsilon}{d\epsilon} \\ &+ \frac{\partial L^\epsilon(k)}{\partial u_{k-\omega_k}^\epsilon}\frac{du_{k-\omega_k}^\epsilon}{d\epsilon} + \frac{\partial L^\epsilon(k)}{\partial \lambda_{k+1}^\epsilon}\frac{d\lambda_{k+1}^\epsilon}{d\epsilon}\Big]\mid_{\epsilon=0}. \end{aligned} \tag{20}$$

Also, according to (16), we have

$$\frac{dx_k^\epsilon}{d\epsilon} = \delta x_k, \quad \frac{dx_{k+1}^\epsilon}{d\epsilon} = \delta x_{k+1}, \quad \frac{du_k^\epsilon}{d\epsilon} = \delta u_k,$$

$$\frac{d\lambda^{\epsilon}_{k+1}}{d\epsilon} = \delta\lambda_{k+1}, \quad \frac{dx^{\epsilon}_{k-\tau_k}}{d\epsilon} = \delta x_{k-\tau_k}, \frac{du^{\epsilon}_{k-\omega_k}}{d\epsilon} = \delta u_{k-\omega_k}. \tag{21}$$

Therefore,

$$\begin{aligned}\delta\bar{J}(u(\cdot)) = \sum_{k=k_0}^{k_f-1} \Big[ & \frac{\partial L(k)}{\partial x_k}\delta x_k + \frac{\partial L(k)}{\partial x_{k+1}}\delta x_{k+1} + \frac{\partial L(k)}{\partial x_{k-\tau_k}}\delta x_{k-\tau_k} \\ & + \frac{\partial L(k)}{\partial u_k}\delta u_k + \frac{\partial L(k)}{\partial u_{k-\omega_k}}\delta u_{k-\omega_k} + \frac{\partial L(k)}{\partial \lambda_{k+1}}\delta\lambda_{k+1}\Big].\end{aligned} \tag{22}$$

Also, we get from (17) that

$$\begin{aligned}&\frac{\partial L(k)}{\partial x_k} = \frac{\partial F(x_k, u_k, k)}{\partial x_k} + \lambda^T_{k+1}\frac{\partial G(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k)}{\partial x_k}, \\ &\frac{\partial L(k)}{\partial x_{k-\tau_k}} = \lambda^T_{k+1}\frac{\partial G(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k)}{\partial x_{k-\tau_k}}, \\ &\frac{\partial L(k)}{\partial u_k} = \frac{\partial F(x_k, u_k, k)}{\partial u_k} + \lambda^T_{k+1}\frac{\partial G(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k)}{\partial u_k}, \\ &\frac{\partial L(k)}{\partial u_{k-\omega_k}} = \lambda^T_{k+1}\frac{\partial G(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k)}{\partial u_{k-\omega_k}}, \\ &\frac{\partial L(k)}{\partial \lambda_{k+1}} = G(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k) - x_{k+1}, \\ &\frac{\partial L(k)}{\partial x_{k+1}} = -\lambda_{k+1}.\end{aligned} \tag{23}$$

Also, we can rearrange the term, including $x_{k+1}$ in (22), as follows:

$$\begin{aligned}&\sum_{k=k_0}^{k_f-1} \frac{\partial \mathcal{L}(x_k, x_{k-\tau_k}, x_{k+1}, u_k, u_{k-\omega_k}, \lambda_{k+1})}{\partial x_{k+1}}\delta x_{k+1} \\ &= \frac{\partial \mathcal{L}\big(x_{k_f-1}, x_{k_f-\tau_{k-1}-1}, x_{k_f}, u_{k_f-1}, u_{k_f-\omega_{k-1}-1}, \lambda_{k_f}\big)}{\partial x_{k_f}}\delta x_{k_f} \\ &\quad - \frac{\partial \mathcal{L}\big(x_{k_0-1}, x_{k_0-\tau_{k-1}-1}, x_{k_0}, u_{k_0-1}, u_{k_0-\omega_{k-1}-1}, \lambda_{k_0}\big)}{\partial x_{k_0}}\delta x_{k_0} \\ &\quad + \sum_{k=k_0}^{k_f-1} \frac{\partial \mathcal{L}\big(x_{k-1}, x_{k-\tau_{k-1}-1}, x_k, u_{k-1}, u_{k-\omega_{k-1}-1}, \lambda_k\big)}{\partial x_k}\delta x_k \\ &= \Big[\frac{\partial \mathcal{L}(x_{k-1}, x_{k-\tau_{k-1}-1}, x_k, u_{k-1}, u_{k-\omega_{k-1}-1}, \lambda_k)}{\partial x_k}\delta x_k\Big] \Big|_{k=k_0}^{k=k_f} \\ &\quad + \sum_{k=k_0}^{k_f-1} \frac{\partial \mathcal{L}\big(x_{k-1}, x_{k-\tau_{k-1}-1}, x_k, u_{k-1}, u_{k-\omega_{k-1}-1}, \lambda_k\big)}{\partial x_k}\delta x_k.\end{aligned} \tag{24}$$

We then conclude the first variation of $\bar{J}(u(\cdot))$ from equations (22)–(24) as

$$\begin{aligned}\delta\bar{J}(u(\cdot)) =& \Big[\frac{\partial\mathcal{L}\big(x_{k-1}, x_{k-\tau_{k-1}-1}, x_k, u_{k-1}, u_{k-\omega_{k-1}-1}, \lambda_k\big)}{\partial x_k}\delta x_k\Big] \,|_{k=k_0}^{k=k_f}\\ &+\sum_{k=k_0}^{k_f-1}\Big(\frac{\partial L(k)}{\partial x_k}\delta x_k + \frac{\partial\mathcal{L}\big(x_{k-1}, x_{k-\tau_{k-1}-1}, x_k, u_{k-1}, u_{k-\omega_{k-1}-1}, \lambda_k\big)}{\partial x_k}\delta x_k\\ &+\frac{\partial L(k)}{\partial x_{k-\tau_k}}\delta x_{k-\tau_k} + \frac{\partial L(k)}{\partial u_k}\delta u_k + \frac{\partial L(k)}{\partial u_{k-\omega_k}}\delta u_{k-\omega_k} + \frac{\partial L(k)}{\partial \lambda_{k+1}}\delta\lambda_{k+1}\Big).\end{aligned} \tag{25}$$

Therefore, the first variation is obtained as follows:

$$\begin{aligned}\delta\bar{J}(u(\cdot)) =& \Big[\frac{\partial\mathcal{L}\big(x_{k-1}, x_{k-\tau_{k-1}-1}, x_k, u_{k-1}, u_{k-\omega_{k-1}-1}, \lambda_k\big)}{\partial x_k}\delta x_k\Big] \,|_{k=k_0}^{k=k_f}\\ &+\sum_{k=k_0}^{k_f-1}\Big[\Big(\frac{\partial F}{\partial x_k} - \lambda_k\Big)\delta x_k + \frac{\partial F}{\partial u_k}\delta u_k\\ &+\delta\lambda_{k+1}\big(G(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k) - x_{k+1}\big)\\ &+\lambda_{k+1}^T\Big(\frac{\partial G}{\partial x_k}\delta x_k + \frac{\partial G}{\partial u_k}\delta u_k + \frac{\partial G}{\partial x_{k-\tau_k}}\delta x_{k-\tau_k} + \frac{\partial G}{\partial u_{k-\omega_k}}\delta u_{k-\omega_k}\Big)\Big].\end{aligned} \tag{26}$$

Let

$$\psi_k = \frac{\partial G}{\partial x_{k-\tau_k}}, \qquad \eta_k = \frac{\partial G}{\partial u_{k-\omega_k}}. \tag{27}$$

Since $x_k$ is specified function for $k \leq k_0$, and $\tau_k : \mathbb{N} \longrightarrow \mathbb{N}$, then

$$\delta x_{k_i-\tau_{k_i}} = 0, \qquad \text{for all } k_i \in [k_0, k_f - 1] \textbf{ and } k_i - \tau_{k_i} \leq 0; \tag{28}$$

otherwise,

$$\lambda_{k_i+1}^T \psi_{k_i} = 0, \qquad k_i - \tau_{k_i} > 0. \tag{29}$$

Similar to equations (28) and (29), we have

$$\delta u_{k_i-\omega_{k_i}} = 0, \qquad \text{for all } k_i \in [k_0, k_f - 1] \textbf{ and } k_i - \omega_{k_i} \leq 0; \tag{30}$$

otherwise,

$$\lambda_{k_i+1}^T \eta_{k_i} = 0, \qquad k_i - \omega_{k_i} > 0. \tag{31}$$

Equation (26) can be rewritten as follows:

$$\begin{aligned}\delta\bar{J}(u(\cdot)) =& \Big[\frac{\partial\mathcal{L}\big(x_{k-1}, x_{k-\tau_{k-1}-1}, x_k, u_{k-1}, u_{k-\omega_{k-1}-1}, \lambda_k\big)}{\partial x_k}\delta x_k\Big] \,|_{k=k_0}^{k=k_f}\\ &+\sum_{k=k_0}^{k_f-1}\Big[\Big(\frac{\partial F}{\partial x_k} - \lambda_k + \lambda_{k+1}^T\frac{\partial G}{\partial x_k}\Big)\delta x_k + \Big(\frac{\partial F}{\partial u_k} + \lambda_{k+1}^T\frac{\partial G}{\partial u_k}\Big)\delta u_k\end{aligned}$$

$$+ \delta\lambda_{k+1}\Big(G(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k) - x_{k+1}\Big)$$
$$+ \lambda_{k+1}^T \psi_k \delta x_{k-\tau_k} + \lambda_{k+1}^T \eta_k \delta u_{k-\omega_k}\big]. \tag{32}$$

In (32), the coefficients $\delta\lambda_k$, $\delta x_k$, and $\delta u_k$ must be zero in order to gain the minimization of $\bar{J}(u(\cdot))$ and $J(u(\cdot))$. Also, Euler–Lagrange equations are derived from (29) and (31) as follows:

$$x_{k+1} = G\Big(x_k, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k\Big), \qquad k_0 \le k \le k_f, \tag{33}$$

$$\begin{cases} \dfrac{\partial F}{\partial x_k} - \lambda_k + \lambda_{k+1}^T \dfrac{\partial G}{\partial x_k} + \lambda_{k+1}^T \psi_k = 0, \; k - \tau_k > 0, \\ \dfrac{\partial F}{\partial x_k} - \lambda_k + \lambda_{k+1}^T \dfrac{\partial G}{\partial x_k} = 0, \qquad\qquad O.W., \end{cases} \tag{34}$$

$$\begin{cases} \dfrac{\partial F}{\partial u_k} + \lambda_{k+1}^T \dfrac{\partial G}{\partial u_k} + \lambda_{k+1}^T \eta_k = 0, \; k - \omega_k > 0, \\ \dfrac{\partial F}{\partial u_k} + \lambda_{k+1}^T \dfrac{\partial G}{\partial u_k} = 0, \qquad\qquad O.W., \end{cases} \tag{35}$$

with the following conditions:

$$x_k = \phi_k, \qquad k_0 - \tau_{k_0} \le k \le k_0, \tag{36}$$
$$u_k = \Theta_k, \qquad k_0 - \omega_{k_0} \le k \le k_0, \tag{37}$$
$$\frac{\partial \mathcal{L}\big(x_{k-1}, x_{k-\tau_{k-1}-1}, x_k, u_{k-1}, u_{k-\omega_{k-1}-1}, \lambda_k\big)}{\partial x_k} \;|_{k=k_f} = 0. \tag{38}$$

□

## 3 Numerical examples

Some of the proposed features, including the efficiency and applicability of the technique, are discussed in this section with numerical examples. Our first example uses a non-autonomous DOCP with a time-varying state variable to implement the suggested method. In the second example, we also present the results of solving an autonomous DOCP with constant delays for state and control variables by the introduced method, indicating that we can solve optimal control problems with delays efficiently by this method.

**Example 1.** Consider the following cost functional:

$$J = \sum_{k=0}^{14} \left( x_k^2 + u_k^2 \right), \tag{39}$$

subject to non-autonomous recursive equation with time-varying delays

$$x_{k+1} = A_k x_k + A_{1_k} x_{k-\tau_k} + B_k u_k, \qquad 0 \le k \le 14, \tag{40}$$

and the following condition

$$x_k = 1, \qquad k \le 0, \tag{41}$$

where $\tau_k$ is the delay function satisfying $\tau_k > 0$ for $0 \le k \le 14$, and $A_k = k$, $A_{1_k} = 1$, and $B_k = 1$. The approach presented in this article has been applied to solve the DOCP with time-varying delays (39)–(41). The numerical results of this example are shown when $\tau_k = 3 - k^2$. The Lagrange function $L$ is defined as follows:

$$\begin{aligned} L(k) =& L\big(x_k, x_{k+1}, x_{k-\tau_k}, u_k, k\big) \\ =& x_k^2 + u_k^2 + \lambda_{k+1}(x_k + k x_{k-3+k^2} + u_k - x_{k+1}). \end{aligned} \tag{42}$$

Therefore, the necessary conditions for the problem (39)–(41) are obtained as follows:

$$\begin{cases} x_{k+1} = x_k + k x_{k-\tau_k} + u_k, & 0 \le k \le 14, \\ \begin{cases} 2x_k - \lambda_k + \lambda_{k+1} k + \lambda_{k+1} = 0, \; k - 3 + k^2 \le 0, \\ 2x_k - \lambda_k + \lambda_{k+1} k = 0, \qquad O.W., \end{cases} \\ 2u_k + \lambda_{k+1} = 0, \quad 0 \le k \le 14. \end{cases} \tag{43}$$

Additionally, the following conditions contribute to obtain the solution:

$$x_k = 1, \qquad k \le 0, \tag{44}$$

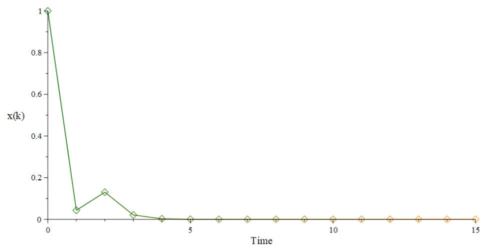
$$\lambda_{15} = 0. \tag{45}$$

The numerical results of state and control variables of Example 1 are shown in Figure 1 when $\tau_k = 3 - k^2$. Also, we show the convergence curve of the performance index function to illustrate the performance of the proposed method, in Figure 2.

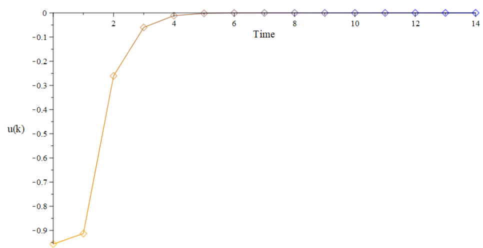
**Example 2.** Consider the following linear multi-delays time invariant problem to minimize the following functional:

$$J(u) = \frac{1}{2} \sum_{k=0}^{100} (x_k^2 + \frac{1}{2} u_k^2), \tag{46}$$

subject to




(a) State variable $x_k$




(b) Control variable $u_k$

Figure 1: Approximation of state and control variable of Example 1.

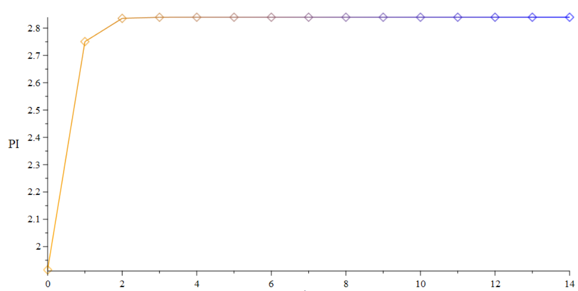



Figure 2: The convergence of performance index function of Example 1.

$$x_k = -x_k + x_{k-\tau} + u_k - \frac{1}{2}u_{k-\omega}, \qquad 0 \le k \le 100, \tag{47}$$

and the following condition

$$x_k = 1, \qquad k \le 0, \tag{48}$$
$$u_k = 0, \qquad k \le 0. \tag{49}$$

Note that in this example,

$$\tau = 6, \qquad \omega = 8.$$

The Lagrange function is defined as follows:

$$\begin{aligned} L(k) =& L\big(x_k, x_{k+1}, x_{k-\tau_k}, u_k, u_{k-\omega_k}, k\big) \\ =& \frac{1}{2}x_k^2 + \frac{1}{4}u_k^2 + \lambda_{k+1}(-x_k + x_{k-6} + u_k - \frac{1}{2}u_{k-8} - x_{k+1}). \end{aligned} \tag{50}$$

The following equations give the optimal solution:

$$\begin{cases} x_{k+1} = -x_k + x_{k-6} + u_k - \frac{1}{2}u_{k-8} \qquad 0 \le k \le 100, \\ \begin{cases} x_k - \lambda_k - \lambda_{k+1} + \lambda_{k+1} = 0, \; k-6 \le 0, \\ x_k - \lambda_k - \lambda_{k+1} = 0, \qquad O.W., \end{cases} \\ \begin{cases} \frac{1}{2}u_k + \lambda_{k+1} - \frac{1}{2}\lambda_{k+1} = 0, \; k-8 \le 0, \\ \frac{1}{2}u_k + \lambda_{k+1} = 0, \qquad O.W., \end{cases} \end{cases} \tag{51}$$

with the boundary conditions:

$$x_k = 1, \qquad k \le 0, \tag{52}$$
$$u_k = 0, \qquad k \le 0, \tag{53}$$
$$\lambda_{100} = 0. \tag{54}$$

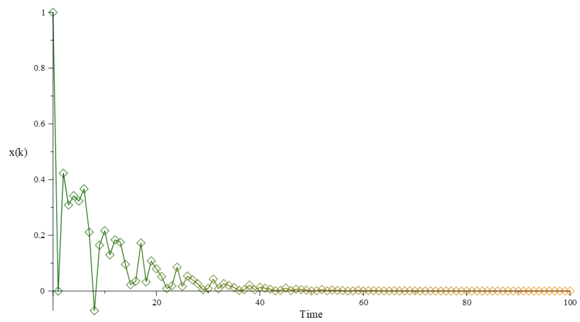
The analytic solution to this problem is not available. In Figure 3, the state and control variables of problem (46)–(48) are depicted. To demonstrate the performance of the proposed method, we show the convergence curve of the performance index function in Figure 4.

**Example 3.** Consider the following two-Dimensional nonlinear time-delays autonomous problem to minimize the following functional:

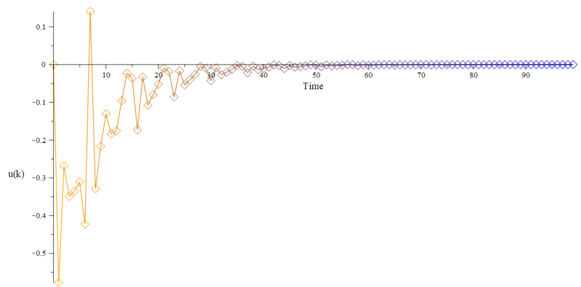
$$J(u_1(\cdot), u_2(\cdot)) = \sum_{k=0}^{k_f-1} (x_1^2(k) + x_2^2(k) + u_1^2(k) + u_2^2(k)), \tag{55}$$

subject to

$$x_1(k+1) = x_2^2(k-2) - 0.2u_1(k), \qquad 0 \le k \le k_f - 1, \tag{56}$$




(a) State variable $x_k$




(b) Control variable $u_k$

Figure 3: Approximation of state and control variable of Example 2

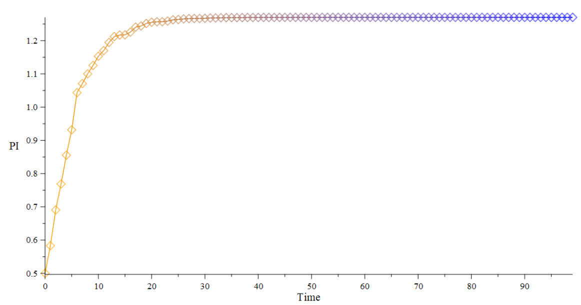



Figure 4: Convergence of performance index function of Example 2.

$$x_2(k+1) = x_1^2(k-2) - 0.2u_2(k), \qquad 0 \le k \le k_f - 1, \tag{57}$$

and the following conditions

$$x_1(k) = 1, \qquad -2 \le k \le 0, \tag{58}$$
$$x_2(k) = -1, \qquad -2 \le k \le 0. \tag{59}$$

The following equations give the optimal solution:

$$\begin{aligned} L(k) =& L\big(x_k, x_{k+1}, x_{k-\tau_k}, u_k, k\big) \\ =& x_1^2(k) + x_2^2(k) + u_1^2(k) + u_2^2(k) + \lambda_1(k+1)(x_2^2(k-2) - 0.2u_1(k)) \\ &+ \lambda_2(k+1)(x_1^2(k-2) - 0.2u_2(k)) - \lambda_1(k+1)x_1(k+1) \\ &- \lambda_2(k+1)x_2(k+1), \end{aligned} \tag{60}$$

From equation (34)–(38), we get

$$\begin{cases} \dfrac{\partial F}{\partial x_1(k)} - \lambda_1(k) + \lambda_1^T(k+1)\dfrac{\partial G}{\partial x_1(k)} = 0, \; k-2 \le 0, \\ \dfrac{\partial F}{\partial x_2(k)} - \lambda_2(k) + \lambda_2^T(k+1)\dfrac{\partial G}{\partial x_2(k)} = 0, \; 0 < k-2, \end{cases} \tag{61}$$

$$\begin{cases} 2x_1(k) - \lambda_1(k) + \lambda_1(k+1)(2x_2(k-2)) = 0, & k-2 \le 0, \\ 2x_1(k) - \lambda_1(k) = 0, & 0 < k-2, \end{cases} \tag{62}$$

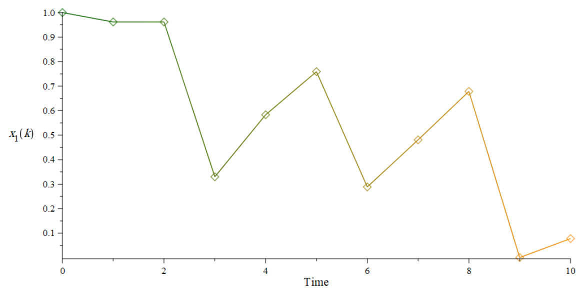
$$\begin{cases} 2x_2(k) - \lambda_2(k) + \lambda_2(k+1)(2x_1(k-2)) = 0, & k-2 \le 0, \\ 2x_2(k) - \lambda_2(k) = 0, & 0 < k-2, \end{cases} \tag{63}$$

$$\begin{cases} 2u_1(k) - 0.2\lambda_1(k+1) = 0, \; k-2 \le 0, \\ 2u_2(k) - 0.2\lambda_2(k+1) = 0, \; 0 < k-2, \end{cases} \tag{64}$$

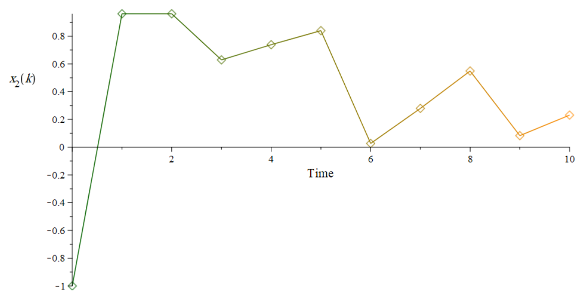
$$\begin{cases} x_1(k+1) = x_2^2(k-2) - 0.2u_1(k), \\ x_2(k+1) = x_1^2(k-2) - 0.2u_1(k), \end{cases} \tag{65}$$

with the boundary conditions:

$$x_1(k) = 1, \qquad -2 \le k \le 0, \tag{66}$$
$$x_2(k) = -1, \qquad -2 \le k \le 0. \tag{67}$$




(a) State variable $x_1(k)$



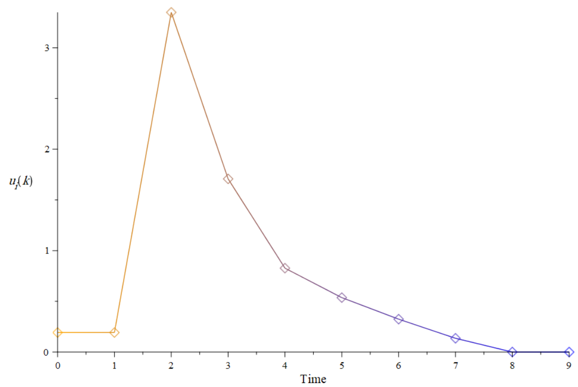

(b) State variable $x_2(k)$

Figure 5: Approximation of state variable of Example 3

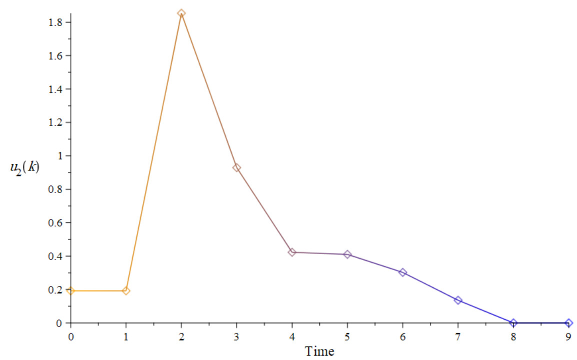
The analytic solution to this problem is not available. In Figure 5, the state variable of the problem (55)–(58) is depicted. Similar to the previous examples, we show the convergence curve of the performance index function in Figure 7. Also, the control variable is illustrated in Figure 6.

## 4 Conclusion

By introducing a new Lagrange multiplier, the original DOCP with time-varying delays problem has been transformed into DOCP problems without time-delay terms to avoid solving the DOCP problem with time-delay terms. In this regard, we utilized the discrete method to derive the new Euler–Lagrange delay formula with a two-point boundary to solve DOCP with time-varying delays. It is important to give a way to solve DOCP with time-varying delays, according to its application. In this technique, we utilized the variation method to construct the Euler–Lagrange formula with a two-point boundary in order to solve DOCP with time-varying delays, which has not been done before. Moreover, two illustrations were supplied to demonstrate how the technique could be used. The performance index influenced the DOCP problem of discrete time-delay systems, and also an approximate




(a) Control variable $u_1(k)$




(b) Control variable $u_2(k)$

Figure 6: Approximation of control variable of Example 3

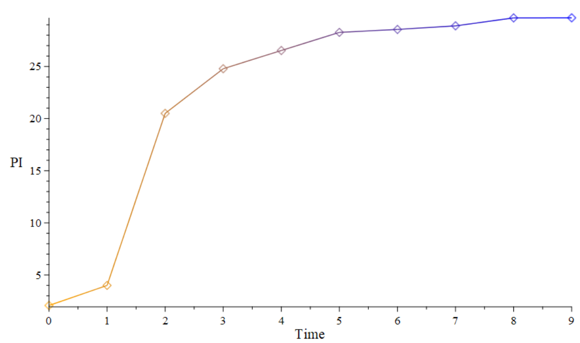



Figure 7: Convergence of performance index function of Example 3.

regulator was proposed. The simulation results showed that it is simple to implement and robust.

## Declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References

1. Al-Tamimi, A., Lewis, F.L., and Abu-Khalaf, M. *Discrete-time nonlinear HJB solution using approximate dynamic programming: Convergence proof.* IEEE Trans. Syst. Man Cybern. Part B (Cybernetics), 38(4), (2008) 943–949.

2. Essounaini, A., Labzai, A., Laarabi, H., and Rachik, M. *Mathematical modeling and optimal control strategy for a discrete time model of COVID-19 variants.* Commun. Math. Biol. Neurosci. (2022) Article-ID 2022.

3. Ferreira, R.A. *Discrete Calculus. In Discrete Fractional Calculus and Fractional Difference Equations* (pp. 1–14). Springer, Cham, 2022.

4. Fisher, M.E., and Jennings, L.S. *Discrete-time optimal control problems with general constraints.* ACM Trans. Math. Softw. (TOMS), 18(4), (1992) 401–413.

5. Haddad, W.M., Lee, J., and Bhat, S.P. *Asymptotic and finite time semistability for nonlinear discrete-time systems with application to network consensus.* IEEE Trans. Automat. Contr. (2022).

6. Hale, J. K., and Lunel, S.V. *Stability and control of feedback systems with time delays.* Int. J. Syst. Sci. 34(8-9), (2003) 497–504.

7. Hasegawa, Y. *Control problems of discrete-time dynamical systems* (Vol. 447) Springer, 2013.

8. Kot, M. *A first course in the calculus of variations* (Vol. 72). American Mathematical Society, 2014.

9. Lewis, F. L., Vrabie, D., and Syrmos, V.L. *Optimal control.* John Wiley and Sons, 2012.

10. Li, B., Teo, K. L., and Duan, G.R. *Optimal control computation for discrete time time-delayed optimal control problem with all-time-step inequality constraints.* Int. J. Innov. Comput. Inf. Control. 6(3), (2010) 521–532.

11. Li, X., Wang, R., Du, S., and Li, T. *An improved exponential stability analysis method for discrete-time systems with a time-varying delay.* Int. J. Robust Nonlinear Control. 32(2), (2022) 669–681.

12. Lu, J., Wei, Q., Wang, Z., Zhou, T., and Wang, F.Y. *Event-triggered optimal control for discrete-time multi-player non-zero-sum games using parallel control.* Inf. Sci. 584, (2022) 519–535.

13. Mariconda, C., and Tonolo, A. *Discrete calculus. Methods for counting* Springer, 2016.

14. Mehraeen, S., Dierks, T., Jagannathan, S., and Crow, M.L. *Zero-sum two-player game theoretic formulation of affine nonlinear discrete-time systems using neural networks.* IEEE Trans. Cybern. 43(6), (2012) 1641–1655.

15. Miller, F.P., Vandome, A.F., and McBrewster, J. *Gâteaux Derivative: Directional Derivative, Differential Calculus, World War I, Locally Convex Topological Vector Space, Topological Vector Space, Banach Space, Fréchet Derivative, Functional Derivative,* Alphascript Publishing, 2010.

16. Naz, R. *A current-value Hamiltonian approach to discrete-time optimal control problems in economic growth theory.* J. Differ. Equ. Appl. (2022) 1–11.

17. Park, J. H., Lee, T.H., Liu, Y., and Chen, J. *Dynamic systems with time delays: Stability and control.* Singapore: Springer, 2019.

18. Rovelli, C., and Zatloukal, V. *Natural discrete differential calculus in physics.* Found. Phys. 49(7), (2019) 693–699.

19. Tang, G., Sun, H., and Pang, H. *Approximately optimal tracking control for discrete time-delay systems with disturbances.* Prog. Nat. Sci. 18(2), (2008) 225–231.

20. Wei, Q., Liu, D., and Lin, H. *Value iteration adaptive dynamic programming for optimal control of discrete-time nonlinear systems.* IEEE Trans. Cybern. 46(3), (2015) 840–853.

21. Xu, J., Wang, J., Rao, J., Zhong, Y., and Wang, H. *Adaptive dynamic programming for optimal control of discrete-time nonlinear system with state constraints based on control barrier function.* Int. J. Robust Nonlinear Control. 32(6), (2022) 3408–3424.

22. Zanma, T., Yamamoto, N., Koiwa, K. and Liu, K.Z. *Optimal control input for discrete-time networked control systems with data dropout.* IET Cyber-Phys. Syst.: Theory Appl. 2022.

23. Zhao, H., Chen, D., and Hou, L. *The optimal control of delay discrete-time linear system with control constraint.* IEEE ICCA 2010 (pp. 1699–1704). IEEE.

**How to cite this article**

S.M. Abdolkhaleghzade, S. Effati and S.A. Rakhshan An efficient design for solving discrete optimal control problem with time-varying multi-delay. *Iranian Journal of Numerical Analysis and Optimization*, 2022; 12(3 (Special Issue), 2022): 719-738. doi: 10.22067/ijnao.2022.78220.1168.