

# A successive iterative approach for two dimensional nonlinear Volterra-Fredholm integral equations

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## Abstract

In this paper, an iterative scheme for extracting approximate solutions of two dimensional Volterra-Fredholm integral equations is proposed. Considering some conditions on the kernel of the integral equation obtained by discretization of the integral equation, the convergence of the approximate solution to the exact solution is investigated. Several examples are provided to demonstrate the efficiency of the approach.

**Keywords:** Volterra-Fredholm integral equation; Iterative method; Discretization; Approximation.

## 1 Introduction

The integral equations provide important tools for modeling a wide range of phenomena and processes [14], and solving many problems in engineering and mechanics which are dependent on finding the solution of their integral equations. They are widely used in plasma physics [10], deblurring of two dimensional images [8, 20], solving applied boundary value problems [1] and Laplace's equations with boundary conditions [16]. Upon the importance of the integral equations, different numerical methods have been developed over the years to tackle them, such as time collocation and time discretization methods [6, 15], trapezoidal Nystrom method [11], Adomian decomposition method [9, 17] and successive iterative scheme [5] but few of them can be used for solving two dimensional integral equations such as two dimensional block pulse functions [3], finite difference inequalities [19], time-stepping methods [7] and block-by-block method [4].

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Studies on iterative approaches play an important role to accelerate convergence rate in solving any system of equations generated by discretizing mathematical models in science and engineering problems [5]. The objective of this study is to present an iterative approach for extracting approximate solutions of two dimensional Volterra-Fredholm integral equations as

$$u(x, t) = f(x, t) + \int_c^t \int_a^b k(x, t, y, z, u(y, z)) dy dz, \quad (x, t) \in D := [a, b] \times [c, d], \quad (1)$$

where  $f(x, t)$  (source function) and  $k(x, t, y, z, u)$  (kernel function) are the given analytical functions defined on  $D$  and  $D \times D \times \mathbb{R}$ , respectively. The existence and uniqueness of the solution for equation (1) are discussed in [12, 15]. This work can be considered as an extension of the method proposed in [5]. Note that, the present approach is applicable to a wide class of integral equations. The structure of the report is as follows. In Section 2 we transform the integral equation into a discretized form. Then, in Section 3, we introduce an successful numerical approach which is used subsequently for making up the solution algorithm in Section 4. Section 5 demonstrates the efficiency and advantages of the proposed algorithm whilst Section 6 concludes the paper.

## 2 Integral equation transformation

Let  $\Delta^{(1)} = \{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$ ,  $\Delta^{(2)} = \{c = t_0, t_1, \dots, t_{m-1}, t_m = d\}$  be equidistance partitions of  $[a, b]$  and  $[c, d]$ , respectively, where  $h_x = x_{i+1} - x_i$ ,  $i = 0, 1, \dots, n-1$  and  $h_t = t_{j+1} - t_j$ ,  $j = 0, 1, \dots, m-1$  are the discretization parameters of the partitions. Now, if  $u^*(x, t)$  be an analytical solution of (1), then for the partitions  $\Delta^{(1)}, \Delta^{(2)}$  on  $[a, b]$  and  $[c, d]$ , we have

$$u^*(x_i, t_j) = f(x_i, t_j) + \int_c^{t_j} \int_a^b k(x_i, t_j, y, z, u^*(y, z)) dy dz, \quad (2)$$

where  $i = 0, 1, \dots, n$  and  $j = 0, 1, \dots, m$ . In (2), the integral term can be estimated by a numerical method of integration, e.g. Newton-Cotes methods. Therefore, by taking equidistance partitions  $\Delta^{(1)}, \Delta^{(2)}$ , as above with  $h_y = y_{i+1} - y_i$ ,  $i = 0, 1, \dots, n-1$ ,  $h_z = z_{j+1} - z_j$ ,  $j = 0, 1, \dots, m-1$ , and also the weights  $w_i$ ,  $i = 0, 1, \dots, n$  and  $w'_{j,r}$ ,  $r = 0, 1, \dots, j$ , equality (2) can be written as,

$$u_{i,j}^* = f_{i,j} + \sum_{r=0}^j \sum_{l=0}^n w'_{j,r} w_l k(x_i, t_j, y_l, z_r, u_{l,r}^*) + O(h_y^\nu) + O(h_z^\mu), \quad (3)$$

where  $u_{i,j}^* = u^*(x_i, t_j)$ ,  $f_{i,j} = f(x_i, t_j)$ ,  $i = 0, 1, \dots, n$ ,  $j = 0, 1, \dots, m$ , and  $\nu, \mu$  depends upon the employed method of Newton-Cotes for estimating the integral in (2).

For partitions  $\Delta^{(1)}, \Delta^{(2)}$ , we consider a nonlinear equations system obtained by neglecting the truncation error of (2), as follows,

$$\xi_{i,j} = f_{i,j} + \sum_{r=0}^j \sum_{l=0}^n w'_{j_r} w_l k(x_i, t_j, y_l, z_r, \xi_{l,r}), \quad i = 0, 1, \dots, n, j = 0, 1, \dots, m, \quad (4)$$

and suppose that the exact solution of nonlinear system (4) are  $\xi_{i,j}^*$ ,  $i = 0, 1, \dots, n$ ,  $j = 0, 1, \dots, m$ . In the following proposition, we seek the conditions of vanishing  $|u_{i,j}^* - \xi_{i,j}^*|$ ,  $i = 0, 1, \dots, n$ ,  $j = 0, 1, \dots, m$ .

**proposition 2.1.** *Suppose,*

- (i)  $|u_{p,q}^* - \xi_{p,q}^*| = \max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} |u_{i,j}^* - \xi_{i,j}^*|$ ,
- (ii)  $k(x, t, y, z, u(y, z)) \in \bar{C}(D \times D \times \mathbb{R})$ ,
- (iii)  $k_u(x, t, y, z, u(y, z))$  exists on  $D \times D \times \mathbb{R}$  and  $\gamma < \frac{1}{(b-a)(d-c)}$ , where

$$\gamma = \sup_{\substack{x, y \in [a, b] \\ t, z \in [c, d]}} |k_u(x, t, y, z, u(y, z))|.$$

Then

$$|u_{p,q}^* - \xi_{p,q}^*| \leq \frac{|O(h_y^\nu)| + |O(h_z^\mu)|}{1 - \gamma(b-a)(d-c)}. \quad (5)$$

*Proof.* By (3) and (4), we have

$$\begin{aligned} u_{p,q}^* - \xi_{p,q}^* &= \sum_{r=0}^q \sum_{l=0}^n w'_{q_r} w_l (k(x_p, t_q, y_l, z_r, u_{l,r}^*) - k(x_p, t_q, y_l, z_r, \xi_{l,r}^*)) \\ &\quad + O(h_y^\nu) + O(h_z^\mu). \end{aligned}$$

According to (iii)

$$k(x_p, t_q, y_l, z_r, u_{l,r}^*) - k(x_p, t_q, y_l, z_r, \xi_{l,r}^*) = \frac{\partial k}{\partial u}(x_p, t_q, y_l, z_r, \eta_{l,r}^*)(u_{l,r}^* - \xi_{l,r}^*), \quad (6)$$

where for each  $l = 0, 1, \dots, n$ ,  $r = 0, 1, \dots, m$ ,  $\eta_{l,r}$  is a real number between  $u_{l,r}^*$  and  $\xi_{l,r}^*$ . Again by (iii) and (6), we conclude that

$$\begin{aligned} |u_{p,q}^* - \xi_{p,q}^*| &\leq \gamma \sum_{r=0}^q \sum_{l=0}^n w'_{q_r} w_l |u_{l,r}^* - \xi_{l,r}^*| + |O(h_y^\nu)| + |O(h_z^\mu)| \\ &\leq \gamma |u_{p,q}^* - \xi_{p,q}^*| \sum_{r=0}^q \sum_{l=0}^n w'_{q_r} w_l + |O(h_y^\nu)| + |O(h_z^\mu)|. \end{aligned}$$

Since in every Newton-Cotes formula  $\sum_{r=0}^q \sum_{l=0}^n w'_{q_r} w_l = (b-a)(d-c)$ ,

$$|u_{p,q}^* - \xi_{p,q}^*| \leq \frac{|O(h_y^\nu)| + |O(h_z^\mu)|}{1 - \gamma(b-a)(d-c)}.$$

□

Inequality (5) leads to the following corollary, **corollary 2.2.**  $|u_{p,q}^* - \xi_{p,q}^*|$  vanishes when  $h_y$  and  $h_z$  tend to zero.

Now, to find the approximate solution, one needs to solve nonlinear equation (4).

### 3 The successive numerical approach

Iterative methods are widely used for finding approximate solution of nonlinear systems of equations [21]. Borzabadi et. al. in [5] presented a successive substitution, similar to Gauss-Seidel method for solving one dimensional Fredholm integral equation. The nonlinear system of equations (4) has also a structure that permits us to approximate its solution by a similar successive iterative approach presented in [5]. Hereby we define an iterative process which leads to the sequence of matrices  $\{\xi^{(k)}\}$ . The components of the matrices satisfy the iteration formula,

$$\xi_{i,j}^{(k+1)} = f_{i,j} + \sum_{r=0}^j \sum_{l=0}^n w'_{j_r} w_l k(x_i, t_j, y_l, z_r, \xi_{l,r}^{(k)}), \quad (7)$$

where  $i = 0, 1, \dots, n, j = 0, 1, \dots, m$  and  $k = 0, 1, \dots$ . Though, the convergence scheme can be constructed for detecting approximate solution (4). However, we first study the conditions that guarantee the convergence of the sequence  $\{\xi^{(k)}\}$ .

**theorem 3.1.** *Considering assumptions of Proposition 2.1, the produced sequence  $\{\xi^{(k)}\}$  from the iteration process (7) tends to the exact solution of (4), say  $\xi^*$ , for any arbitrary initial matrix  $\xi^{(0)}$ .*

*Proof.* By (4) and (7) we have,

$$\xi_{i,j}^{(k+1)} - \xi_{i,j}^* = \sum_{r=0}^j \sum_{l=0}^n w'_{j_r} w_l (k(x_i, t_j, y_l, z_r, \xi_{l,r}^{(k)}) - k(x_i, t_j, y_l, z_r, \xi_{l,r}^*)),$$

and according to condition (iii) of Proposition 2.1,

$$\xi_{i,j}^{(k+1)} - \xi_{i,j}^* = \sum_{r=0}^j \sum_{l=0}^n w'_{j_r} w_l \frac{\partial k}{\partial u}(x_i, t_j, y_l, z_r, \eta_{l,r}^{(k)})(\xi_{l,r}^{(k)} - \xi_{l,r}^*),$$

where  $\eta_{l,r}^{(k)}$  is a real number between  $\xi_{l,r}^{(k)}$  and  $\xi_{l,r}^*$  for  $l = 0, 1, \dots, n$  and  $r = 0, 1, \dots, m$ . Thus one may obtain the following inequalities

$$\begin{aligned} |\xi_{i,j}^{(k+1)} - \xi_{i,j}^*| &\leq \max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} |\xi_{i,j}^{(k)} - \xi_{i,j}^*| \sum_{r=0}^j \sum_{l=0}^n w'_{j_r} w_l \left| \frac{\partial k}{\partial u}(x_p, t_q, y_l, z_r, \eta_{l,r}^{(k)}) \right| \\ &\leq \gamma \max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} |\xi_{i,j}^{(k)} - \xi_{i,j}^*| \sum_{r=0}^m \sum_{l=0}^n w'_r w_l, \end{aligned}$$

where  $i = 0, 1, \dots, n, j = 0, 1, \dots, m$ . By setting  $\lambda = \gamma(b-a)(d-c)$  we conclude that

$$\max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} |\xi_{i,j}^{(k+1)} - \xi_{i,j}^*| \leq \lambda \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} |\xi_{i,j}^{(k)} - \xi_{i,j}^*|.$$

By mathematical induction on  $k$ , we get

$$\max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} |\xi_{i,j}^{(k+1)} - \xi_{i,j}^*| \leq \lambda^k \max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} |\xi_{i,j}^{(0)} - \xi_{i,j}^*|,$$

for each  $k = 0, 1, \dots$ . Since  $0 < \lambda < 1$ , then,  $k \rightarrow +\infty$  implies that  $\max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} |\xi_{i,j}^{(k+1)} - \xi_{i,j}^*|$  vanishes.  $\square$

## 4 Algorithm of the approach

In this section, we propose an algorithm on the basis of the above discussions to solve the Volterra-Fredholm integral equation (1). This algorithm is presented in two stages, the initialization and the main steps.

### Initialization

Choose  $\epsilon > 0$ , and equidistance partitions  $\Delta^{(1)} = \{a = x_0 = y_0, x_1 = y_1, \dots, x_{n-1} = y_{n-1}, x_n = y_n = b\}$  on  $[a, b]$  with the step size  $h_x = x_{i+1} - x_i$ ,  $i = 0, 1, \dots, n-1$ , plus  $\Delta^{(2)} = \{c = t_0 = z_0, t_1 = z_1, \dots, t_{n-1} = z_{n-1}, t_n = z_n = d\}$  on  $[c, d]$  with the step size  $h_t = t_{j+1} - t_j$ ,  $j = 0, 1, \dots, m-1$  and an initial matrix  $\xi^{(0)}$ . Set  $k = 0$  and go to the main steps.

### Main steps

Step 1. Compute  $\xi^{(k+1)}$  by (6), and go to Step 2.

Step 2. Compute  $\max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} |\xi_{i,j}^{(k+1)} - \xi_{i,j}^{(k)}|$  and go to Step 3.

Step 3. If  $\max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} |\xi_{i,j}^{(k+1)} - \xi_{i,j}^{(k)}| < \epsilon$ , stop; Otherwise, set  $k = k + 1$  and go to step 1.

In the next section the advantages and the influence of the proposed approach in thrilling convergence rate of the solution for the problems is demonstrated via some examples.

## 5 Numerical examples

Suppose  $u^*(x, t)$  is the exact solution of Volterra-Fredholm integral equation (1) and  $\hat{\xi}_{i,j}$ ,  $i = 0, 1, \dots, n$ ,  $j = 0, 1, \dots, m$  is a solution obtained by applying the given algorithm with a known  $\epsilon > 0$  and partitions  $\Delta^{(1)}$  and  $\Delta^{(2)}$ . To compare the precision of the approximate solution, the discrete error function

$$e(x_i, t_j) = |u^*(x_i, t_j) - \hat{\xi}(x_i, t_j)|, \quad i = 0, 1, \dots, n, \quad j = 0, 1, \dots, m, \quad (8)$$

is established.

**Example 5.1.** In this example, we apply the developed method to a two dimensional Fredholm integral equation as follows [13],

$$u(x, t) = \frac{1}{(1+x+t)^2} - \frac{x}{6(1+t)} + \int_0^1 \int_0^1 \frac{x}{1+t} (1+y+z) u^2(y, z) dy dz.$$

This integral equation has analytical solution  $u(x, t) = \frac{1}{(1+x+t)^2}$  on  $[0, 1] \times [0, 1]$ . We take  $\epsilon = 10^{-6}$  and partitions with the discretization parameters  $h_x = \frac{1}{100}$  and  $h_t = \frac{1}{100}$ . The initial matrix  $\xi^{(0)} = \mathbf{0}$  is considered first to start the algorithm. In Table 1, one can see all acceptable values for error estimation (8) which is obtained by applying the developed algorithm to the illustrated equation.

Table 1: Error estimation for Example 5.1

$x \ t$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.000	0.000	0.000	0.000	0.000	0.000
0.2	$3.980 \times 10^{-6}$	$3.317 \times 10^{-6}$	$2.843 \times 10^{-6}$	$2.488 \times 10^{-6}$	$2.211 \times 10^{-6}$	$1.990 \times 10^{-6}$
0.4	$7.958 \times 10^{-6}$	$6.631 \times 10^{-6}$	$5.684 \times 10^{-6}$	$4.973 \times 10^{-6}$	$4.421 \times 10^{-6}$	$3.0979 \times 10^{-6}$
0.6	$1.193 \times 10^{-5}$	$9.943 \times 10^{-6}$	$8.522 \times 10^{-6}$	$7.457 \times 10^{-6}$	$6.629 \times 10^{-6}$	$5.966 \times 10^{-6}$
0.8	$1.590 \times 10^{-5}$	$1.325 \times 10^{-5}$	$1.136 \times 10^{-5}$	$9.940 \times 10^{-6}$	$8.835 \times 10^{-6}$	$7.952 \times 10^{-6}$
1.0	$1.987 \times 10^{-5}$	$1.656 \times 10^{-5}$	$1.420 \times 10^{-5}$	$1.242 \times 10^{-5}$	$1.104 \times 10^{-5}$	$9.937 \times 10^{-6}$

**Example 5.2.** In this example, we apply our method for the following two dimensional Fredholm integral equation [2],

$$u(x, t) = xe^{-t} - \frac{1}{2}t - \frac{7}{12}x + \frac{1}{3}xe^{-1} + \int_0^1 \int_0^1 (xy + te^z)u(y, z)dydz.$$

The analytical solution of this integral equation is  $u(x, t) = xe^{-t} + t$  on  $[0, 1] \times [0, 1]$ . By solving this equation, we observe that the proposed algorithm does not give rise to a convergent sequence. So, to overcome this shortcoming, we put  $[0, 0.1] \times [0, 0.1]$  in place of  $[0, 1] \times [0, 1]$ , where the conditions of Theorem 3.1 hold. Then

$$u(x, t) = xe^{-t} + \frac{1799}{2000}t - \frac{43}{120000}x + \frac{9}{100}te^{-0.1} + \frac{1}{3000}xe^{-1} + \int_0^{0.1} \int_0^{0.1} (xy + te^z)u(y, z)dydz.$$

Table 2 shows that in this region of integration, approximate solution tracks the exact one, almost precise.

Table 2: Error estimation for Example 5.2

$x t$	0.0	0.02	0.04	0.06	0.08	0.1
0.0	0.000	$3.598 \times 10^{-11}$	$7.197 \times 10^{-11}$	$1.080 \times 10^{-10}$	$1.439 \times 10^{-10}$	$1.799 \times 10^{-10}$
0.02	$3.178 \times 10^{-11}$	$6.776 \times 10^{-11}$	$1.037 \times 10^{-10}$	$1.397 \times 10^{-10}$	$1.757 \times 10^{-10}$	$2.117 \times 10^{-10}$
0.04	$6.356 \times 10^{-11}$	$9.954 \times 10^{-11}$	$1.355 \times 10^{-10}$	$1.715 \times 10^{-10}$	$2.075 \times 10^{-10}$	$2.435 \times 10^{-10}$
0.06	$9.533 \times 10^{-11}$	$1.313 \times 10^{-10}$	$1.673 \times 10^{-10}$	$2.033 \times 10^{-10}$	$2.393 \times 10^{-10}$	$2.752 \times 10^{-10}$
0.08	$1.271 \times 10^{-10}$	$1.631 \times 10^{-10}$	$1.991 \times 10^{-10}$	$2.351 \times 10^{-10}$	$2.710 \times 10^{-10}$	$3.070 \times 10^{-10}$
0.1	$1.589 \times 10^{-10}$	$1.949 \times 10^{-10}$	$2.308 \times 10^{-10}$	$2.668 \times 10^{-10}$	$3.028 \times 10^{-10}$	$3.388 \times 10^{-10}$

**Example 5.3.** In this example, we apply the proposed method to the following two dimensional Volterra integral equation [18],

$$u(x, t) = xsin(t)\left(1 - \frac{x^2 sin^2(t)}{9}\right) + \frac{x^6}{10}\left(\frac{sin(2t)}{2} - t\right) + \int_0^t \int_0^x (xy^2 + cos(z))u^2(y, z)dydz.$$

This integral equation has analytical solution  $u(x, t) = xsin(t)$  on  $[0, 1] \times [0, 1]$ . We take  $\epsilon = 10^{-6}$  and partitions with the discretization parameters  $h_x = \frac{1}{100}$  and  $h_t = \frac{1}{100}$ . The initial matrix  $\xi^{(0)} = \mathbf{0}$  is considered for starting the algorithm. Table 3 illustrates the precision of the approximate solution by showing the error criteria (7) corresponding to the given partition.

Table 3: Error estimation for Example 5.3

$x t$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	$3.906 \times 10^{-31}$	$2.651 \times 10^{-33}$	$1.716 \times 10^{-33}$	$7.961 \times 10^{-34}$	$2.432 \times 10^{-34}$	$4.102 \times 10^{-35}$
0.2	$3.227 \times 10^{-33}$	$1.722 \times 10^{-8}$	$8.025 \times 10^{-8}$	$2.172 \times 10^{-7}$	$4.268 \times 10^{-7}$	$6.769 \times 10^{-7}$
0.4	$4.711 \times 10^{-33}$	$8.889 \times 10^{-8}$	$2.616 \times 10^{-7}$	$5.724 \times 10^{-7}$	$1.025 \times 10^{-6}$	$1.569 \times 10^{-6}$
0.6	$1.199 \times 10^{-32}$	$2.928 \times 10^{-7}$	$7.087 \times 10^{-7}$	$1.338 \times 10^{-6}$	$2.213 \times 10^{-6}$	$3.293 \times 10^{-6}$
0.8	$5.513 \times 10^{-32}$	$7.761 \times 10^{-7}$	$1.742 \times 10^{-6}$	$3.063 \times 10^{-6}$	$4.852 \times 10^{-6}$	$7.135 \times 10^{-6}$
1.0	$3.906 \times 10^{-31}$	$1.833 \times 10^{-6}$	$4.003 \times 10^{-6}$	$6.847 \times 10^{-6}$	$1.067 \times 10^{-5}$	$1.571 \times 10^{-5}$

**Example 5.4.** In this example, we apply our method to a Volterra-Fredholm integral equation as follows [3],

$$u(x, t) = x^2 + xt - \frac{1}{15}xt^4 - \frac{1}{16}xt^5 + \int_0^t \int_0^1 xty^2z^2u(y, z)dydz.$$

This integral equation has analytical solution  $u(x, t) = x^2 + xt$  on  $[0, 1] \times [0, 1]$ . We take  $\epsilon = 10^{-6}$  and partitions with the discretization parameters  $h_x = \frac{1}{100}$  and  $h_t = \frac{1}{100}$ . The initial matrix  $\xi^{(0)} = \mathbf{0}$  is considered for starting the algorithm. Table 4 exhibits good error values by applying the developed algorithm.

Table 4: Error estimation for Example 5.4

$x \ t$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.000	0.000	0.000	0.000	0.000	0.000
0.2	0.000	$4.063 \times 10^{-8}$	$2.567 \times 10^{-7}$	$9.000 \times 10^{-7}$	$2.425 \times 10^{-6}$	$5.598 \times 10^{-6}$
0.4	0.000	$8.126 \times 10^{-8}$	$5.133 \times 10^{-7}$	$1.800 \times 10^{-6}$	$4.850 \times 10^{-6}$	$1.119 \times 10^{-5}$
0.6	0.000	$1.219 \times 10^{-7}$	$7.699 \times 10^{-7}$	$2.700 \times 10^{-6}$	$7.276 \times 10^{-7}$	$1.679 \times 10^{-5}$
0.8	0.000	$1.625 \times 10^{-7}$	$1.027 \times 10^{-6}$	$3.600 \times 10^{-6}$	$9.701 \times 10^{-6}$	$2.239 \times 10^{-5}$
1.0	0.000	$2.031 \times 10^{-7}$	$1.283 \times 10^{-6}$	$4.500 \times 10^{-6}$	$1.213 \times 10^{-5}$	$2.799 \times 10^{-5}$

## 6 Conclusions

In this paper, an iterative approach for obtaining approximate solutions for two dimensional Volterra-Fredholm integral equations, considering some special conditions on the kernel, as continuous differentiability of kernel, is proposed. Theorem 3.1 provides a sufficient condition for convergence of the approach, but it is not necessary. Therefore, Examples 5.1, 5.3, and 5.4 show that, despite the lack of conditions, convergence of the proposed method holds for a class of two dimensional Volterra-Fredholm integral equations. Also the changing in problem for holding conditions of Theorem 3.1 lead to the convergence of the method, as it is described in Example 5.2. The validity and efficiency of the proposed scheme is demonstrated on the examples included.

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