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Research Article

A fuzzy solution approach to multi-objective fully triangular fuzzy optimization problem

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Abstract

Numerous optimization problems comprise uncertain data in practical circumstances and such uncertainty can be suitably addressed using the concept of fuzzy logic. This paper proposes a computationally efficient solution methodology to generate a set of fuzzy non-dominated solutions of a fully fuzzy multi-objective linear programming problem, which incorporates all its parameters and decision variables expressed in form of triangular fuzzy

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numbers. The fuzzy parameters associated with the objective functions are transformed into interval forms by utilizing the fuzzy-cuts, which subsequently generates the equivalent interval valued objective functions. The concept of centroid of triangular fuzzy numbers derives the deterministic form of the constraints. Furthermore, the scalarization process of weighting sum approach and certain concepts of interval analysis are used to generate the fuzzy non-dominated solutions from which the compromise solution can be determined based on the corresponding real valued expressions of fuzzy optimal objective values resulted due to the ranking function. Three numerical problems and one practical problem are solved for illustration and validation of the proposed approach. The computational results are also discussed as compared to some existing methods.

AMS subject classifications (2020): 90C05; 90C29; 90C70.

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1 Introduction

Linear optimization has several applications in context of decision making real world problems but the associated data often exist in ambiguous state, which cannot be defined or categorized exactly. To manage such data with uncertainty, the concept of fuzzy logic is appropriately useful in formulating practical problems into suitable fuzzy optimization models. The concept of fuzzy was first introduced by Zadeh [31], which is rapidly implemented in numerous fields of applications including business, economics, management, engineering, health care, and many more. A crisp form of linear programming problem (LPP) usually exists in the following form:

$Max\ Z = \sum_{j=1}^{n} c_j x_j$

subject to

$\sum_{j=1}^{n} a_{ij} x_j (\leq, =, \geq) b_i, \quad i = 1, 2, \ldots, m$

$x_j \geq 0$, where $x = (x_j) \in \mathbb{R}^n, c_j, a_{ij}, b_i \in \mathbb{R}$.

The linear optimization that comprises all the technological coefficients, resource constants, and decision variables expressed in form of fuzzy numbers, is interpreted as fully fuzzy linear programming problem (FFLPP) [18, 19]. Practical problems of several fields incorporate multiple conflicting objective functions that need simultaneous optimization over some common constraints; such problems are mathematically modeled as multi-objective optimization problems (MOOP) [7]. An MOOP comprises a set of non-dominated solutions [21] from which the decision-maker (DM) determines the compromise solution. Lotfi et al. [19] proposed a method to solve FFLPP using the lexicography method where the associated symmetric triangular fuzzy numbers (TFNs) are approximated by nearest symmetric triangular numbers to derive its symmetric fuzzy solution. Das, Mandal and Edalatpanah [10] derived a method to solve FFLPP based on multi-objective LPP, and lexicographic ordering approach where all its parameters are considered as trapezoidal fuzzy numbers. Ezzati, Khorram, and Enayati [12] developed a method to solve FFLPP, which converts the fuzzy problem into a multi-objective LPP (MOLPP) and found its solution using lexicographic method. Hosseinzadeh and Edalatpanah [15] proposed a method of solution for FFLPP using lexicography method where the parameters are considered as L-R fuzzy numbers. Kumar, Kaur, and Singh [18] developed a method of solution for FFLPP with equality constraints using the ranking functions to derive the fuzzy optimal solution whereas Allahviranloo et al.[2] proposed another solution approach for FFLPP using a new linear ranking function for defuzzification and found its equivalent form based on some proposed theoretical results. Ebrahimnejad [11] developed a solution approach for FFLPP, assuming the parameters as non-negative TFNs where FFLPP is formulated into tri-objective LPP. Khan, Ahmad, and Maan [17] proposed a modified version of simplex method for solving FFLPP, which utilizes both the concepts of ranking function and gauss elimination method to derive the optimal solution. Khalili, Nasseri, and Taghi-Nezhad [16] developed a fuzzy interactive approach based on the concept of membership function to solve a fully fuzzy mixed integer LPP. Daneshrad and Jafari [8] proposed an algorithm to find out the non-dominated solutions of an FFLPP by considering its parameters as symmetric trapezoidal fuzzy numbers. Das [9] proposed a modified algorithm to solve and derive fuzzy optimal solutions of an FFLPP containing equality constraints. Arana-Jiménez [3] proposed an algorithm to determine the non-dominated solutions of an FFLPP involving TFNs without applying ranking functions but an equivalent MOLPP is formulated, which generates the corresponding fuzzy optimal solutions. Pérez-Cañedo et al. [24] developed a solution approach for fully fuzzy linear regression problem comprising LR fuzzy numbers where this problem is transformed into an equivalent fully fuzzy MOLPP and fuzzy goal programming is used to find its solution. Bas and Ozkok [6] developed an iterative approach to solve a fully fuzzy linear fractional programming problem (LFPP) i.e., objective function exists as the ratio of linear functions, by transforming it to crisp MOLPP.

Some practical problems can be designed as optimization models with multiple goals that can be mathematically expressed as the problems of multi-objective optimization by the DMs. Fully fuzzy multi-objective LPP(FFMOLPP) has multiple conflicting linear objective functions with all its parameters and variables both in the objective functions and constraints are expressed as fuzzy numbers. Jimenez [4] proposed a method to solve FFMOLPP and derived fuzzy Pareto optimal solutions without using any ranking functions. Hop [28] studied the relationship between fuzzy numbers and solved a fully fuzzy multi-objective decision making problem based on the concepts of absolute fuzzy and relative fuzzy dominant degrees. Temelcan, Gonce Kocken, and Albayrak [27] also proposed an algorithm to solve an FFMOLPP where game theory approach was used for determining proper weights and solving the weighted LPP to find a fuzzy compromise solution. Pérez-Cañedo, Verdegay, and Miranda Perez [25] proposed fuzzy epsilon constraint method to derive fuzzy Pareto optimal solutions of FFMOLPP using lexicographic ranking process. Hamadameen and Hassan [14] developed a compromise solution algorithm to solve FFMOLPP by using revised simplex and Gaussian elimination methods. Sharma and Aggarwal [26] proposed a solution algorithm for FFMOLPP involving LR flat fuzzy numbers as the coefficients and decision variables, which converts the fuzzy problem into a crisp LPP using the concepts of scalarization methods and nearest interval approximation for fuzzy numbers. Yang, Cao, and Lin [30] proposed a method based on lexicographic order relation to solve an FFMOLPP with TFNs by converting it into multi-level MOLPP. Arana-Jiménez [5] developed an algorithm to obtain the fuzzy Pareto solutions of an FFMOLPP by using fuzzy arithmetics and partial orders to formulate its associated crisp MOLPP instead of applying ranking functions. Fathy and Hassanien [13] used a fuzzy harmonic approach to solve a fully fuzzy multi-level MOLPP by converting it into crisp MOLPP and derived its compromise solution. Malik and Gupta [20] proposed a method to solve fully triangular intuitionistic fuzzy multi-objective LFPP using goal programming approach and provided an application in e-education system. Niksirat [23] used nearest interval approximation concept to solve a fully fuzzy multi-objective transportation problem considering its parameters in uncertain forms.

MOLPP often arises in numerous practical problems of several domains comprising ambiguous information. In view of this, the paper studies FFMOLPP with the following major objectives:

- Develop a computationally efficient solution methodology to solve FFMOLPP in triangular fuzzy environment.
- Generate a set of fuzzy Pareto optimal(fuzzy non-dominated) solutions of FFMOLPP so that DM gets a choice to determine the compromise solution.

As it is observed, most of the existing methods deal with single objective fully fuzzy LPP or solve FFMOLPP using ranking function for converting it into crisp MOLPP or derives only one solution as the compromise solution of FFMOLPP. However, this paper studies a multi-objective FFLPP with all triangular fuzzy parameters and derives multiple fuzzy non-dominated solutions from which DM can choose the compromise solution. The proposed solution approach is also simple to formulate and computationally efficient, which is verified through comparisons of the computational results of numerical problems with some existing methods. The proposed study utilizes multiple concepts to solve FFMOLPP such as fuzzy α-cuts and centroid concept of TFNs, concept of interval analysis, which converts interval valued to real valued objective functions, weighting sum approach and ranking function for TFNs. Different values of fuzzy cuts maintain different degrees of satisfaction for the fuzzy parameters involved in the objective functions whereas the centroid concept of TFNs makes a better comparison in between both sides of the fuzzy valued constraints. This is a novel approach in context of utilizing the aforesaid concepts to efficiently solve and derive fuzzy Pareto optimal solutions of FFMOLPP.

This paper is organized as follows: Including introduction in Section 1, Section 2 incorporates the discussions on some preliminary concepts. Section 3 comprises the problem formulation of FFMOLPP and a description of the proposed methodology along with the presentation of its algorithm and flowchart. Section 4 contains the solutions of three numerical and one practical problems of FFMOLPP using the proposed methodology and their comparative result analysis. Finally, Section 5 contains the concluding remarks of this paper.

2 Certain preliminary concepts

Definition 1. [32] Let U be the universal set; then the fuzzy set $\widetilde{T}$ on U is defined as follows:

$$\widetilde{T} = \{(x, \mu_{\widetilde{T}}(x)) : x \in U\},$$

where $\mu_{\widetilde{T}} : U \to [0,1]$ and $\mu_{\widetilde{T}}(x) \in [0,1]$ represents the degree of membership of x in $\widetilde{T}$.

Definition 2. [32] A fuzzy set $\widetilde{T}$ is characterized as a fuzzy number if it satisfies the following properties:

- $\widetilde{T}$ must be defined over the real line $\mathbb{R}$.
- $\widetilde{T}$ must be normal, that is, here exists at least one element $x_0 \in \mathbb{R}$ such that $\mu_{\widetilde{T}}(x_0)$=1.
- $\widetilde{T}$ is a convex fuzzy set, that is, for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, $\mu_{\widetilde{T}}(\lambda x_1 + (1-\lambda)\, x_2) \geq \min\{\mu_{\widetilde{T}}(x_1), \mu_{\widetilde{T}}(x_2)\}$
- The membership function $\mu_{\widetilde{T}}$ of the fuzzy set $\widetilde{T}$ in $\mathbb{R}$ is piecewise continuous.
- Support of the fuzzy set $\widetilde{T}$ in $\mathbb{R}$, $S(\widetilde{T}) = \{x | \mu_{\widetilde{T}}(x) > 0\}$ must be bounded.

Definition 3. [32] A TFN on $\mathbb{R}^+$ remains in the form of $\widetilde{T} = (t_1, t_2, t_3)$, where $t_1 \leq t_2 \leq t_3$ and $t_1, t_2, t_3 \in \mathbb{R}^+$. The associated membership function $\mu_{\widetilde{T}}$ can be mathematically defined as follows with its geometrical interpretation stated in Figure 1:

$$\mu_{\widetilde{T}}(x) = \begin{cases} \dfrac{x - t_1}{t_2 - t_1}, & t_1 \leq x \leq t_2 \ , \\ \dfrac{t_3 - x}{t_3 - t_2}, & t_2 \leq x \leq t_3, \\ 0, & otherwise. \end{cases} \tag{1}$$

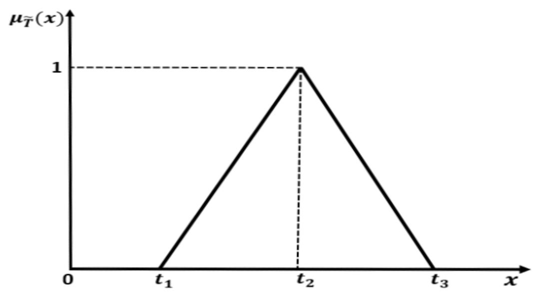

Figure 1: TFNs

Definition 4. Let $\widetilde{T}_1 = (t_1^l, t_1^m, t_1^n)$, $\widetilde{T}_2 = (t_2^l, t_2^m, t_2^n)$ be the TFNs on $\mathbb{R}$; then the arithmetic operations can be defined as follows:

- $\widetilde{T}_1 \oplus \widetilde{T}_2 = (t_1^l + t_2^l, t_1^m + t_2^m, t_1^n + t_2^n)$,

- $\widetilde{T}_1 \ominus \widetilde{T}_2 = (t_1^l - t_2^n, t_1^m - t_2^m, t_1^n - t_2^l)$,

- $\alpha \widetilde{T}_1 = \begin{cases} (\alpha t_1^l, \alpha t_1^m, \alpha t_1^n), & \alpha \geq 0, \\ (\alpha t_1^n, \alpha t_1^m, \alpha t_1^l), & \alpha < 0, \end{cases}$

- $\widetilde{T}_1 \otimes \widetilde{T}_2 = \begin{cases} (t_1^l t_2^l, t_1^m t_2^m, t_1^n t_2^n), & \widetilde{T}_1, \widetilde{T}_2 \in \mathbb{R}^+, \\ (t_1^n t_2^n, t_1^m t_2^m, t_1^l t_2^l), & \widetilde{T}_1, \widetilde{T}_2 \in \mathbb{R}^-, \\ (t_1^n t_2^l, t_1^m t_2^m, t_1^l t_2^n), & \widetilde{T}_1 \in \mathbb{R}^+, \widetilde{T}_2 \in \mathbb{R}^-, \\ (t_1^l t_2^n, t_1^m t_2^m, t_1^n t_2^l), & \widetilde{T}_1 \in \mathbb{R}^-, \widetilde{T}_2 \in \mathbb{R}^+, \end{cases}$

- $$\frac{\widetilde{T}_1}{\widetilde{T}_2} = \begin{cases} \left(\frac{t_1^l}{t_2^n}, \frac{t_1^m}{t_2^m}, \frac{t_1^n}{t_2^l}\right), & \widetilde{T}_1, \widetilde{T}_2 \in \mathbb{R}^+, \\ \left(\frac{t_1^n}{t_2^l}, \frac{t_1^m}{t_2^m}, \frac{t_1^l}{t_2^n}\right), & \widetilde{T}_1, \widetilde{T}_2 \in \mathbb{R}^-, \\ \left(\frac{t_1^n}{t_2^n}, \frac{t_1^m}{t_2^m}, \frac{t_1^l}{t_2^l}\right), & \widetilde{T}_1 \in \mathbb{R}^+, \widetilde{T}_2 \in \mathbb{R}^-, \\ \left(\frac{t_1^l}{t_2^l}, \frac{t_1^m}{t_2^m}, \frac{t_1^n}{t_2^n}\right), & \widetilde{T}_1 \in \mathbb{R}^-, \widetilde{T}_2 \in \mathbb{R}^+, \end{cases}$$

where the symbols $\oplus, \ominus$, and $\otimes$ denote addition, subtraction, and multiplication of fuzzy numbers, respectively.

Definition 5. [32] Fuzzy α-cut $\widetilde{T}_\alpha$ of the fuzzy set $\widetilde{T}$ can be defined as $\widetilde{T}_\alpha = \{x \in U : \mu_{\widetilde{T}}(x) \geq \alpha, \alpha \in [0,1]\}$. The fuzzy α-cut of a TFN $\widetilde{T} = (t_1, t_2, t_3)$ is defined as $\widetilde{T}_\alpha = [t_1 + \alpha(t_2 - t_1), t_3 - \alpha(t_3 - t_2)]$.

Definition 6. [22] Ranking function over the set of TFNs $\widetilde{T} = (t_1, t_2, t_3)$ in $\mathbb{R}$ can be defined as, $\mathcal{R}(\widetilde{T}) = \frac{t_1+2t_2+t_3}{4}$ for all $\widetilde{T} = (t_1, t_2, t_3) \in TFN(\mathbb{R})$.

Definition 7. [22] Let ABC be a triangle in xy-plane with its vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Then its centroid is the point of intersections of the medians, which can be defined as $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$. The centroid of a TFN $\widetilde{T} = (t_1, t_2, t_3)$ is defined as $\left(\frac{t_1 + t_2 + t_3}{3}, \frac{1}{3}\right)$.

Theorem 1. [29] The optimal solution of the real valued optimization, $Max_{x\in\delta}\ F^L(x) + F^U(x)$ is a non-dominated solution of the interval valued optimization, $Max_{x\in\delta}\ [F^L(x), F^U(x)]$.

3 Problem formulation and methodology developed for FFMOLPP

FFMOLPP frequently arises in many real world problems. The use of TFNs are common and often required to address the informational ambiguity in these optimization models.

3.1 FFMOLPP formulation:

Consider the following FFMOLPP with p number of objective functions:

$$\begin{aligned}
&Max\ \widetilde{Z}_1(\widetilde{X}) = \sum_{j=1}^{r} \widetilde{F}_{1j} \otimes \tilde{x}_j = \widetilde{F}_{11} \otimes \tilde{x}_1 \oplus \widetilde{F}_{12} \otimes \tilde{x}_2 \oplus \cdots \oplus \widetilde{F}_{1r} \otimes \tilde{x}_r \\
&Max\ \widetilde{Z}_2(\widetilde{X}) = \sum_{j=1}^{r} \widetilde{F}_{2j} \otimes \tilde{x}_j = \widetilde{F}_{21} \otimes \tilde{x}_1 \oplus \widetilde{F}_{22} \otimes \tilde{x}_2 \oplus \cdots \oplus \widetilde{F}_{2r} \otimes \tilde{x}_r \\
&\quad \vdots \\
&Max\ \widetilde{Z}_p(\widetilde{X}) = \sum_{j=1}^{r} \widetilde{F}_{pj} \otimes \tilde{x}_j = \widetilde{F}_{p1} \otimes \tilde{x}_1 \oplus \widetilde{F}_{p2} \otimes \tilde{x}_2 \oplus \cdots \oplus \widetilde{F}_{pr} \otimes \tilde{x}_r
\end{aligned} \tag{2}$$

subject to

$$\widetilde{\Delta}(\widetilde{X}) = \{\tilde{A}\widetilde{X}(\preccurlyeq, \approx, \succcurlyeq)\tilde{b}\} = \Big\{ \sum_{j=1}^{r} \tilde{a}_{ij}\tilde{x}_j(\preccurlyeq, \approx, \succcurlyeq)\tilde{b}_i \Big\},$$

where $\widetilde{F}_{kj}, \widetilde{X}=(\tilde{x}_j), \widetilde{a}_{ij}, \widetilde{b}_i \in TFN(\mathbb{R}^+)$ for $k = 1, 2, \ldots, p,\ j = 1, 2, \ldots, r,\ i = 1, 2, \ldots, q$ and $\preccurlyeq, \succcurlyeq, \approx$, represent the fuzziness in the inequalities and equality, respectively. Moreover, $\widetilde{Z}, \widetilde{X}, \widetilde{\Delta}$ represent the fuzzy or interval valued characteristics whereas Z, X, Δ represent the real valued characteristics throughout this paper.

Definition 8. [22] $\widetilde{X}^* \in \widetilde{\Delta}(\widetilde{X})$ is a fuzzy Pareto optimal solution or fuzzy non-dominated solution of the FFMOLPP (2) if and only if there is no another feasible solution $\widetilde{X}^{**} \in \widetilde{\Delta}(\widetilde{X})$ such that $\widetilde{Z}_r(\widetilde{X}^{**}) \succcurlyeq \widetilde{Z}_r(\widetilde{X}^*)$ for all $(r = 1, 2, 3, \ldots, p)$ and $\widetilde{Z}_s(\widetilde{X}^{**}) \succ \widetilde{Z}_s(\widetilde{X}^*)$ for at least one $s \in \{1, 2, \ldots, p\}$.

3.2 Developed solution methodology:

On substituting the values of decision variables and coefficients/constants, which exist in form of TFNs, the objective functions and constraints of the model (2) can be expressed as follows:

$$\begin{aligned}
&Max\ \widetilde{Z}_1(\widetilde{X}) = \sum_{j=1}^{n} (c^l_{1j}, c^m_{1j}, c^n_{1j}) \otimes (x^l_j, x^m_j, x^n_j) \\
&Max\ \widetilde{Z}_2(\widetilde{X}) = \sum_{j=1}^{n} (c^l_{2j}, c^m_{2j}, c^n_{2j}) \otimes (x^l_j, x^m_j, x^n_j) \\
&\quad \vdots
\end{aligned}$$

$$Max\ \widetilde{Z}_p(\widetilde{X}) = \sum_{j=1}^{n}(c_{pj}^l, c_{pj}^m, c_{pj}^n) \otimes (x_j^l, x_j^m, x_j^n) \tag{3}$$

subject to

$$\widetilde{\Delta}(\widetilde{X}) = \begin{cases} \sum\limits_{j=1}^{r}(a_{ij}^l, a_{ij}^m, a_{ij}^n)\otimes(x_j^l, x_j^m, x_j^n)(\preccurlyeq, \approx, \succcurlyeq)(b_i^l, b_i^m, b_i^n),\ i = 1, 2, \ldots, q, \\ \tilde{x}_j \succcurlyeq 0,\ j = 1, 2, \ldots, r. \end{cases}$$

In $\widetilde{\Delta}(\widetilde{X})$, $\tilde{x}_j = (x_j^l, x_j^m, x_j^n) \succcurlyeq 0$, that is , $\tilde{x}_j \in TFN(\mathbb{R}^+)$ implies $x_j^l \geq 0$, $x_j^m \geq x_j^l$, $x_j^n \geq x_j^m$ and $\tilde{a}_{ij} = (a_{ij}^l, a_{ij}^m, a_{ij}^n), \tilde{x}_j = (x_j^l, x_j^m, x_j^n) \in TFN(\mathbb{R}^+)$ implies $\tilde{a}_{ij} \otimes \tilde{x}_j = (a_{ij}^l x_j^l, a_{ij}^m x_j^m, a_{ij}^n x_j^n)$ based on the arithmetic operations in Definition 4. So, $\widetilde{\Delta}(\widetilde{X})$ is converted into $\widetilde{\Delta}_1(\widetilde{X})$ and the fuzzy model (3) is equivalently transformed into the following MOLPP using fuzzy α-cuts of TFNs in the objective functions:

$$\begin{aligned} Max\ \widetilde{Z}_1(\widetilde{X}) &= \sum_{j=1}^{n}[c_{1j}^l + \alpha(c_{1j}^m - c_{1j}^l), c_{1j}^n - \alpha(c_{1j}^n - c_{1j}^m)] \\ &\quad \times [x_j^l + \alpha(x_j^m - x_j^l), x_j^n - \alpha(x_j^n - x_j^m)] \\ Max\ \widetilde{Z}_2(\widetilde{X}) &= \sum_{j=1}^{n}[c_{2j}^l + \alpha(c_{2j}^m - c_{2j}^l), c_{2j}^n - \alpha(c_{2j}^n - c_{2j}^m)] \\ &\quad \times [x_j^l + \alpha(x_j^m - x_j^l), x_j^n - \alpha(x_j^n - x_j^m)] \\ &\vdots \\ Max\ \widetilde{Z}_p(\widetilde{X}) &= \sum_{j=1}^{n}[c_{pj}^l + \alpha(c_{pj}^m - c_{pj}^l), c_{pj}^n - \alpha(c_{pj}^n - c_{1j}^m)] \\ &\quad \times [x_j^l + \alpha(x_j^m - x_j^l), x_j^n - \alpha(x_j^n - x_j^m)] \end{aligned} \tag{4}$$

subject to

$$\widetilde{\Delta}_1(\widetilde{X}) = \begin{cases} \sum\limits_{j=1}^{r}(a_{ij}^l x_j^l, a_{ij}^m x_j^m, a_{ij}^n x_j^n)(\preccurlyeq, \approx, \succcurlyeq)(b_i^l, b_i^m, b_i^n), & i = 1, 2, \ldots, q, \\ x_j^l, x_j^m - x_j^l, x_j^n - x_j^m \geq 0, & j = 1, 2, \ldots, r. \end{cases}$$

To convert the triangular fuzzy valued inequalities $\widetilde{\Delta}_1(\widetilde{X})$ to its equivalent deterministic form, the concept of centroid of TFNs discussed in Definition 7 is utilized in the set of constraints $\widetilde{\Delta}_1(\widetilde{X})$.

$\sum\limits_{j=1}^{r}(a_{ij}^l x_j^l, a_{ij}^m x_j^m, a_{ij}^n x_j^n)(\preccurlyeq, \approx, \succcurlyeq)(b_i^l, b_i^m, b_i^n)$,

that is, $\Big(\sum\limits_{j=1}^{r} a_{ij}^l x_j^l, \sum\limits_{j=1}^{r} a_{ij}^m x_j^m, \sum\limits_{j=1}^{r} a_{ij}^n x_j^n\Big)(\preccurlyeq, \approx, \succcurlyeq)\Big(b_i^l, b_i^m, b_i^n\Big)$, that is, $\left(\frac{\sum\limits_{j=1}^{r} a_{ij}^l x_j^l + \sum\limits_{j=1}^{r} a_{ij}^m x_j^m + \sum\limits_{j=1}^{r} a_{ij}^n x_j^n}{3}, \frac{1}{3}\right)(\preccurlyeq, \approx, \succcurlyeq)\left(\frac{b_i^l + b_i^m + b_i^n}{3}, \frac{1}{3}\right)$,

that is, $\Big(\sum\limits_{j=1}^{r} a_{ij}^l x_j^l + \sum\limits_{j=1}^{r} a_{ij}^m x_j^m + \sum\limits_{j=1}^{r} a_{ij}^n x_j^n\Big)(\preccurlyeq, \approx, \succcurlyeq)\Big(b_i^l + b_i^m + b_i^n\Big)$.

Consequently, the following deterministic form of inequalities $\Delta_2(X)$ is derived for the constraints:

$$\Delta_2(X) = \begin{cases} \Big(\sum\limits_{j=1}^{r} a_{ij}^l x_j^l + \sum\limits_{j=1}^{r} a_{ij}^m x_j^m + \sum\limits_{j=1}^{r} a_{ij}^n x_j^n\Big)(\leq, =, \geq)(b_i^l + b_i^m + b_i^n), \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, q \\ x_j^l, x_j^m - x_j^l, x_j^n - x_j^m \geq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \ldots, r. \end{cases}$$

The objective functions of the model (4) can be transformed into interval valued forms. The mathematical formulation of $\widetilde{Z}_1(\widetilde{X})$ as interval valued function, is expressed below:

$$\begin{aligned} \widetilde{Z}_1(\widetilde{X}) =& \sum_{j=1}^{n}[(1-\alpha)c_{1j}^l + \alpha c_{1j}^m, (1-\alpha)c_{1j}^n + \alpha c_{1j}^m] \\ & \times [(1-\alpha)x_{1j}^l + \alpha x_{1j}^m, (1-\alpha)x_j^n + \alpha x_j^m] \\ =& \Big[\sum_{j=1}^{n}\{(1-\alpha)c_{1j}^l + \alpha c_{1j}^m\}\{(1-\alpha)x_{1j}^l + \alpha x_{1j}^m\}, \\ & \sum_{j=1}^{n}\{(1-\alpha)c_{1j}^n + \alpha c_{1j}^m\}\{(1-\alpha)x_j^n + \alpha x_j^m\}\Big] \\ =& [Z_1^L(X), Z_1^U(X)]. \end{aligned} \tag{5}$$

Similarly, the remaining objective functions can be expressed in interval valued forms,

$$\widetilde{Z}_2(\widetilde{X}) = [Z_2^L(X), Z_2^U(X)], \widetilde{Z}_3(\widetilde{X}) = [Z_3^L(X), Z_3^U(X)], \ldots, \widetilde{Z}_p(\widetilde{X}) = [Z_p^L(X), Z_p^U(X)]. \tag{6}$$

The FFMOLPP (2) can be expressed now as the following optimization model with interval valued objective functions and deterministic constraints:

$$\begin{aligned} &Max\ \widetilde{Z}_1(\widetilde{X}) = [Z_1^L(X), Z_1^U(X)]\\ &Max\ \widetilde{Z}_2(\widetilde{X}) = [Z_2^L(X), Z_2^U(X)]\\ &\quad\vdots\\ &Max\ \widetilde{Z}_p(\widetilde{X}) = [Z_p^L(X), Z_p^U(X)] \end{aligned} \tag{7}$$

subject to

$$\Delta_2(X) = \begin{cases} (\sum_{j=1}^{r} a_{ij}^l x_j^l + \sum_{j=1}^{r} a_{ij}^m x_j^m + \sum_{j=1}^{r} a_{ij}^n x_j^n)(\leq,=,\geq)(b_i^l + b_i^m + b_i^n), \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1,2,\dots,q, \\ x_j^l, x_j^m - x_j^l, x_j^n - x_j^m \geq 0, \qquad\qquad j = 1,2,\dots,r. \end{cases}$$

The multi-objective interval valued optimization (7) can be scalarized into the following single objective interval valued optimization using weighting sum approach with the weight vectors $(w_1, w_2, \dots, w_p)$:

$$\begin{aligned} Max\ \widetilde{Z}(\widetilde{X}) =& w_1[Z_1^L(X), Z_1^U(X)] + w_2[Z_2^L(X), Z_2^U(X)]\\ &+ \dots + w_p[Z_p^L(X), Z_p^U(X)] \end{aligned} \tag{8}$$

subject to

$$\Delta_2(X) = \begin{cases} (\sum_{j=1}^{r} a_{ij}^l x_j^l + \sum_{j=1}^{r} a_{ij}^m x_j^m + \sum_{j=1}^{r} a_{ij}^n x_j^n)(\leq,=,\geq)(b_i^l + b_i^m + b_i^n), \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1,2,\dots,q, \\ x_j^l, x_j^m - x_j^l, x_j^n - x_j^m \geq 0, w_k > 0, \sum_{k=1}^{p} w_k = 1, \\ \qquad\qquad\qquad\qquad\qquad\qquad j = 1,2,\dots,r. \end{cases}$$

The objective function of the optimization model (8) can be expressed in the following form.:

$$\begin{aligned} \widetilde{Z}(\widetilde{X}) =& \Big[w_1 Z_1^L(X) + w_2 Z_2^L(X) + \dots + w_p Z_p^L(X),\ w_1 Z_1^U(X)\\ &+ w_2 Z_2^U(X) + \dots + w_p Z_p^U(X)\Big]. \end{aligned}$$

That is, $\tilde{Z}(\widetilde{X}) = \Big[\sum_{i=1}^{p} w_i Z_i^L(X),\ \sum_{i=1}^{p} w_i Z_i^U(X)\Big]$.

The optimization model (8) can be further simplified as,

$$Max_{X\in\Delta_2(X)} \ \ \widetilde{Z}(\widetilde{X}) = \Big[\sum_{i=1}^{p} w_i Z_i^L(X), \ \sum_{i=1}^{p} w_i Z_i^U(X)\Big]. \tag{9}$$

Based on Theorem 1, it can be clearly proved that the optimal solution of the following model (10) is a non-dominated solution of the optimization model (9):

$$Max_{X\in\Delta_2(X)} \ \ Z(X) = \sum_{i=1}^{p} w_i Z_i^L(X) + \sum_{i=1}^{p} w_i Z_i^U(X). \tag{10}$$

The FFMOLPP (2) is equivalently converted into the optimization model (8), which is next simplified as the model (9). Since an optimal solution of (10) is a non-dominated solution of the model (9), so it is also considered as non-dominated for the FFMOLPP (2).

Finally, the FFMOLPP (2) can be transformed into the following single objective deterministic LPP on applying Theorem 1 and simplifying the model (10):

$$\begin{aligned} Max \ Z(X) =& w_1[Z_1^L(X) + Z_1^U(X)] + w_2[Z_2^L(X) + Z_2^U(X)] \\ &+ \cdots + w_p[Z_p^L(X) + Z_p^U(X)] \end{aligned} \tag{11}$$

subject to

$$\Delta_2(X) = \begin{cases} (\sum_{j=1}^{r} a_{ij}^l x_j^l + \sum_{j=1}^{r} a_{ij}^m x_j^m + \sum_{j=1}^{r} a_{ij}^n x_j^n)(\leq, =, \geq)(b_i^l + b_i^m + b_i^n), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, q, \\ x_j^l, x_j^m - x_j^l, x_j^n - x_j^m \geq 0, \\ w_k > 0, \sum_{k=1}^{p} w_k = 1, \qquad\qquad\qquad\qquad j = 1, 2, \ldots, r. \end{cases}$$

Solving the model (11), a set of non-dominated solutions can be generated for the FFMOLPP (2) by substituting different values of $\alpha \in [0, 1]$ and weights $w_k > 0$ with $\sum_{k=1}^{p} w_k = 1$.

3.3 Algorithm and flowchart presentation of the developed methodology:

An algorithm and a flowchart Figure 2 are presented below, describing the details of the proposed methodology.

Step 1. Use fuzzy α-cuts and arithmetic operations of TFNs in the objective functions and constraints of FFMOLPP (3), respectively.

Step 2. Simplify the objective functions in the form of the model (4) and the set of constraints as $\widetilde{\Delta}_1(\widetilde{X})$.

Step 3. Linearize the constraints of $\widetilde{\Delta}_1(\widetilde{X})$ as $\Delta_2(X)$ by using the concept of centroid of TFNs.

Step 4. Express the FFMOLPP as the optimization model (7) with interval valued objective functions and deterministic constraints.

Step 5. Use weighting sum method to scalarize the model (7) in form of single objective optimization (8).

Step 6. Apply Theorem 1 in order to transform the problem (8) to its equivalent model (11) comprising real valued objective function $Z(X)$ and deterministic constraints $\Delta_2(X)$.

Step 7. Solve the optimization model (11) substituting different values of $\alpha \in [0,1]$ and weights $w_k > 0$, $\sum_{k=1}^{p} w_k = 1$. The optimal solutions obtained are considered as the non-dominated solutions of the FFMOLPP (2).

Step 8. If DM is unsatisfied with the obtained non-dominated solutions, then reformulate and solve model (11) by changing $\alpha \in [0,1]$ and weight vector (w_k).

4 Numerical examples

In order to illustrate and validate the feasibility of the proposed methodology, two existing numerical and one practical problems in form of FFMOLPP are solved and the comparative study on its result analysis is incorporated.

Example 1. Consider the following FFMOLPP, which is initially solved by Aggarwal and Sharma [1], Temelcan, Gonce Kocken, and Albayrak [27]:

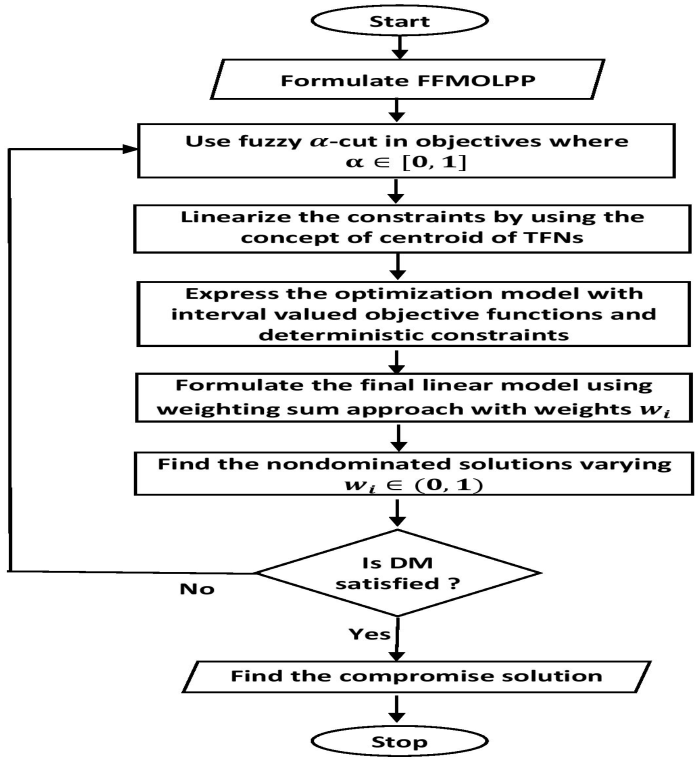

Figure 2: Flowchart of proposed solution approach

$$
\begin{aligned}
&Max\ \widetilde{Z}_1(\widetilde{X}) = (1,2,3)\otimes\tilde{x}_1 \oplus (2,4,5)\otimes\tilde{x}_2\\
&Max\ \widetilde{Z}_2(\widetilde{X}) = (2,3,4)\otimes\tilde{x}_1 \oplus (3,4,5)\otimes\tilde{x}_2\\
&\text{subject to}\\
&(0,1,2)\otimes\tilde{x}_1 \oplus (1,2,3)\otimes\tilde{x}_2 \preccurlyeq (1,10,27)\\
&(1,2,3)\otimes\tilde{x}_1 \oplus (0,1,2)\otimes\tilde{x}_2 \preccurlyeq (2,11,28)\\
&\tilde{x}_1, \tilde{x}_2 \succcurlyeq 0,
\end{aligned}
\tag{12}
$$

Solution:

According to the proposed methodology, the FFMOLPP (12) is equivalently transformed into the following optimization model with fuzzy constraints based on fuzzy α-cuts in objective functions and arithmetic operations of TFNs:

$$\begin{aligned}
Max\ \widetilde{Z}_1(\widetilde{X}) =&[1+\alpha, 3-\alpha][(1-\alpha)x_1^l+\alpha x_1^m, (1-\alpha)x_1^n+\alpha x_1^m]\\
&+[2+2\alpha, 5-\alpha][(1-\alpha)x_2^l+\alpha x_2^m, (1-\alpha)x_2^n+\alpha x_2^m]\\
Max\ \widetilde{Z}_2(\widetilde{X}) =&[2+\alpha, 4-\alpha][(1-\alpha)x_1^l+\alpha x_1^m, (1-\alpha)x_1^n+\alpha x_1^m]\\
&+[3+\alpha, 5-\alpha][(1-\alpha)x_2^l+\alpha x_2^m, (1-\alpha)x_2^n+\alpha x_2^m]
\end{aligned} \tag{13}$$

subject to

$$\begin{aligned}
&(x_2^l, x_1^m+2x_2^m, 2x_1^n+3x_2^n) \preccurlyeq (1,10,27)\\
&(x_1^l, 2x_1^m+x_2^m, 3x_1^n+2x_2^n) \preccurlyeq (2,11,28)\\
&x_1^l, x_2^l, x_1^m-x_1^l, x_2^m-x_2^l, x_1^n-x_1^m, x_2^n-x_2^m \geq 0.
\end{aligned}$$

Using centroid concept of TFNs in constraints and arithmetic operations, the model (13) is expressed as the following interval valued optimization with deterministic linear constraints:

$$\begin{aligned}
Max\ \widetilde{Z}_1(\widetilde{X}) =&[(1-\alpha^2)x_1^l+\alpha(1+\alpha)x_1^m+(1-\alpha)(2+2\alpha)x_2^l+\alpha(2+2\alpha)x_2^m,\\
&(3-\alpha)(1-\alpha)x_1^n+\alpha(3-\alpha)x_1^m+(1-\alpha)(5-\alpha)x_2^n+\alpha(5-\alpha)x_2^m]\\
Max\ \widetilde{Z}_2(\widetilde{X}) =&[(1-\alpha)(2+\alpha)x_1^l+\alpha(2+\alpha)x_1^m+(1-\alpha)(3+\alpha)x_2^l+\alpha(3+\alpha)x_2^m,\\
&(1-\alpha)(4-\alpha)x_1^n+\alpha(4-\alpha)x_1^m+(1-\alpha)(5-\alpha)x_2^n+\alpha(5-\alpha)x_2^m]
\end{aligned} \tag{14}$$

subject to

$$\begin{aligned}
&x_1^m+2x_1^n+x_2^l+2x_2^m+3x_2^n \leq 38\\
&x_1^l+2x_1^m+3x_1^n+x_2^m+2x_2^n \leq 41\\
&x_1^l, x_2^l, x_1^m-x_1^l, x_2^m-x_2^l, x_1^n-x_1^m, x_2^n-x_2^m \geq 0.
\end{aligned}$$

The FFMOLPP (14) is finally formulated as the following single objective optimization using the weight vector (w_1, w_2):

$$\begin{aligned}
Max\ Z(X) =&w_1[(1-\alpha^2)x_1^l+4\alpha x_1^m+(3-\alpha)(1-\alpha)x_1^n\\
&+(1-\alpha)(2+2\alpha)x_2^l+\alpha(7+\alpha)x_2^m+(1-\alpha)(5-\alpha)x_2^n]\\
&+w_2[(1-\alpha)(2+\alpha)x_1^l+6\alpha x_1^m+(1-\alpha)(4-\alpha)x_1^n\\
&+(1-\alpha)(3+\alpha)x_2^l+8\alpha x_2^m+(1-\alpha)(5-\alpha)x_2^n]
\end{aligned}$$

subject to (15)

$$\begin{aligned}
&x_1^m+2x_1^n+x_2^l+2x_2^m+3x_2^n \leq 38\\
&x_1^l+2x_1^m+3x_1^n+x_2^m+2x_2^n \leq 41\\
&x_1^l, x_2^l, x_1^m-x_1^l, x_2^m-x_2^l, x_1^n-x_1^m, x_2^n-x_2^m \geq 0\\
&w_1, w_2 > 0, w_1+w_2 = 1.
\end{aligned}$$

On solving problem (15) with different $\alpha \in [0,1]$ and weights $w_1, w_2 > 0, w_1 + w_2 = 1$, a set of optimal solutions are derived, which are considered as the non-dominated solutions of the FFMOLPP (12).
The triangular fuzzy optimal objective values are computed along with their corresponding real valued expressions using ranking function and these values are shown in the following Table 1.

Example 2. Consider the following FFMOLPP, which is initially solved by Yang, Cao, and Lin [30]:

$$\begin{aligned}
&Min\ \widetilde{Z}_1(\widetilde{X}) = (7,10,11)\otimes\tilde{x}_1 \oplus (8,10,13)\otimes\tilde{x}_2\\
&Min\ \widetilde{Z}_2(\widetilde{X}) = (2,3,4)\otimes\tilde{x}_1 \oplus (4,7,12)\otimes\tilde{x}_2\\
&\text{subject to}\\
&(1,2,4)\otimes\tilde{x}_1 \oplus (2,8,10)\otimes\tilde{x}_2 \approx (3,22,36)\\
&(2,3,6)\otimes\tilde{x}_1 \oplus (4,10,15)\otimes\tilde{x}_2 \approx (6,29,54)\\
&\tilde{x}_1, \tilde{x}_2 \succcurlyeq 0.
\end{aligned} \tag{16}$$

Solution:
The FFMOLPP (16) is formulated into the following optimization model using fuzzy α-cuts in objective functions and arithmetic operations of TFNs in constraints:

$$\begin{aligned}
Min\ \widetilde{Z}_1(\widetilde{X}) =& [7+3\alpha, 11-\alpha][(1-\alpha)x_1^l + \alpha x_1^m, (1-\alpha)x_1^n + \alpha x_1^m]\\
&+ [8+2\alpha, 13-3\alpha][(1-\alpha)x_2^l + \alpha x_2^m, (1-\alpha)x_2^n + \alpha x_2^m]\\
Min\ \widetilde{Z}_2(\widetilde{X}) =& [2+\alpha, 4-\alpha][(1-\alpha)x_1^l + \alpha x_1^m, (1-\alpha)x_1^n + \alpha x_1^m]\\
&+ [4+3\alpha, 12-5\alpha][(1-\alpha)x_2^l + \alpha x_2^m, (1-\alpha)x_2^n + \alpha x_2^m]\\
\text{subject to}&\\
&(x_1^l + 2x_2^l, 2x_1^m + 8x_2^m, 4x_1^n + 10x_2^n) \approx (3,22,36)\\
&(2x_1^l + 4x_2^l, 3x_1^m + 10x_2^m, 6x_1^n + 15x_2^n) \approx (6,29,54)\\
&x_1^l, x_2^l, x_1^m - x_1^l, x_2^m - x_2^l, x_1^n - x_1^m, x_2^n - x_2^m \geq 0.
\end{aligned} \tag{17}$$

The above model (17) is equivalently expressed into the following model comprising interval valued objective functions and deterministic linear constraints based on the centroid concept:

Table 1: Non-dominated solutions, fuzzy valued optimal objective functions and their real valued expressions of Example 1

α	w	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{Z}_1$	$\tilde{Z}_2$	$R(\tilde{Z}_1)$	$R(\tilde{Z}_2)$
0.1	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4)	(3.916667,3.916667, 3.916667)	(0,0,8.75)	(3.916667,7.8333, 55.5)	(7.8333,11.75, 59.4167)	18.7708	22.6875
	(0.8,0.2)	(0,0,0)	(0,0,12.667)	(0,0,63.335)	(0,0,63.335)	15.8338	15.8338
0.2	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4)	(4.88889,4.88889, 4.88889)	(3.88889,3.88889, 3.88889)	(12.6667,25.3333, 34.1111)	(21.4445,30.2222, 39)	24.3611	30.2222
	(0.8,0.2)	(3.916667,3.916667, 3.916667)	(0,0,8.75)	(3.916667,7.8333, 55.5)	(7.8333,11.75, 59.4167)	18.7708	22.6875
0.3	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(4.88889,4.88889, 4.88889)	(3.88889,3.88889, 3.88889)	(12.6667,25.3333, 34.1111)	(21.4445,30.2222, 39)	24.3611	30.2222
0.4	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(4.88889,4.88889, 4.88889)	(3.88889,3.88889, 3.88889)	(12.6667,25.3333, 34.1111)	(21.4445,30.2222, 39)	24.3611	30.2222
0.5	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(4.88889,4.88889, 4.88889)	(3.88889,3.88889, 3.88889)	(12.6667,25.3333, 34.1111)	(21.4445,30.2222, 39)	24.3611	30.2222
0.6	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(4.88889,4.88889, 4.88889)	(3.88889,3.88889, 3.88889)	(12.6667,25.3333, 34.1111)	(21.4445,30.2222, 39)	24.3611	30.2222
0.7	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(4.333333,4.333333 ,4.333333)	(0,5,5)	(4.333333,28.6667, 38)	(8.6667,33, 42.3333)	24.9167	29.25
0.8	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(4.333333,4.333333 ,4.333333)	(0,5,5)	(4.333333,28.6667, 38)	(8.6667,33, 42.3333)	24.9167	29.25
0.9	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4)	(4.333333,4.333333 ,4.333333)	(0,5,5)	(4.333333,28.6667, 38)	(8.6667,33, 42.3333)	24.9167	29.25
	(0.8,0.2)	(0,0,0)	(0,7.6,7.6)	(0,30.4,38)	(0,30.4,38)	24.7	24.7

$$
\begin{aligned}
Min\ \widetilde{Z}_1(\widetilde{X}) =& [(7+3\alpha)(1-\alpha)x_1^l + \alpha(7+3\alpha)x_1^m + (8+2\alpha)(1-\alpha)x_2^l + \alpha(8+2\alpha)x_2^m,\\
& (11-\alpha)(1-\alpha)x_1^n + \alpha(11-\alpha)x_1^m + (13-3\alpha)(1-\alpha)x_2^n + \alpha(13-3\alpha)x_2^m]\\
Min\ \widetilde{Z}_2(\widetilde{X}) =& [(2+\alpha)(1-\alpha)x_1^l + \alpha(2+\alpha)x_1^m + (4+3\alpha)(1-\alpha)x_2^l + \alpha(4+3\alpha)x_2^m,\\
& (4-\alpha)(1-\alpha)x_1^n + \alpha(4-\alpha)x_1^m + (12-5\alpha)(1-\alpha)x_2^n + \alpha(12-5\alpha)x_2^m]\\
&\text{subject to}\\
& x_1^l + 2x_1^m + 4x_1^n + 2x_2^l + 8x_2^m + 10x_2^n = 61\\
& 2x_1^l + 3x_1^m + 6x_1^n + 4x_2^l + 10x_2^m + 15x_2^n = 89\\
& x_1^l, x_2^l, x_1^m - x_1^l, x_2^m - x_2^l, x_1^n - x_1^m, x_2^n - x_2^m \geq 0.
\end{aligned} \tag{18}
$$

The FFMOLPP (16) is finally formulated into the following single objective optimization according to the proposed methodology:

$$
\begin{aligned}
Min\ Z(X) = & w_1[(7+3\alpha)(1-\alpha^2)x_1^l + \alpha(18+2\alpha)x_1^m + (11-\alpha)(1-\alpha)x_1^n\\
& + (8+2\alpha)(1-\alpha)x_2^l + \alpha(21-\alpha)x_2^m + (13-3\alpha)(1-\alpha)x_2^n]\\
& + w_2[(2+\alpha)(1-\alpha)x_1^l + 6\alpha x_1^m + (4-\alpha)(1-\alpha)x_1^n\\
& + (4+3\alpha)(1-\alpha)x_2^l + \alpha(16-2\alpha)x_2^m + (12-5\alpha)(1-\alpha)x_2^n]\\
\text{subject to}\\
& x_1^l + 2x_1^m + 4x_1^n + 2x_2^l + 8x_2^m + 10x_2^n = 61\\
& 2x_1^l + 3x_1^m + 6x_1^n + 4x_2^l + 10x_2^m + 15x_2^n = 89\\
& x_1^l, x_2^l, x_1^m - x_1^l, x_2^m - x_2^l, x_1^n - x_1^m, x_2^n - x_2^m \geq 0\\
& w_1, w_2 > 0, w_1 + w_2 = 1.
\end{aligned} \tag{19}
$$

Substituting different $\alpha \in [0,1]$, weights $(w_1, w_2) > 0$ with $w_1 + w_2 = 1$ in (19) and solving each resulted optimization model, a set of optimal solutions are obtained, which are considered as the non-dominated solutions of the FFMOLPP (16). The triangular fuzzy optimal objective values and their corresponding real valued expressions are computed based on the concept of ranking function, which are shown in the following Table 2.

Example 3. Consider the following FFMOLPP, which is initially solved by Arana-Jiménez [5]:

Table 2: Non-dominated solutions, fuzzy valued optimal objective functions and their real valued expressions of Example 2

α	w	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{Z}_1$	$\tilde{Z}_2$	$R(\tilde{Z}_1)$	$R(\tilde{Z}_2)$
0.1	(0.2,0.8)	(0,6.41667,6.41667)	(0,1.25,1.25)	(0,76.6667,86.8333)	(0,28,40.6667)	60.0417	24.1667
	(0.4,0.6)	(0,0,0)	(2.5,2.5,3.6)	(20,25,46.8)	(10,17.5,43.2)	29.2	22.05
	(0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,1.25,5.1)	(0,12.5,66.3)	(0,8.75,61.2)	22.825	19.675
0.2	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,1.25,5.1)	(0,12.5,66.3)	(0,8.75,61.2)	22.825	19.675
0.3	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,1.25,5.1)	(0,12.5,66.3)	(0,8.75,61.2)	22.825	19.675
0.4	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,1.25,5.1)	(0,12.5,66.3)	(0,8.75,61.2)	22.825	19.675
0.5	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,1.25,5.1)	(0,12.5,66.3)	(0,8.75,61.2)	22.825	19.675
0.6	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,1.25,5.1)	(0,12.5,66.3)	(0,8.75,61.2)	22.825	19.675
0.7	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,1.25,5.1)	(0,12.5,66.3)	(0,8.75,61.2)	22.825	19.675
0.8	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,1.25,5.1)	(0,12.5,66.3)	(0,8.75,61.2)	22.825	19.675
0.9	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,1.25,5.1)	(0,12.5,66.3)	(0,8.75,61.2)	22.825	19.675

$$Max\ \widetilde{Z}_1(\widetilde{X}) = (\frac{7}{5}, 4, \frac{43}{7}) \otimes \tilde{x}_1 \oplus (5, 7, 12) \otimes \tilde{x}_2 \oplus (\frac{39}{4}, 11, \frac{33}{2}) \otimes \tilde{x}_3$$
$$Max\ \widetilde{Z}_2(\widetilde{X}) = (3, 4, 6) \otimes \tilde{x}_1 \oplus (\frac{10}{3}, 5, 9) \otimes \tilde{x}_2 \oplus (4, 5, 10) \otimes \tilde{x}_3$$

subject to

$$(2, 5, 8) \otimes \tilde{x}_1 \oplus (3, \frac{41}{6}, 10) \otimes \tilde{x}_2 \oplus (5, \frac{31}{3}, 18) \otimes \tilde{x}_3 \preceq (6, \frac{50}{3}, 30) \quad (20)$$
$$(4, \frac{32}{3}, 12) \otimes \tilde{x}_1 \oplus (5, \frac{73}{6}, 20) \otimes \tilde{x}_2 \oplus (7, \frac{105}{6}, 30) \otimes \tilde{x}_3 \preceq (10, 30, 50)$$
$$(3, 5, 7) \otimes \tilde{x}_1 \oplus (5, 15, 20) \otimes \tilde{x}_2 \oplus (5, 10, 15) \otimes \tilde{x}_3 \preceq (2, \frac{145}{6}, 30)$$
$$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \succcurlyeq 0.$$

Solution:

The FFMOLPP (20) is formulated into the following model using fuzzy α-cuts in the objective functions and arithmetic operations of TFNs in the constraints:

$$\begin{aligned} Max\ \widetilde{Z}_1(\widetilde{X}) = & [\frac{7}{5} + 2.6\alpha, \frac{43}{7} - 2.1429\alpha][(1-\alpha)x_1^l + \alpha x_1^m, (1-\alpha)x_1^n + \alpha x_1^m] \\ & + [5 + 2\alpha, 12 - 5\alpha][(1-\alpha)x_2^l + \alpha x_2^m, (1-\alpha)x_2^n + \alpha x_2^m] \\ & + [\frac{39}{4} + 1.25\alpha, \frac{33}{2} - 5.5\alpha][(1-\alpha)x_3^l + \alpha x_3^m, (1-\alpha)x_3^n + \alpha x_3^m] \\ Max\ \widetilde{Z}_2(\widetilde{X}) = & [3 + \alpha, 6 - 2\alpha][(1-\alpha)x_1^l + \alpha x_1^m, (1-\alpha)x_1^n + \alpha x_1^m] \\ & + [\frac{10}{3} + 1.6667\alpha, 9 - 4\alpha][(1-\alpha)x_2^l + \alpha x_2^m, (1-\alpha)x_2^n + \alpha x_2^m] \\ & + [4 + \alpha, 10 - 5\alpha][(1-\alpha)x_3^l + \alpha x_3^m, (1-\alpha)x_3^n + \alpha x_3^m] \end{aligned}$$

subject to

$$(2x_1^l + 3x_2^l + 5x_3^l, 5x_1^m + \frac{41}{6}x_2^m + \frac{31}{3}x_3^m, 8x_1^n + 10x_2^n + 18x_3^n) \preceq (6, \frac{50}{3}, 30)$$
$$(4x_1^l + 5x_2^l + 7x_3^l, \frac{32}{3}x_1^m + \frac{73}{6}x_2^m + \frac{105}{6}x_3^m, 12x_1^n + 20x_2^n + 30x_3^n) \preceq (10, 30, 50)$$
$$(3x_1^l + 5x_2^l + 5x_3^l, 5x_1^m + 15x_2^m + 10x_3^m, 7x_1^n + 20x_2^n + 15x_3^n) \preceq (2, \frac{145}{6}, 30)$$
$$x_1^l, x_2^l, x_3^l, x_1^m - x_1^l, x_2^m - x_2^l, x_3^m - x_3^l, x_1^n - x_1^m, x_2^n - x_2^m, x_3^n - x_3^m \geq 0. \quad (21)$$

The model (21) is equivalently converted into the following interval valued optimization model, which contains the following deterministic linear constraints using the concept of centroid of TFNs:

$$
\begin{aligned}
Max\ \widetilde{Z}_1(\widetilde{X}) =&[(\frac{7}{5}+2.6\alpha)(1-\alpha)x_1^l+\alpha(\frac{7}{5}+2.6\alpha)x_1^m\\
&+(5+2\alpha)(1-\alpha)x_2^l+\alpha(5+2\alpha)x_2^m+(\frac{39}{4}+1.25\alpha)(1-\alpha)x_3^l\\
&+\alpha(\frac{39}{4}+1.25\alpha)x_3^m,\\
&(\frac{43}{7}-2.1429\alpha)(1-\alpha)x_1^n+\alpha(\frac{43}{7}-2.1429\alpha)x_1^m+(12-5\alpha)(1-\alpha)x_2^n\\
&+\alpha(12-5\alpha)x_2^m+(\frac{33}{2}-5.5\alpha)(1-\alpha)x_3^n+\alpha(\frac{33}{2}-5.5\alpha)x_3^m]\\
Max\ \widetilde{Z}_2(\widetilde{X}) =&[(3+\alpha)(1-\alpha)x_1^l+\alpha(3+\alpha)x_1^m+(\frac{10}{3}+1.6667\alpha)(1-\alpha)x_2^l\\
&+\alpha(\frac{10}{3}+1.6667\alpha)x_2^m+(4+\alpha)(1-\alpha)x_3^l+\alpha(4+\alpha)x_3^m,\\
&(6-2\alpha)(1-\alpha)x_1^n+\alpha(6-2\alpha)x_1^m+(9-4\alpha)(1-\alpha)x_2^n\\
&+\alpha(9-4\alpha)x_2^m+(10-5\alpha)(1-\alpha)x_3^n+\alpha(10-5\alpha)x_3^m]
\end{aligned}
\tag{22}
$$

subject to

$$
\begin{aligned}
&2x_1^l+5x_1^m+8x_1^n+3x_2^l+\frac{41}{6}x_2^m+10x_2^n+5x_3^l+\frac{31}{3}x_3^m+18x_3^n\le 52.6667\\
&4x_1^l+\frac{32}{3}x_1^m+12x_1^n+5x_2^l+\frac{73}{6}x_2^m+20x_2^n+7x_3^l+\frac{105}{6}x_3^m+30x_3^n\le 90\\
&3x_1^l+5x_1^m+7x_1^n+5x_2^l+15x_2^m+20x_2^n+5x_3^l+10x_3^m+15x_3^n\le 56.1667\\
&x_1^l,x_2^l,x_3^l,x_1^m-x_1^l,x_2^m-x_2^l,x_3^m-x_3^l,x_1^n-x_1^m,x_2^n-x_2^m,x_3^n-x_3^m\ge 0.
\end{aligned}
$$

The FFMOLPP (22) is finally expressed as the following single objective deterministic linear optimization model based on the concepts of the proposed methodology:

$$
\begin{aligned}
&Max\ Z(X)\\
&=w_1[(\frac{7}{5}+2.6\alpha)(1-\alpha)x_1^l+\alpha(7.5429+0.4571\alpha)x_1^m+(\frac{43}{7}-2.1429\alpha)(1-\alpha)x_1^n\\
&\quad+(5+2\alpha)(1-\alpha)x_2^l+\alpha(17-3\alpha)x_2^m+(12-5\alpha)(1-\alpha)x_2^n\\
&\quad+(\frac{39}{4}+1.25\alpha)(1-\alpha)x_3^l+\alpha(26.25-4.25\alpha)x_3^m+(\frac{33}{2}-5.5\alpha)(1-\alpha)x_3^n]\\
&\quad+w_2[(3+\alpha)(1-\alpha)x_1^l+\alpha(9-\alpha)x_1^m+(6-2\alpha)(1-\alpha)x_1^n\\
&\quad+(\frac{10}{3}+1.6667\alpha)(1-\alpha)x_2^l+\alpha(12.3333-2.3333\alpha)x_2^m+(9-4\alpha)(1-\alpha)x_2^n\\
&\quad+(4+\alpha)(1-\alpha)x_3^l+\alpha(14-4\alpha)x_3^m+(10-5\alpha)(1-\alpha)x_3^n]
\end{aligned}
$$

subject to

$$2x_1^l+5x_1^m+8x_1^n+3x_2^l+\frac{41}{6}x_2^m+10x_2^n+5x_3^l+\frac{31}{3}x_3^m+18x_3^n\leq 52.6667$$
$$4x_1^l+\frac{32}{3}x_1^m+12x_1^n+5x_2^l+\frac{73}{6}x_2^m+20x_2^n+7x_3^l+\frac{105}{6}x_3^m+30x_3^n\leq 90$$
$$3x_1^l+5x_1^m+7x_1^n+5x_2^l+15x_2^m+20x_2^n+5x_3^l+10x_3^m+15x_3^n\leq 56.1667$$
$$x_1^l,x_2^l,x_3^l,x_1^m-x_1^l,x_2^m-x_2^l,x_3^m-x_3^l,x_1^n-x_1^m,x_2^n-x_2^m,x_3^n-x_3^m\geq 0,$$
$$w_1,w_2>0,w_1+w_2=1. \tag{23}$$

On solving the problem (23) with different values of $\alpha\in[0,1]$ and different weights $w_1,\ w_2>0$ satisfying $w_1+w_2=1$, a set of solutions are generated, which are considered as the non-dominated solutions of the FFMOLPP (20). The triangular fuzzy valued optimal objective values are computed at the non-dominated solutions, which are shown in the Table 3.

Example 4 (Practical problem)**.** A company manufactures two types of products A and B with the profit amount of Rs. 5 and Rs. 3 per unit, respectively. The import of the products A and B are required as two units and eight units, respectively, in a day. The processing time required to manufacture the products A and B are 2hr. and 3hr. per unit, respectively, in a day not exceeding 5hr and the raw materials required are 4kg. and 5kg. per unit, respectively, not exceeding 10kg. The company aims to maximize the profit and maximize the import of products. The parameters and variables are considered as fuzzy numbers because the data varies due to insufficient labourers, supply of raw materials, defective machine, bad weather condition, etc.

Consider the formulation of the above problem as the following FFMOLPP:

$$\begin{aligned}
&Max\ \widetilde{Z}_1(\widetilde{X})=(4,5,6)\otimes\tilde{x}_1\oplus(2,3,4)\otimes\tilde{x}_2\\
&Max\ \widetilde{Z}_2(\widetilde{X})=(1,2,3)\otimes\tilde{x}_1\oplus(7,8,9)\otimes\tilde{x}_2\\
&\text{subject to}\\
&(3,4,5)\otimes\tilde{x}_1\oplus(4,5,7)\otimes\tilde{x}_2\preceq(9,10,11)\\
&(1,2,4)\otimes\tilde{x}_1\oplus(1,3,5)\otimes\tilde{x}_2\preceq(3,5,7)\\
&\tilde{x}_1,\tilde{x}_2\succcurlyeq 0
\end{aligned} \tag{24}$$

Solution:

According to the proposed methodology, FFMOLPP (24) is expressed into

Table 3: Non-dominated solutions and fuzzy valued optimal objective values of Example 3

α	w	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{Z}_1$	$\tilde{Z}_2$	$R(\tilde{Z}_1)$	$R(\tilde{Z}_2)$
0.1	(0.2,0.8) (0.4,0.6) (0.5,0.5)	(0,0,5.462967)	(0,0,0.8962967)	(0,0,0)	(0,0,44.3138)	(0,0,40.8445)	11.0785	10.2111
	(0.6,0.4)	(0,0,0.90004)	(0,0,1.026666)	(0,0,1.95554)	(0,0,50.1152)	(0,0,34.1956)	12.5288	8.5489
	(0.8,0.2)	(0,0,0)	(0,0,1.11667)	(0,0,2.255553)	(0,0,50.6167)	(0,0,32.6056)	12.6542	8.1514
0.2	(0.2,0.8) (0.4,0.6)	(0,0,5.462967)	(0,0,0.8962967)	(0,0,0)	(0,0,44.3138)	(0,0,40.8445)	11.0785	10.2111
	(0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,0,0.79697)	(1.34091,1.34091 ,1.34091)	(13.0739,14.75 ,31.6887)	(5.3636,6.7046 ,20.5818)	18.5657	9.8387
0.3	(0.2,0.8)	(2.970483,2.970483 ,2.970483)	(0.2902363,0.2902363 ,0.2902363)	(0,0,0)	(5.6099,13.9136 ,21.7301)	(9.8789,13.3331 ,20.4350)	13.7918	14.2450
	(0.4,0.6)	(1.106286,1.106286 ,1.106286)	(0.3208687,0.3208687 ,0.3208687)	(0.8912555,0.8912555 ,0.8912555)	(11.8429,16.475 ,25.3519)	(7.9534,10.4858 ,18.4381)	17.5362	11.8408
	(0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0.3957865,0.3957865 ,0.3957865)	(1.344508,1.344508 ,1.344508)	(15.0879,17.5601 ,26.9338)	(6.6973,8.7015 ,17.0072)	19.2855	10.2769
0.4	(0.2,0.8)	(2.970483,2.970483 ,2.970483)	(0.2902363,0.2902363 ,0.2902363)	(0,0,0)	(5.6099,13.9136 ,21.7301)	(9.8789,13.3331 ,20.4350)	13.7918	14.2450
	(0.4,0.6)	(1.106286,1.106286 ,1.106286)	(0.3208687,0.3208687 ,0.3208687)	(0.8912555,0.8912555 ,0.8912555)	(11.8429,16.475 ,25.3519)	(7.9534,10.4858 ,18.4381)	17.5362	11.8408
	(0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0.3957865,0.3957865 ,0.3957865)	(1.344508,1.344508 ,1.344508)	(15.0879,17.5601 ,26.9338)	(6.6973,8.7015 ,17.0072)	19.2855	10.2769
0.5	(0.2,0.8)	(2.970483,2.970483 ,2.970483)	(0.2902363,0.2902363 ,0.2902363)	(0,0,0)	(5.6099,13.9136 ,21.7301)	(9.8789,13.3331 ,20.4350)	13.7918	14.2450
	(0.4,0.6)	(1.106286,1.106286 ,1.106286)	(0.3208687,0.3208687 ,0.3208687)	(0.8912555,0.8912555 ,0.8912555)	(11.8429,16.475 ,25.3519)	(7.9534,10.4858 ,18.4381)	17.5362	11.8408
	(0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0.3957865,0.3957865 ,0.3957865)	(1.344508,1.344508 ,1.344508)	(15.0879,17.5601 ,26.9338)	(6.6973,8.7015 ,17.0072)	19.2855	10.2769
0.6	(0.2,0.8) (0.4,0.6)	(2.970483,2.970483 ,2.970483)	(0.2902363,0.2902363 ,0.2902363)	(0,0,0)	(5.6099,13.9136 ,21.7301)	(9.8789,13.3331 ,20.4350)	13.7918	14.2450
	(0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0.3957865,0.3957865 ,0.3957865)	(1.344508,1.344508 ,1.344508)	(15.0879,17.5601 ,26.9338)	(6.6973,8.7015 ,17.0072)	19.2855	10.2769
0.7	(0.2,0.8)	(0,3.970588 ,3.970588)	(0,0,0)	(0,0,0)	(0,15.8824 ,24.3908)	(0,15.8824 ,23.8235)	14.0389	13.8971
	(0.4,0.6)	(0,3.297802 ,3.297802)	(0,0.4740881 ,0.4740881)	(0,0,0)	(0,16.5098 ,25.9470)	(0,15.5616 ,24.0536)	14.7417	13.7942
	(0.5,0.5) (0.6,0.4) (0.8,0.2)	(0,0,0)	(0,0.4774971 ,0.4774971)	(0.1290657,1.552359 ,1.552359)	(1.2584,20.4184 ,31.3439)	(0.5163,10.1493 ,19.8211)	18.3598	10.1590
0.8	(0.2,0.8)	(0,3.970588 ,3.970588)	(0,0,0)	(0,0,0)	(0,15.8824 ,24.3908)	(0,15.8824 ,23.8235)	14.0389	13.8971
	(0.4,0.6) (0.5,0.5)	(0,3.297802 ,3.297802)	(0,0.4740881 ,0.4740881)	(0,0,0)	(0,16.5098 ,25.9470)	(0,15.5616 ,24.0536)	14.7417	13.7942
	(0.6,0.4)	(0,0,0)	(0,0.486895 ,0.486895)	(0,1.565015 ,1.565015)	(0,20.6234 ,31.6655)	(0,10.2596 ,20.0322)	18.2281	10.1379
	(0.8,0.2)	(0,0,0)	(0,0.432284 ,0.432284)	(0,1.601997 ,1.601997)	(0,20.648 ,31.6204)	(0,10.1714 ,19.9105)	18.2291	10.0633
0.9	(0.2,0.8)	(0,3.970588 ,3.970588)	(0,0,0)	(0,0,0)	(0,15.8824 ,24.3908)	(0,15.8824 ,23.8235)	14.0389	13.8971
	(0.4,0.6) (0.5,0.5)	(0,3.297802 ,3.297802)	(0,0.4740881 ,0.4740881)	(0,0,0)	(0,16.5098 ,25.9470)	(0,15.5616 ,24.0536)	14.7417	13.7942
	(0.6,0.4)	(0,0,0)	(0,0.486895 ,0.486895)	(0,1.565015 ,1.565015)	(0,20.6234 ,31.6655)	(0,10.2596 ,20.0322)	18.2281	10.1379
	(0.8,0.2)	(0,0,0)	(0,0.432284 ,0.432284)	(0,1.601997 ,1.601997)	(0,20.648 ,31.6204)	(0,10.1714 ,19.9105)	18.2291	10.0633

the following model using fuzzy α-cuts in the objective functions and arithmetic operations of TFNs in the constraints:

$$\begin{aligned} Max\ \widetilde{Z}_1(\widetilde{X}) &= [4+\alpha, 6-\alpha][(1-\alpha)x_1^l + \alpha x_1^m, (1-\alpha)x_1^n + \alpha x_1^m] \\ &\quad + [2+\alpha, 4-\alpha][(1-\alpha)x_2^l + \alpha x_2^m, (1-\alpha)x_2^n + \alpha x_2^m] \\ Max\ \widetilde{Z}_2(\widetilde{X}) &= [1+\alpha, 3-\alpha][(1-\alpha)x_1^l + \alpha x_1^m, (1-\alpha)x_1^n + \alpha x_1^m] \\ &\quad + [7+\alpha, 9-\alpha][(1-\alpha)x_2^l + \alpha x_2^m, (1-\alpha)x_2^n + \alpha x_2^m] \end{aligned} \tag{25}$$

subject to

$$\begin{aligned} &(3x_1^l + 4x_2^l, 4x_1^m + 5x_2^m, 5x_1^n + 7x_2^n) \preceq (9, 10, 11) \\ &(x_1^l + x_2^l, 2x_1^m + 3x_2^m, 4x_1^n + 5x_2^n) \preceq (3, 5, 7) \\ &x_1^l, x_2^l, x_1^m - x_1^l, x_2^m - x_2^l, x_1^n - x_1^m, x_2^n - x_2^m \geq 0. \end{aligned}$$

The model (25) is equivalently converted into the following interval valued optimization model with deterministic linear constraints based on the centroid concept as discussed:

$$\begin{aligned} Max\ \widetilde{Z}_1(\widetilde{X}) &= [(4+\alpha)(1-\alpha)x_1^l + \alpha(4+\alpha)x_1^m + (2+\alpha)(1-\alpha)x_2^l + \alpha(2+\alpha)x_2^m, \\ &\quad (6-\alpha)(1-\alpha)x_1^n + \alpha(6-\alpha)x_1^m + (4-\alpha)(1-\alpha)x_2^n + \alpha(4-\alpha)x_2^m] \\ Max\ \widetilde{Z}_2(\widetilde{X}) &= [(1-\alpha^2)x_1^l + \alpha(1+\alpha)x_1^m + (7+\alpha)(1-\alpha)x_2^l + \alpha(7+\alpha)x_2^m, \\ &\quad (3-\alpha)(1-\alpha)x_1^n + \alpha(3-\alpha)x_1^m + (9-\alpha)(1-\alpha)x_2^n + \alpha(9-\alpha)x_2^m] \end{aligned}$$

subject to

$$\begin{aligned} &3x_1^l + 4x_1^m + 5x_1^n + 4x_2^l + 5x_2^m + 7x_2^n \leq 30 \\ &x_1^l + 2x_1^m + 4x_1^n + x_2^l + 3x_2^m + 55x_2^n \leq 15 \\ &x_1^l, x_2^l, x_1^m - x_1^l, x_2^m - x_2^l, x_1^n - x_1^m, x_2^n - x_2^m \geq 0. \end{aligned} \tag{26}$$

The FFMOLPP (24) is finally modeled as the following single objective deterministic linear optimization model based on the weighting sum approach and Theorem 1:

$$\begin{aligned} Max\ Z(X) &= w_1[(4+\alpha)(1-\alpha^2)x_1^l + 10\alpha x_1^m + (6-\alpha)(1-\alpha)x_1^n \\ &\quad + (2+\alpha)(1-\alpha)x_2^l + 6\alpha x_2^m + (4-\alpha)(1-\alpha)x_2^n] \\ &\quad + w_2[(1-\alpha^2)x_1^l + 4\alpha x_1^m + (3-\alpha)(1-\alpha)x_1^n \\ &\quad + (7+\alpha)(1-\alpha)x_2^l + 16\alpha x_2^m + (9-\alpha)(1-\alpha)x_2^n] \end{aligned}$$

subject to (27)

$$3x_1^l + 4x_1^m + 5x_1^n + 4x_2^l + 5x_2^m + 7x_2^n \leq 30$$
$$x_1^l + 2x_1^m + 4x_1^n + x_2^l + 3x_2^m + 55x_2^n \leq 15$$
$$x_1^l, x_2^l, x_1^m - x_1^l, x_2^m - x_2^l, x_1^n - x_1^m, x_2^n - x_2^m \geq 0$$
$$w_1, w_2 > 0, w_1 + w_2 = 1$$

On solving problem (27) by substituting different values of $\alpha \in [0, 1]$ and different weights $w_1,\ w_2 > 0$ satisfying $w_1 + w_2 = 1$, a set of optimal solutions are generated, which are considered as the non-dominated solutions of the FFMOLPP (24). The triangular fuzzy valued optimal objective values are computed at the non-dominated solutions and also their real valued expressions are calculated using ranking function of TFNs, which are shown in Table 4.

4.1 Comparative result discussions

The FFMOLPP considered in the numerical examples are existing problems, which are solved using our proposed methodology. The computational results of Examples 1, 2, and 3 are incorporated in Tables 1, 2, and 3, respectively. The triangular fuzzy valued optimal objective values are computed at the fuzzy Pareto optimal solutions and their equivalent real valued expressions are determined using the concept of ranking function.

As it is observed, the ranking function values calculated for the optimal objective functions due to our proposed method and existing methods are closely approaching towards each other in certain cases. Besides, the methodology proposed in this paper derives a set of fuzzy Pareto optimal solutions from, which some of the solutions generate better optimal objective values based on the values of their ranking function.

In this context, some of the comparative results are incorporated in Tables 5, 6, and 7 for Examples 1, 2, and 3, respectively. The real-valued expressions of fuzzy optimal objective values are evaluated using ranking functions in Tables 5, 6, and 7 which are further comparatively discussed with existing methods through the following figures, Figures 3, 4, and 5, respectively. Since Exam-

Table 4: Non-dominated solutions, fuzzy valued optimal objective functions and their real valued expressions of Example 4

α	w	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{Z}_1$	$\tilde{Z}_2$	$R(\tilde{Z}_1)$	$R(\tilde{Z}_2)$
0.1	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4)	(0,0,0)	(1.666667,1.666667, 1.666667)	(3.3333,5,6.6667)	(11.6667,13.3333,15)	5	13.3333
	(0.8,0.2)	(2.142857,2.142857, 2.142857)	(0,0,0)	(8.5714,10.7143, 12.8571)	(2.142857,4.2857, 6.4286)	10.7143	4.2857
0.2	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4)	(0,0,0)	(1.666667,1.666667, 1.666667)	(3.3333,5,6.6667)	(11.6667,13.3333,15)	5	13.3333
	(0.8,0.2)	(2.142857,2.142857, 2.142857)	(0,0,0)	(8.5714,10.7143, 12.8571)	(2.142857,4.2857, 6.4286)	10.7143	4.2857
0.3	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4)	(0,0,0)	(1.666667,1.666667, 1.666667)	(3.3333,5,6.6667)	(11.6667,13.3333,15)	5	13.3333
	(0.8,0.2)	(2.142857,2.142857, 2.142857)	(0,0,0)	(8.5714,10.7143, 12.8571)	(2.142857,4.2857, 6.4286)	10.7143	4.2857
0.4	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4)	(0,0,0)	(1.666667,1.666667, 1.666667)	(3.3333,5,6.6667)	(11.6667,13.3333,15)	5	13.3333
	(0.8,0.2)	(2.142857,2.142857, 2.142857)	(0,0,0)	(8.5714,10.7143, 12.8571)	(2.142857,4.2857, 6.4286)	10.7143	4.2857
0.5	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4)	(0,0,0)	(1.666667,1.666667, 1.666667)	(3.3333,5,6.6667)	(11.6667,13.3333,15)	5	13.3333
	(0.8,0.2)	(2.142857,2.142857, 2.142857)	(0,0,0)	(8.5714,10.7143, 12.8571)	(2.142857,4.2857, 6.4286)	10.7143	4.2857
0.6	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4)	(0,0,0)	(1.666667,1.666667, 1.666667)	(3.3333,5,6.6667)	(11.6667,13.3333,15)	5	13.3333
	(0.8,0.2)	(2.142857,2.142857, 2.142857)	(0,0,0)	(8.5714,10.7143, 12.8571)	(2.142857,4.2857, 6.4286)	10.7143	4.2857
0.7	(0.2,0.8) (0.4,0.6) (0.5,0.5) (0.6,0.4)	(0,0,0)	(1.666667,1.666667, 1.666667)	(3.3333,5,6.6667)	(11.6667,13.3333,15)	5	13.3333
	(0.8,0.2)	(0,2.5,2.5)	(0,0,0)	(0,12.5,15)	(0,5,7.5)	10	4.375
0.8	(0.2,0.8) (0.4,0.6) (0.5,0.5)	(0,0,0)	(0,1.875,1.875)	(0,5.625,7.5)	(0,15,16.875)	4.6875	11.7188
	(0.6,0.4) (0.8,0.2)	(0,2.5,2.5)	(0,0,0)	(0,12.5,15)	(0,5,7.5)	10	4.375
0.9	(0.2,0.8) (0.4,0.6) (0.5,0.5)	(0,0,0)	(0,1.875,1.875)	(0,5.625,7.5)	(0,15,16.875)	4.6875	11.7188
	(0.6,0.4) (0.8,0.2)	(0,2.5,2.5)	(0,0,0)	(0,12.5,15)	(0,5,7.5)	10	4.375

ples 1, 2, and 3 are of maximization, minimization and maximization types, respectively, the figures validate and justify the feasibility and effectiveness of the proposed solution approach.

Table 5: Comparative results of Example 1

Methods	$\tilde{Z}_1$	$\tilde{Z}_2$	$R(\tilde{Z}_1)$	$R(\tilde{Z}_2)$
Agarwal and Sharma [1]	(4, 20, 43)	(7, 24, 49)	21.75	26
Temelcan, Gonce Kocken, and Albayrak [27]	(12, 24, 32.33)	(20.33, 28.67, 37)	23.08	28.67
Proposed method	(4.3333,28.6667,38)	(8.6667,33,42.3333)	24.9167	29.25
Proposed method	(3.8889,3.8889,3.8889)	(12.6667,25.3333,34.1111)	24.3611	30.2222

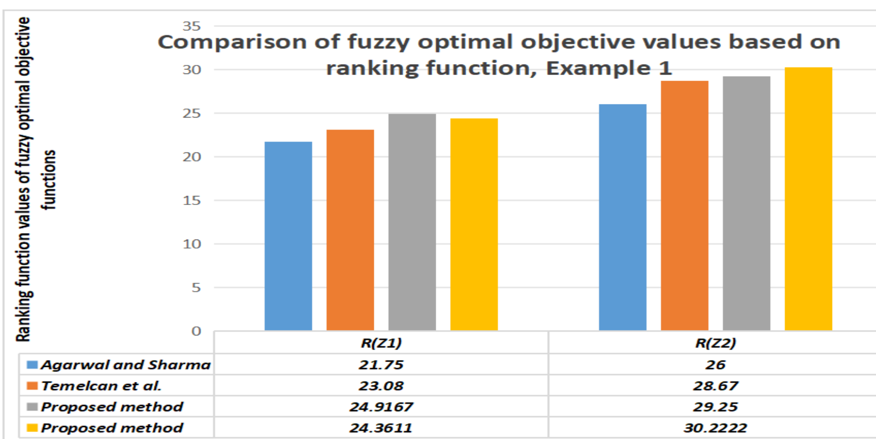

Figure 3: Comparative results on ranking function, Example 1 and Table 5

Table 6: Comparative results of Example 2

Methods	$\tilde{Z}_1$	$\tilde{Z}_2$	$R(\tilde{Z}_1)$	$R(\tilde{Z}_2)$
Yang, Cao, and Lin [30]	(12, 50, 66.3712)	(6, 23, 40.5008)	44.5928	23.1252
Proposed method	(20,25,46.8)	(10,17.5,43.2)	29.2	22.05

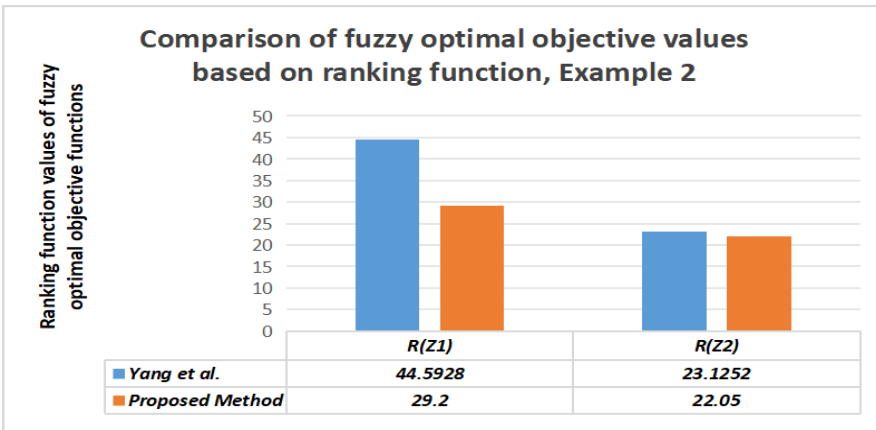

Figure 4: Comparative results on ranking function, Example 2 and Table 6

Table 7: Comparative results of Example 3

Methods	$\tilde{Z}_1$	$\tilde{Z}_2$	$R(\tilde{Z}_1)$	$R(\tilde{Z}_2)$
Arana-Jiménez [5]	(2.457143, 16.343755, 33.540027)	(2.4, 11.49366, 15.47613)	17.17117	10.2158625
Proposed method	(15.0879, 17.5601, 26.9338)	(6.6973, 8.7015, 17.0072)	19.2855	10.2769
Proposed method	(1.2584, 20.4184, 31.3439)	(0.5163, 10.1493, 19.8211)	18.3598	10.1590

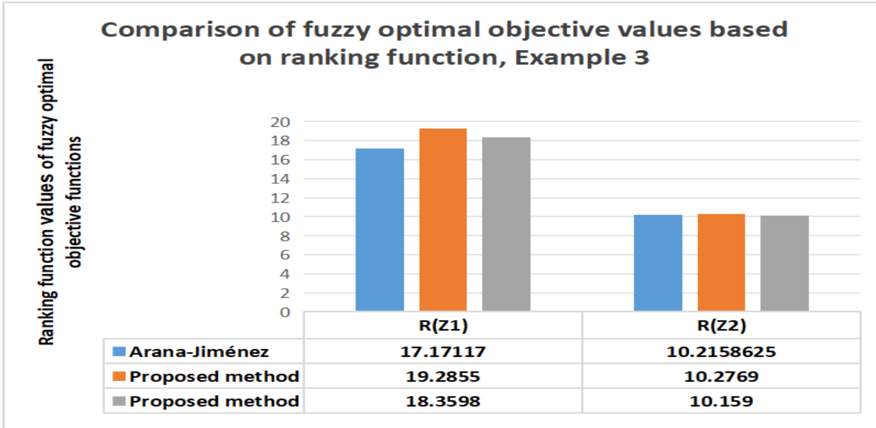

Figure 5: Comparative results on ranking function, Example 3 and Table 7

5 Conclusions

This paper developed a methodology that generates a set of fuzzy Pareto optimal solutions of an FFMOLPP, which is considered in a triangular fuzzy environment. The concepts of fuzzy cuts and centroid of TFNs are used to develop an equivalent interval valued MOLPP, which is further solved using weighting sum approach with different weight vectors and a theorem of interval analysis. Different fuzzy cuts and positive weight vectors derive the set of fuzzy Pareto optimal solutions. The solution approach developed is computationally simple and efficient, which provides a set of fuzzy valued solutions comprising the compromise solution of the FFMOLPP whereas most of the existing methods generate only one solution as the compromise solution. DM compares the resulted fuzzy optimal objective values using ranking function concept to decide the compromise solution. The incorporated algorithm and flowchart narrate the detailed steps of the proposed solution approach. In the numerical section, three existing problems along with one practical problem are solved and the results computed are discussed as compared to the existing methods, which shows the feasibility, effectiveness and acceptability of the proposed methodology. LINGO software is used for the computational works of the numerical section. The developed approach can also be applicable for FFMOLPP designed with trapezoidal, pentagonal and hexagonal fuzzy numbers etc. As the future scope of research, fully fuzzy nonlinear MOOP, fully fuzzy bi-level MOOP in various fuzzy environment can be studied to develop some new efficient solution methodologies.

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